HYDRAULICS AND FLUID MECHANICS

A TEXT-BOOK COVERING THE SYLLABUSES OF THE B.Sc. (ENG.), I.C.E., AND I.MECH.E. EXAMINATIONS IN THIS SUBJECT

BY

E. H. LEWITT, B.Sc., Ph.D.

M.I.AERO.E., A.M.I.MECH.E.

Formerly Lecturer in the Department of Mechanical Engineering of the Imperial College of Science and Technology, South Kensington

FIRST EDITION

LONDON

SIR ISAAC PITMAN & SONS, LTD.

PREFACE TO FIRST EDITION

THIS book is intended for students working for an engineering degree. Although particularly written for the internal and external degrees of the University of London, it will be found to cover the course of other universities and to be suitable for the final examinations of the professional engineering institutions.

The book deals from first principles with the theory of hydraulics and its applications; no attempt has been made to deal with design problems, which are beyond the scope of the engineering degree. The mathematics have been kept as elementary as possible and many of the standard proofs have been simplified. Numerous worked-out examples from past B.Sc. examinations of the University of London are included, and the majority of the exercises given at the end of each chapter are taken from this source. The illustrations, with the exception of those from actual photographs, are intended as diagrammatic only.

This book forms the first of a series intended to cover the various subjects included in the engineering degree examinations. It will also, it is hoped, be found eminently suitable for students taking the advanced courses for the National Certificates which are now being issued jointly by the Board of Education and the Engineering Institutions.

The author wishes to thank the Hydro-Electric Department of Sir W. G. Armstrong, Whitworth and Co., and Messrs. Boving and Co., for supplying photographs of turbines, etc., and Mr. C. E. Rusbridge, B.Sc., for kindly checking the proofs and examples.

E. H. LEWITT

SOUTH KENSINGTON
1923

CONTENTS

LIST OF SYMBOLS

A	area
a	area
B	breadth
	centre of buoyancy
b	breadth
C	centre of pressure
	Chezy constant
	strength of vortex
C_c	coefficient of contraction
C_d	coefficient of discharge
C_f	frictional drag coefficient
C_v	coefficient of velocity
C_D	drag coefficient $= 2k_D$
C_L	lift coefficient $= 2k_L$
c	chord length
c_p	specific heat at constant pressure
c_V	specific heat at constant volume
D	diameter
	drag
D_f	frictional drag
d	differential coefficient
d	diameter
E	linear elastic modulus
	specific energy
	total energy
e	base of natural logarithms
e	efficiency
F_r	Froude number
f	Darcy's frictional coefficient
f'	Froude's frictional coefficient
f_c	circumferential stress
f_L	longitudinal stress
G	centre of area
	shear modulus
g	gravitational acceleration
H	enthalpy (total heat)
	height or head
H_d	heat drop
h	height or head
h_f	head lost due to friction
I	moment of inertia
	second moment of area
i	slope
J	Joule's equivalent
K	bulk modulus
k	adiabatic efficiency
	radius of gyration
k_D	drag coefficient
k_L	lift coefficient
L	length
	lift
l	length
M_a	Mach number
m	hydraulic mean depth
	Poisson's ratio
n	speed
P	power
	propulsive force
	total pressure
p	intensity of pressure
Q	heat absorbed
	quantity
q	viscous stress
R	radius
	total resistance
	viscous stress
R_e	Reynolds number
R_f	frictional resistance
R_w	wave resistance
r	radius
	resistance
S	entropy
s	specific gravity
T	temperature (absolute)
	tension
	time
	torque
t	temperature
	thickness
	time
U	internal energy
V	volume
V_s	specific volume of a gas

V velocity
V_f velocity of flow (turbines)
V_r relative velocity
V_w velocity of whirl
v velocity
v_c circumferential velocity
v_r radial velocity
v_s velocity of sound
W weight
 work
w weight per unit volume
$\bar{x}$ depth of centre of area

α angle of incidence
 angular acceleration
γ ratio of specific heats
η coefficient of viscosity
λ wavelength
μ coefficient of friction
ν kinematic viscosity
ρ absolute density (mass per unit volume)
σ surface tension
τ drag stress
ω angular velocity

UNITS

The units used will be mainly engineers' units, but British absolute units and c.g.s. absolute units will be used in certain problems. The units used will be given in the same order in which the three fundamental dimensions, mass, length and time (M, L and T), are generally stated. The following are the units of the three systems.

Engineers' Units (slug-ft-sec units)

Unit of mass = 1 slug
Unit of length = 1 ft
Unit of time = 1 sec
Unit of force = 1 Lb

The imperial Lb is the gravitational force in London on a certain mass of platinum which is kept in the Standards Department of the Board of Trade, Westminster.

The slug may be defined as a mass on which a force of 1 Lb produces an acceleration of 1 ft/sec^2.

If M is the mass of a body of weight W, the gravitational force on the body due to the earth's attraction produces an acceleration of g ft/sec^2; then, force $= W = M \times g$ Lb

or $$M = \frac{W}{g} \text{ slugs}$$

Hence, weight of 1 slug = 32·2 lb in the latitude of London.

British Absolute Units (lb-ft-sec units)

Unit of mass = 1 lb
Unit of length = 1 ft
Unit of time = 1 sec
Unit of force = 1 poundal

C.G.S. Absolute Units (g-cm-sec units)

Unit of mass = 1 g
Unit of length = 1 cm
Unit of time = 1 sec
Unit of force = 1 dyne

CHAPTER 1

STATIC PRESSURE OF A FLUID

1.1. Introduction. Mechanics of Fluids deals with the behaviour of fluids when subjected to changes of pressure, frictional resistance, flow through various types of duct, orifice and nozzle, impact of jets and production of power. It also includes the development and testing of theories devised to explain the various phenomena which occur.

In Fluid Mechanics a liquid is usually treated as an incompressible fluid, thus ignoring the small amount of compression due to its volumetric elasticity. The portion of the subject dealing with liquids is called Hydraulics, and, on account of its relative simplicity, it will be dealt with mainly in the first part of the book.

Gases and vapours are known as compressible fluids and present a more complex problem than liquids on account of their changes in density due to alteration in temperature and pressure. For this reason this part of the subject will be dealt with in the latter part of the book. The treatment of vapours differs from that of pure gases because the former are subjected to further condensation or evaporation of the moisture which may be in suspension. Problems on compressible fluids require some knowledge of Thermodynamics for their solution.

The chapters dealing with hydraulic machinery have been inserted at the end of the book.

All formulae, examples and solutions given in the text are in engineers' units (slug-ft-sec) unless otherwise stated.

1.2. Properties of Fluids. The term fluid is applied to all substances which offer no resistance to change of shape. Fluids may be divided into three classes: liquids, vapours and gases. Liquids offer great resistance to compression and are not greatly affected by change of temperature; vapour and gases are easily compressed and are more susceptible to temperature changes. Liquids have a bulk elastic modulus when under compression, and will store up energy in the same manner as a solid. The value of the bulk elastic modulus of water under compression is 300,000 Lb/in.^{2} As the contraction of volume of a liquid under compression is extremely small, it is usually ignored and the liquid is assumed to be incompressible.

A liquid will withstand a slight amount of tension due to the molecular attraction between the particles, which will cause an apparent shear resistance between two adjacent layers; this phenomenon is known as viscosity.

The coefficients of expansion of liquids are small, and, as the science of hydraulics deals with liquids under small ranges of temperature only, the effect of temperature changes may be ignored; consequently, the density of a liquid may be assumed to be constant.

The weight of water is 62·4 Lb/ft³.

The ratio between the density of any liquid and the density of water is known as the specific gravity of that liquid.

No liquid can exist as a liquid at zero pressure; in fact all known liquids vaporize at various pressures above zero, depending on the temperature.

Water vaporizes at a pressure of 0·34 Lb/in.² at 20°C; below this pressure it cannot exist as a liquid. There are also dissolved gases in water which are given off at low pressures and cause great inconvenience in hydraulic problems. For this reason care must be taken to prevent the pressure of water getting below 8 ft of water absolute, at which pressure the dissolved gases are given off and vaporization is also liable to commence.

1.3. Pressure of a Fluid. The intensity of pressure of a fluid is the pressure per unit area. If the pressure is measured in pounds weight (Lb) and the area in inches, the intensity of pressure will be in pounds per square inch (Lb/in.²).

Let a fluid be under a uniform pressure and let its total pressure on an area of a in.² be P Lb. Let p be the intensity of pressure. Then,

$$p = \frac{P}{a} \text{ Lb/in.}^2$$

At any point in a fluid the intensity of pressure acts equally in all directions. If a fluid is contained in a vessel and is under a uniform intensity of pressure throughout, a slight increase in the intensity of pressure at one part will be immediately transmitted to all parts of the vessel.

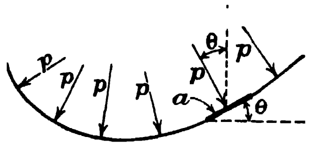


FIG. 1

The pressure of a fluid on a surface will always act normal to the surface.

Suppose a curved surface be under a uniform pressure p (Fig. 1); the direction of p at any point will be at right angles to the surface at that point. Consider a small element of the surface of area a, inclined to the horizontal at an angle θ. Then, as p acts normal to a, its inclination to the vertical will be θ.

Total pressure on area $a = pa$

Vertical component of total pressure on $a = pa \cos \theta$

But, $a \cos \theta =$ horizontal projection of area a

Therefore,

vertical component of pressure on $a = p \times$ horizontal projection of area a

From this it will be seen that, if any shaped surface is under uniform pressure, the total pressure acting on it in any given direction is the intensity of pressure multiplied by the projected area normal to the given direction.

EXAMPLE 1

A hemispherical dome (Fig. 2) of 2 ft radius contains a fluid under a pressure of 120 Lb/ft². Find the total force tending to lift the dome.

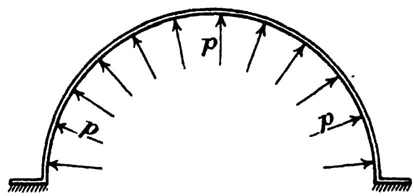


FIG. 2

$$\begin{aligned}\text{Total vertical force} &= p \times \text{horizontal projected area}\\ &= p \times \pi\,(\text{radius})^2\\ &= 120 \times \pi \times 2^2\\ &= 1{,}509 \text{ Lb}\end{aligned}$$

1.4. The Hydraulic Press. The hydraulic press is a machine by which large weights may be lifted by the application of a much

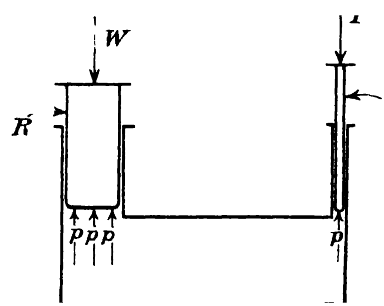


FIG. 3

smaller force. A diagrammatic view of a hydraulic press is shown in Fig. 3. The weight W is lifted by the large ram R, which is raised by the pressure of the fluid. This pressure is produced by the force F acting on the plunger P.

Let A = area of ram,
a = area of plunger,
p = intensity of pressure of fluid.

As the intensity of pressure is the same throughout the chamber,

$$W = pA$$

and $$F = pa$$

Equating the values of p from these two equations,

$$\frac{W}{A} = \frac{F}{a}$$

Therefore $$W = \frac{FA}{a}$$

Thus, the mechanical advantage obtained by means of this press is equal to the ratio of the areas of the ram and plunger.

This is the principle of the hydraulic lifting jack.

EXAMPLE 2

A hydraulic press has a ram of 5 in. diameter and a plunger of ½ in. diameter. What force would be required on the plunger to raise a weight of 1 ton on the ram? If the plunger had a stroke of 10 in., how many strokes would be necessary to lift the weight 3 ft? If the time taken to lift the weight is 12 min, what horse-power would be required to drive the plunger? Neglect all losses, and assume the motion of the weight is continuous.

$$\text{Force on plunger} = \frac{Wa}{A}$$

$$= 2{,}240 \times \left(\frac{\frac{1}{2}}{5}\right)^2 = 22{\cdot}4 \text{ Lb}$$

As the work done by plunger equals work done by ram,

$$22{\cdot}4 \times \frac{10}{12} \times n = 2{,}240 \times 3$$

where n = number of strokes of plunger.

Therefore $$n = \frac{2{,}240 \times 3}{22{\cdot}4} \times \frac{12}{10}$$

$$= 360$$

$$\text{Horse-power required} = \frac{22{\cdot}4 \times \frac{10}{12} \times 360}{12 \times 33{,}000}$$

$$= 0{\cdot}017$$

1.5. Pressure Head of a Liquid. A liquid is subjected to pressure due to its own weight; this pressure increases as the depth of the liquid increases. Consider a vessel containing a liquid of a depth H ft (Fig. 4). Let w be the weight in pounds of 1 ft^3 of the liquid. Then

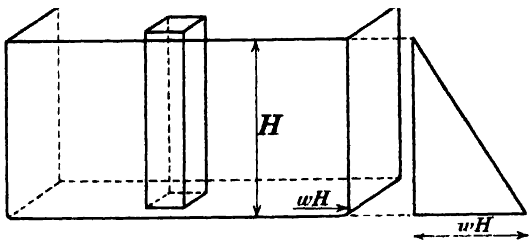


FIG. 4

the pressure at any point in the liquid will depend on the weight of liquid above that point.

Consider an area of 1 ft^2 on the bottom of the vessel. The pressure on this square foot is equal to the weight of the column of liquid above it, which it is supporting. This column is in the shape of a square prism of a height H, standing on its end. Then,

$$\text{total pressure on base of prism} = \text{weight of prism}$$
$$= wH$$

As the base has an area of 1 ft^2, this is the intensity of pressure p. Therefore,

$$p = wH \text{ Lb/ft}^2 \quad . \quad . \quad . \quad (1)$$

If w were the weight of 1 in.^{3} and H the height in inches, p would then be the intensity of pressure in pounds per square inch.

As $p = wH$, the intensity of pressure in a liquid due to its depth will vary directly with the depth.

As the pressure at any point in a liquid depends on the height of the free surface above that point, it is sometimes convenient to express a liquid pressure by the height of the free surface which would cause that pressure, i.e.

$$H = \frac{p}{w} \text{ (from eq. (1))}$$

The height of the free surface above any point is known as the static head at that point. In this case the static head is denoted by H.

Hence, the intensity of pressure of a liquid may be expressed as a pressure in pounds per square inch, or as an equivalent static head in feet of liquid; and one form may be converted to the other by means of the equation

$$p = wH$$

In using this equation care should be taken that the units of one side are the same as those of the other.

When dealing with fresh water w may be taken as 62·4 Lb/ft^3.

Referring to Fig. 4, the intensity of pressure at any point due to the weight of liquid above that point is in a vertical direction; but, as the pressure of a liquid acts equally in all directions, this pressure will cause an equal horizontal pressure on the side of the vessel. The intensity of pressure on the sides of the vessel will, therefore, be equal to wH at the bottom and decrease uniformly to zero at the free surface, as shown by the pressure diagram to the right of the figure.

Then, total pressure of liquid on side of vessel

$$\begin{aligned} &= \text{average pressure} \times \text{area of side} \\ &= \frac{wH}{2} \times \text{area of side} \end{aligned}$$

EXAMPLE 3

Find the pressure in tons per square inch at the bottom of the sea at a point where the depth is 7 miles. The weight of 1 ft^3 of sea water is 64 Lb.

$$\begin{aligned} p &= wH \\ &= 64 \times 7 \times 5{,}280 \text{ Lb/ft}^2 \\ &= \frac{64 \times 7 \times 5{,}280}{144 \times 2{,}240} \text{ tons/in.}^2 \\ &= 7{\cdot}34 \text{ tons/in.}^2 \end{aligned}$$

EXAMPLE 4

A rectangular tank 14 ft long and 5 ft wide contains water to a depth of 6 ft. Find the intensity of pressure on the base of the tank and the total pressure on the end.

$$\begin{aligned} \text{Pressure on base} &= wH \\ &= 62{\cdot}4 \times 6 \text{ Lb/ft}^2 \\ &= \frac{62{\cdot}4 \times 6}{144} \text{ Lb/in.}^2 \\ &= 2{\cdot}6 \text{ Lb/in.}^2 \\ \text{Maximum pressure on end} &= wH \\ \text{Average pressure on end} &= \frac{wH}{2} \\ \text{Total pressure on end} &= \text{average pressure} \times \text{area} \\ &= \frac{62{\cdot}4 \times 6}{2} \times 5 \times 6 \\ &= 5{,}620 \text{ Lb} \end{aligned}$$

1.6. Pressure of Atmosphere. The pressure of the atmosphere at the earth's surface is due to the weight of the column of air above. This cannot be calculated in the same way as a liquid because the air is compressible, and, consequently, the density will vary. The pressure of the atmosphere is measured by the height of the column of liquid it will support. This varies slightly according to the amount of moisture in the atmosphere; the average value may be taken as 14·7 Lb/in.^{2}, which is equivalent to a static head of 34 ft of water.

The pressure of a fluid is measured by some type of gauge. A gauge registers the pressure above atmosphere, and the pressure thus measured is termed gauge pressure. To convert gauge pressure to absolute pressure the reading of the barometer must be added.

If the pressure of the fluid is below atmospheric pressure it is measured by means of a vacuum gauge. A vacuum gauge gives the amount the pressure is below atmosphere; this must be subtracted from the atmospheric pressure in order to obtain absolute pressure. Thus, if the reading of the vacuum gauge is 24 ft of water, the absolute pressure will be $34 - 24 = 10$ ft of water.

EXAMPLE 5

The reading of the barometer is 76 cm of mercury. If the specific gravity of mercury is 13·6, convert this pressure to feet of water and pounds per square inch.

$$\text{Centimetres of water} = 76 \times 13{\cdot}6 = 1{,}033$$

$$\text{Inches of water} = \frac{1{,}033}{2{\cdot}54} = 407$$

$$\text{Feet of water} = \frac{407}{12} = 33{\cdot}92$$

$$\text{Pounds per square foot} = wH = 62{\cdot}4 \times 33{\cdot}92 = 2{,}117$$

$$\text{Pounds per square inch} = \frac{2{,}117}{144} = 14{\cdot}7$$

1.7. Pressure Gauges.* 1. PIEZOMETER TUBE. The pressure in a pipe or vessel, full of a liquid, may be measured by inserting, vertically, a glass tube with open ends, into the vessel. The liquid will rise in the tube to a height equal to the equivalent static head of the pressure in the vessel. This simple type of pressure gauge is known as a piezometer tube.

2. U-TUBE. The pressure of a fluid may be measured with a glass U-tube containing a heavier fluid which does not mix with the fluid of which the pressure is required.

* For Chattock Tilting Gauge, see § 13.14.

Let the pipe in Fig. 5 contain water under a pressure of h in. of water, and let the U-tube contain a liquid of specific gravity s. If the left limb of the U-tube be open to the atmosphere and the right limb, containing water, be connected to the pipe, the pressure in the pipe will force the heavy liquid in the right limb of the U-tube

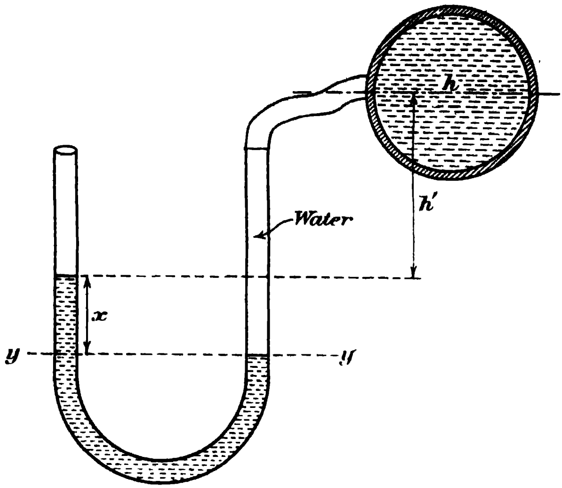


FIG. 5

downwards; this will cause it to rise by a corresponding amount in the left limb. The surface of contact between the heavy liquid and the water is known as the common surface.

Consider the horizontal section yy through the common surface.

Let h' = height of centre of pipe above liquid surface in open limb in inches,

x = height of heavy liquid in left limb above yy in inches.

As the liquid below the common surface is homogeneous, the pressure at yy in the left limb must equal the pressure at yy in the right limb.

Pressure in left limb at yy, above atmosphere $= xs$ in. of water.

Pressure in right limb at yy, above atmosphere $= x + h' + h$ in. of water

Equating these pressures,

$$x + h' + h = xs$$

from which $$h = x(s - 1) - h' \text{ in. of water}$$

If the heavy liquid in the U-tube is mercury, $s = 13{\cdot}6$. Then,

$$h = x(13{\cdot}6 - 1) - h'$$
$$= (12{\cdot}6x - h') \text{ in. of water}$$

If the pressure being measured is large, mercury should be used in the U-tube. For small pressures the liquid should be a little heavier than water.

3. Inverted U-tube. The difference of pressure between two sections of a pipe containing water may be measured by an inverted

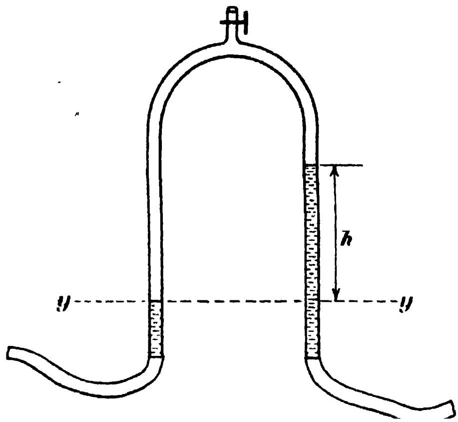


Fig. 6

U-tube (Fig. 6). The upper part of the tube contains air, whilst the water from the two sections of the pipe being measured passes into the left and right limb respectively.

The heights of the water columns may be adjusted to convenient heights by letting out air through the valve at the top.

As the air trapped in the upper part of the tube is under constant pressure, the difference of pressure between the sections of the pipe is equal to the difference in the heights of the two water columns.

Let h = difference of height of water columns in inches.

Then difference of pressure $= h$ in. of water

4. Differential Gauge. The inverted U-tube may be made very sensitive by having a liquid lighter than water in the upper part of the tube in place of the air.

Let s = specific gravity of liquid used.

Consider pressures above section yy (Fig. 6).

$$\begin{aligned} \text{Difference of pressure} &= h \text{ in. of water} - h \text{ in. of liquid} \\ &= h - hs \\ &= h(1 - s) \end{aligned}$$

The nearer s is to unity, the more sensitive the instrument becomes.

All gauges which are made sensitive by using liquids of different specific gravity are known as differential gauges.

5. Oil Gauge with Enlarged Ends. A sensitive type of gauge may be obtained by using a U-tube with enlarged ends (Fig. 7); this type is used for measuring small differences of pressure of gases.

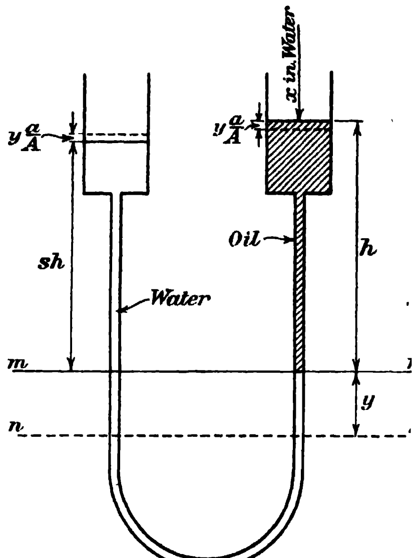


Fig. 7

Water and oil are placed in the limbs, the free surface of each liquid being in the enlarged ends.

Let A = area of enlarged end,
a = area of tube,
s = specific gravity of oil used,
mm = common surface when both limbs are subjected to equal pressures.

Assume both ends of the U-tube are exposed to the same pressure, and that h in. be the height of free surface of oil above the common surface mm. Then

$$\text{height of free surface of water above } mm = sh$$

Now let the surface of the oil be subjected to an additional pressure equal to x in. of water. This will cause the common surface to fall by the amount y in. to the level nn. The level of the oil in the enlarged end will consequently fall by $y(a/A)$, whilst the level of the water in the other limb will rise by the same amount.

Consider the total pressure in both limbs at the new common surface nn.

$$\text{Height of oil surface above } nn = h + y - y\frac{a}{A}$$

$$= h + y\left(1 - \frac{a}{A}\right)$$

$$\text{Height of water surface above } nn = sh + y + y\frac{a}{A}$$

$$= sh + y\left(1 + \frac{a}{A}\right)$$

Then, as pressures in both limbs at nn are equal,

$$sh + y\left(1 + \frac{a}{A}\right) = s\left[h + y\left(1 - \frac{a}{A}\right)\right] + x$$

both being in inches of water.

Therefore
$$y\left(1 + \frac{a}{A}\right) = sy\left(1 - \frac{a}{A}\right) + x$$

or
$$x = y\left[1 + \frac{a}{A} - s\left(1 - \frac{a}{A}\right)\right]$$

EXAMPLE 6

A U-tube containing mercury has its right limb open to the atmosphere. The left limb is full of water and is connected to a pipe containing water under pressure, the centre of which is level with the free surface of the mercury. Find the pressure of the water in the pipe, above atmosphere, if the difference of level of the mercury in the limbs is 2 in.

Consider a horizontal section through the common surface and consider the pressure in inches of water in each limb above this section.

Let x = pressure of water in pipe above atmosphere in inches of water.

$$\text{Pressure in left limb} = \text{pressure in right limb}$$

$$x + 2 = 13{\cdot}6 \times 2$$

$$x = (13{\cdot}6 - 1)2$$

$$= 25{\cdot}2 \text{ in. of water}$$

$$\text{Pressure in Lb/in.}^2 = wH$$

$$= \frac{62{\cdot}4 \times 25{\cdot}2}{144 \times 12}$$

$$= 0{\cdot}91$$

Example 7

A pressure gauge consists of two cylindrical bulbs A and B, each of 1 in.^{2} cross-sectional area, which are connected by a U-tube with vertical limbs, each of 0·025 in.^{2} cross-sectional area. A red liquid of specific gravity 0·95 is filled into A and a clear liquid of specific gravity 0·9 is filled into B, the surface of separation being in the limb attached to B. Find the displacement of the surface of separation when the pressure on the surface in A is greater than that in B by an amount equivalent to 1 in. head of water. (*Lond. Univ.*)

Consider gauge when pressure in bulb A equals pressure in bulb B.

Let h = height of liquid in A above common surface. Then, height of liquid in B above common surface

$$= \frac{0{\cdot}95}{0{\cdot}9}\,h$$

Now let pressure of 1 in. of water act on liquid in A, and let this cause the common surface to rise x in.

Then, surface of liquid in A will fall $x/40$ in. and surface of liquid in B will rise $x/40$ in.

Consider pressures in each limb above new common surface.

Height of liquid in bulb A above new common surface

$$= h - x - \frac{x}{40} \text{ in.}$$

Height of liquid in bulb B above new common surface

$$= \frac{0{\cdot}95}{0{\cdot}9}\,h - x + \frac{x}{40} \text{ in.}$$

Total pressure of A = total pressure of B

$$0{\cdot}95\left(h - x - \frac{x}{40}\right) + 1 = 0{\cdot}9\left(\frac{0{\cdot}95}{0{\cdot}9}\,h - x + \frac{x}{40}\right)$$

$$-\,0{\cdot}95x\left(1 + \frac{1}{40}\right) + 1 = -\,0{\cdot}9x\left(1 - \frac{1}{40}\right)$$

$$x = 10{\cdot}41 \text{ in.}$$

1.8. Total Pressure on a Surface. As the static pressure of a liquid varies with the depth, the intensity of pressure on a surface will not be uniform, but will vary with the depth.

Consider any vertical surface in a liquid (Fig. 8).

Let A = area of surface,

G = centre of area of surface,

$\bar{x}$ = depth of centre of area,

oo be the free surface of the liquid,

P = total pressure of liquid on the surface.

Consider a thin horizontal strip of the surface of thickness dx and breadth b. Let the depth of this strip be x.

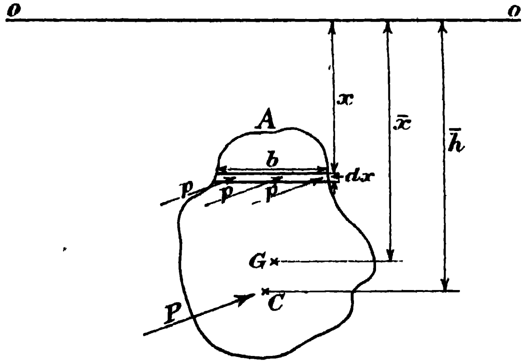


Fig. 8

Let intensity of pressure on strip be p; this may be taken as uniform as the strip is extremely narrow. Then,

$$p = wx$$

where w is the density of the liquid.

$$\text{Area of strip} = b \,.\, dx$$

$$\text{Total pressure on strip} = p \,.\, b \,.\, dx$$

$$= w \,.\, x \,.\, b \,.\, dx$$

$$\text{Total pressure on whole area} = P = w\int b \,.\, dx \,.\, x$$

But $$\int b \,.\, dx \,.\, x = \text{1st moment of area}$$

$$= A\bar{x}$$

Therefore $$P = wA\bar{x} \quad \ldots \quad (2)$$

Or, the total pressure on a surface is equal to the area multiplied by the intensity of pressure at the centre of area of the figure.

This equation will hold for all surfaces, whether flat or curved.

Example 8

A vertical square sluice is situated with its top edge 10 ft below the level of the water; the sluice is 3 ft square. Find the total pressure on the sluice.

$$\text{Depth of centre of area} = 10 + 1\tfrac{1}{2} = 11\tfrac{1}{2} \text{ ft}$$

$$\text{Area of sluice} = 9 \text{ ft}^2$$

$$\text{Total pressure} = wA\bar{x} \text{ (from eq. 2)}$$

$$= 62{\cdot}4 \times 9 \times 11\tfrac{1}{2}$$

$$= 6{,}460 \text{ Lb}$$

1.9. Centre of Pressure. The intensity of pressure on a surface is not uniform but increases with the depth. As the pressure will be greatest over the lower portion of the figure it follows that the resultant pressure will act at some point towards the lower edge of the figure. The problem is to find the point of application of the resultant pressure on the surface; this point is known as the centre of pressure.

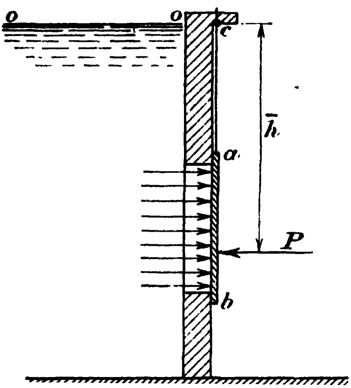


FIG. 9

As an example of the meaning of centre of pressure, consider the diagram in Fig. 9. This represents a wall with water on the left-hand side only, the surface of the water being at *oo*. The wall contains an opening below the water level through which the water is prevented from flowing by the gate *ab*, which is freely suspended by a cord at *c*. The point *c* is on the water level. The pressure of the water is tending to swing the gate outwards about the pivot *c*; to prevent this let a force P be applied to the gate as shown in the figure. P will equal the total water pressure on the gate. There is only one point on the gate at which P may be applied which will keep the gate perfectly closed; that point is the centre of pressure. If P were applied above this point the gate would open outwards at the bottom; if P were applied below the centre of pressure the gate would open outwards at the top. Thus, the moment of P about the pivot c must equal the sum of all the moments of the water pressures on the gate about the water surface. Therefore, the depth of the centre of pressure may be found by taking moments about the water surface.

Referring to Fig. 8, let C be the centre of pressure of the immersed figure. Then the resultant pressure P will act through this point.

Let $\bar{h}$ = depth of centre of pressure below free surface,

I_0 = 2nd moment of area of figure about *oo*.

Consider the horizontal strip of thickness dx.

$$\text{Force on strip} = w\,.\,x\,.\,b\,.\,\mathrm{d}x \text{ (as in § 1.8)}$$

Moment of force on strip about free surface *oo*

$$= w\,.\,x^2\,.\,b\,.\,\mathrm{d}x$$

Total moment of forces for whole area

$$= w\int b\,.\,\mathrm{d}x\,.\,x^2$$

But
$$\int b\,.\,\mathrm{d}x\,.\,x^2 = \text{2nd moment of area}$$
$$= I_0$$

Therefore total moment $= wI_0$

But moment due to resultant pressure about *oo*

$$= P\bar{h}$$

Therefore $$P\bar{h} = wI_0$$

or $$\bar{h} = \frac{wI_0}{P}$$

But $$P = wA\bar{x} \text{ (eq. (2))}$$

Therefore $$\bar{h} = \frac{I_0}{A\bar{x}} \qquad \text{. . . . . . (3)}$$

Or, the depth of centre of pressure

$$= \frac{\text{2nd moment of area}}{\text{1st moment of area}}$$ $\bar{h}-\bar{x} = \frac{I_G}{A\bar{x}}$

The value of I_0 may be obtained from the theorem—

$$I_0 = I_G + A\bar{x}^2$$

where I_G is the 2nd moment of area of the figure about a horizontal axis through its centre of area.

EXAMPLE 9

A circular plate 6 ft in diameter is placed vertically in water so that the centre of the plate is 4 ft below the surface. Find the depth of the centre of pressure and the total pressure on the plate.

$$I_G = \frac{\pi d^4}{64}$$

$$= \frac{\pi 6^4}{64} = 63{\cdot}6 \text{ ft}^4$$

$$\text{Area of plate} = \frac{\pi}{4}6^2 = 28{\cdot}25 \text{ ft}^2$$

$$\bar{x} = 4 \text{ ft}$$

Also $$I_0 = I_G + A\bar{x}^2$$

$$\bar{h} = \frac{I_0}{A\bar{x}}$$

$$= \frac{515{\cdot}6}{28{\cdot}25 \times 4} = 4{\cdot}56$$

$$\text{Total pressure on plate} = wA\bar{x}$$

$$= 62{\cdot}4 \times 28{\cdot}25 \times 4$$

$$= 7{,}050 \text{ Lb}$$

1.10. Centre of Pressure for an Inclined Surface. The depth of the centre of pressure for a surface inclined to the liquid surface may be found by taking moments about the point of intersection of the plane of the inclined surface and the liquid surface.

Referring to Fig. 10, let *mn* represent the inclined surface, the view of which is shown projected.

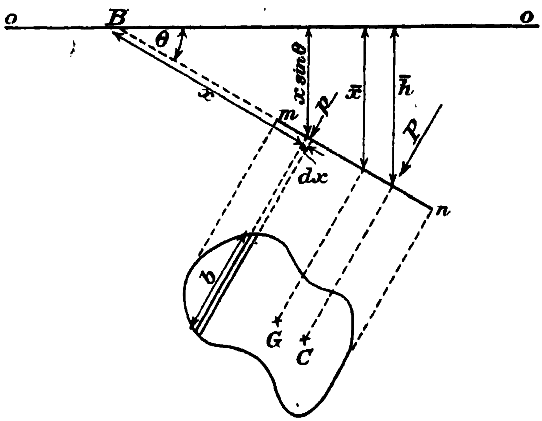


FIG. 10

Let B = point of intersection of surface produced with liquid surface,

θ = angle of inclination of immersed surface.

Consider a thin horizontal strip of area at a distance x from B. Then, using the same notation as in §1.8 and §1.9,

$$p = wx \,.\, \sin\theta, \text{ and acts normal to surface}$$

$$\text{Area of strip} = b \,.\, \mathrm{d}x$$

$$\text{Force on strip} = p \,.\, b \,.\, \mathrm{d}x$$

$$= wx \,.\, \sin\theta \,.\, b \,.\, \mathrm{d}x$$

$$\text{Total force on surface} = P = w\int b \,.\, \mathrm{d}x \,.\, x \,.\, \sin\theta$$

But $\int b \,.\, \mathrm{d}x \,.\, x$ = 1st moment of area about B

$$= \frac{A\bar{x}}{\sin\theta}$$

Therefore $$P = wA\bar{x} \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Taking moments about B,

$$\text{Moment of force on strip} = w \,.\, x \,.\, \sin\theta \,.\, b \,.\, \mathrm{d}x \,.\, x$$

$$\text{Sum of moments of forces on strip} = w \sin\theta \int b \,.\, \mathrm{d}x.\, x^2$$

Let I_B = 2nd moment of area of surface about B. Then,

$$I_B = \int b \,.\, dx \,.\, x^2$$

Therefore $\qquad$ total moment $= w \sin\theta\, I_B$

But $\qquad$ total moment $= \dfrac{P\bar{h}}{\sin\theta}$

Therefore

$$\frac{P\bar{h}}{\sin\theta} = w \sin\theta\, I_B$$

or

$$\bar{h} = \frac{wI_B \sin^2\theta}{P}$$

Substituting for P from eq. (4),

$$\bar{h} = \frac{I_B \sin^2\theta}{A\bar{x}} \quad . \tag{5}$$

where

$$I_B = I_G + \frac{A\bar{x}^2}{\sin^2\theta}$$

It will be noticed that, if $\theta = 90°$, eq. (5) becomes the same as eq. (3).

EXAMPLE 10

Find (*a*) the total pressure, and (*b*) the position of the centre of pressure on one side of an immersed rectangular plate, 6 ft long and 3 ft wide, when the plane of the plate makes an angle of 60° with the surface of the water and the 3 ft edge of the plate is parallel to, and at a depth of $2\frac{1}{2}$ ft below, the surface level of the water.

If you employ any formula you must prove its correctness. (*Lond. Univ.*)

(*b*) Using eq. (5),

$$\bar{h} = \frac{I_B \sin^2\theta}{A\bar{x}}$$

where $A = 18$ ft^2,

$\theta = 60°$,

$\bar{x} = 2{\cdot}5 + 3 \sin 60° = 5{\cdot}1$ ft,

$$I_B = I_G + \frac{A\bar{x}^2}{\sin^2 60°}$$

$$= \frac{3 \times 6^3}{12} + \frac{18 \times 5{\cdot}1^2}{0{\cdot}75} = 678 \text{ ft}^4.$$

$$\bar{h} = \frac{678 \times 0{\cdot}75}{18 \times 5{\cdot}1} = 5{\cdot}53 \text{ ft}$$

(*a*) Using eq. (4),

$$P = wA\bar{x}$$

$$= 62{\cdot}4 \times 18 \times 5{\cdot}1 = 5{,}725 \text{ Lb}$$

1.11. Fluid Pressure on a Curved Surface. The total fluid pressure on a curved surface and the position of the centre of pressure can be obtained by drawing the force polygon for the forces causing equilibrium. Consider the curved surface AB of Fig. 11. The total force on the surface and its point of application can be obtained by considering the equilibrium of the volume of water ABC. Consider unit length of the surface in a direction perpendicular to the plane of the figure.

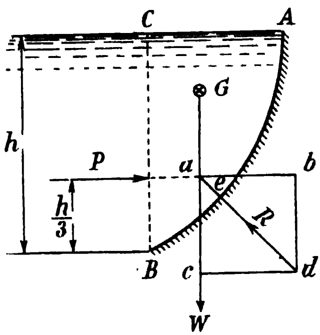


FIG. 11

Let P = total fluid pressure on rectangular area CB,

W = weight of volume ABC of fluid acting at its centre of gravity G,

R = total reaction to fluid pressure of surface AB.

As the three forces P, W and R maintain the fluid ABC in equilibrium, they will intersect at a common point; this point will be a, the point of intersection of P and W. Consider the rectangular water face CB,

$$P = wA\bar{x}$$

$$= w \times BC \times \frac{BC}{2}$$

$$= \frac{wh^2}{2}$$

The point of application of P is at $h/3$ from the base. Choosing a convenient scale, draw ab to represent P, and ac to represent W; then, diagonal ad gives the resultant force on the surface R. The centre of pressure on the surface AB is at e, the point at which the line of action of R cuts the surface.

1.12. The Pressure on Lock Gates. A practical problem on the centre of pressure is encountered in finding the forces on a lock gate. The plan of a pair of lock gates is shown in Fig. 12. AB and BC represent the gates which are held in contact at B by the water pressure, the water level being higher on the left-hand side of the gates. The gates are hinged at top and bottom at A and C.

Consider the forces acting on the gate AB.

The water pressure acts with a resultant force P at the centre of the gate and normal to it. The gate BC acts on it with a pressure T which is normal to the surface of contact of the two gates. The two hinges on the side A will react with a total force R, the direction of

which is not yet known. As the gate is in equilibrium under these three forces, they will all intersect at one point. Let P and T intersect at D; then R must pass through this point. Thus, the

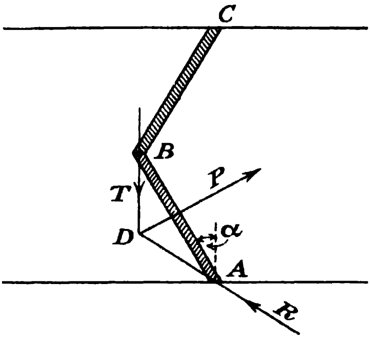


Fig. 12

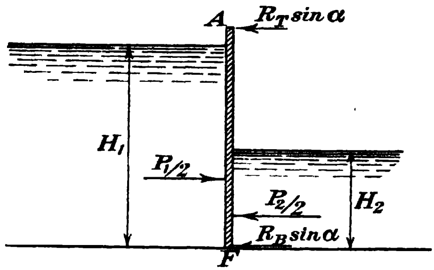


Fig. 13

gate is in equilibrium under the action of three forces intersecting at D. Let α = angle of inclination of gate to the normal of side of lock. Then, triangle ADB will be isosceles, as angles DBA and DAB equal α.

Resolving the forces at D in a direction parallel to gate,

$$R \cos \alpha = T \cos \alpha$$

Therefore $$R = T$$

Resolving normal to the gate,

$$P = (R + T) \sin \alpha$$
$$= 2R \sin \alpha$$

or $$R = \frac{P}{2 \sin \alpha}$$

Also, inclination of R to centre line of gate $= \alpha$.

Next consider the water pressure on the gate. Fig. 13 is a view of the gate in the direction AB.

Let H_1 = height of water to left of gate,
H_2 = height of water to right of gate,
H = height of top hinge from bottom of gate,
P_1 = total pressure of water to left of gate,
P_2 = total pressure of water to right of gate,
R_T = reaction of top hinge,
R_B = reaction of bottom hinge,

Then $$R_T + R_B = R$$

Also $$P_1 = \frac{wH_1}{2} \times \text{wetted area of gate}$$

and $$P_2 = \frac{wH_2}{2} \times \text{wetted area of gate}$$

Then $$P = P_1 \quad P_2$$

P_1 will act at the centre of pressure which is $H_1/3$ from the bottom. Also, P_2 will act at $H_2/3$ from the bottom.

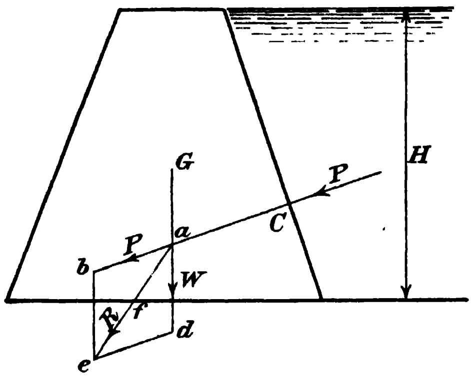


FIG. 14

It will be noticed that only half the water pressure may be taken as acting on the hinge edge of the gate; the remaining half will be taken by the reaction of the gate BC.

Taking moments about F (Fig. 13)

$$R_T \sin \alpha \, H = \left(\frac{P_1}{2} \times \frac{H_1}{3} \quad \left(\frac{P_2}{2} \times \frac{H_2}{3}\right) \quad . \qquad (6)$$

Resolving horizontally,

$$\frac{P_1}{2} - \frac{P_2}{2} = R_B \sin \alpha + R_T \sin \alpha \quad . \quad . \qquad (7)$$

Then, from eqs. (6) and (7), R_T and R_B may be found.

1.13. Water Pressure on Masonry Dams. Fig. 14 shows the section of a masonry dam having a sloping back; let the height of the water be H. The total pressure P on the dam will act at the centre of pressure C, the height of C being one-third H, and it will act normal to the surface. The weight of the masonry W will act at the centre of area of the cross-section of the dam.

Produce the forces P and W to intersect at a.

Let ab represent P and let ad represent W to the same scale.

These are the only forces acting on the dam. Complete the parallelogram and draw the diagonal *ae*. Then *ae* gives the magnitude and direction of the resultant force *R*.

Let the point at which the resultant force cuts the base of the dam be *f*. Then, in order to keep the stresses on the base of the dam

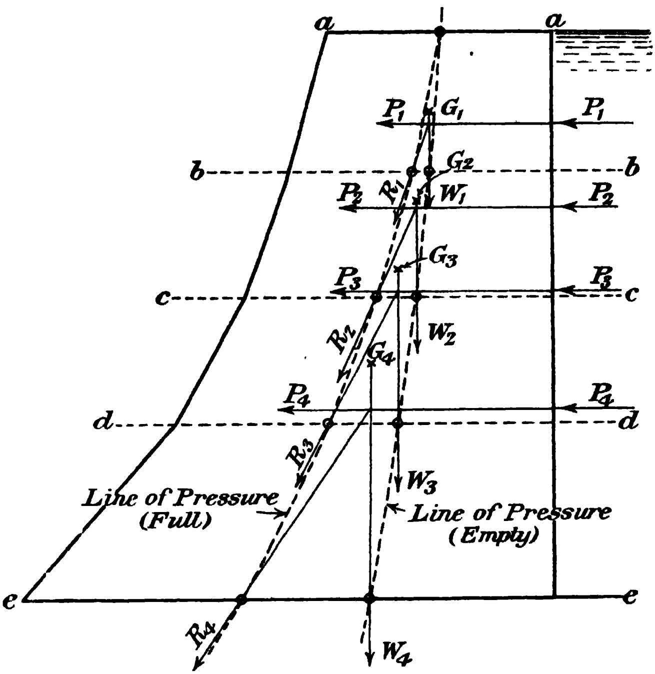


FIG. 15

within certain limits, *f* must fall within a certain distance from the centre of the base.

The above method investigates the strength of the base of the dam only; it is necessary to extend this method to other horizontal sections and so test the strength of the dam at all heights, and for all depths of water.

Consider the section of the dam in Fig. 15, and assume, in the first case, the dam to be full. Consider the horizontal section line *bb* as the base of the dam, as in the previous problem, and find the point at which the resultant force cuts the section line. To do this, let G_1 be the centre of area above *bb* and W_1 its weight; let P_1 be the water pressure above *bb* acting at one-third of the height above *bb*.

Next consider the whole section of the dam above *cc*. This gives a new centre of area G_2, a new weight W_2, and a new pressure P_2.

Find the point on *cc* at which the resultant of W_2 and P_2 cuts the line. Repeat this by considering the whole section above *dd* and *ee* in turn. Mark clearly the point at which each resultant cuts its own section line and draw a smooth curve through these points. This curve is known as the line of pressure for the dam when full;

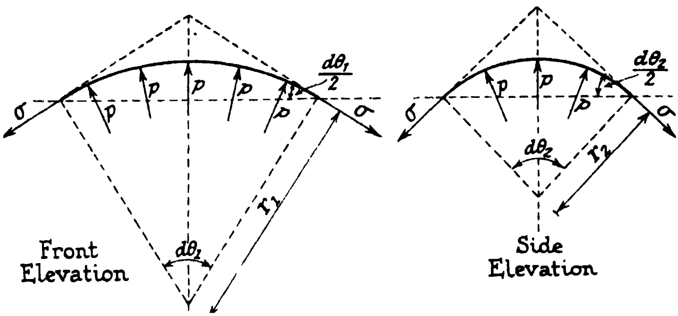


FIG. 16

and for any horizontal section line, this curve must cut the line within a given distance from the centre.

It is next required to draw the line of pressure for the dam when empty. In this case there will be no water pressure acting, the only force being the weight of the masonry. The point at which the resultant cuts the base is now where W_1 cuts *bb*, where W_2 cuts *cc*, etc. By drawing a smooth curve to pass through these points, the line of pressure for the dam when empty is obtained. This curve, also, must cut any horizontal section line within a given distance from the centre.

It will be noticed that the centre point of the top of the dam, *aa*, will be the required point for both lines of pressure at this section.

In all problems dealing with masonry dams it is usual to calculate all the forces on a length of 1 ft.

1.14. Surface Tension. When two liquids of different density, or when a liquid and a gas, are in contact, the surface of contact forms a curve called the meniscus. The formation of the curved surface is due to the attraction of the molecules, and it is found that there is a slight pressure difference between the fluids on either side of the surface. The surface appears to act as an elastic skin which is in tension in both directions; this tension is called the *surface tension*.

This phenomenon may be observed in the curved surface of a liquid in a tube, in a bubble of a light liquid immersed in a heavier

liquid, or a bubble of gas immersed in a liquid. It is familiar in the bubble of a spirit level and in the soap film of a soap bubble.

Let the curves of Fig. 16 represent the front elevation and end view of a small rectangular portion of a meniscus.

Let p = excess of inside pressure over outside pressure, pounds per square inch,

r_1 and r_2 = radius, in inches, of surface in front and end view respectively,

$d\theta_1$ and $d\theta_2$ = angle in radians subtended by surface in front and end view respectively,

σ = surface tension in pounds per inch length in both perpendicular directions.

Then
$$\text{length of arc in front view} = r_1\, d\theta_1$$
$$\text{length of arc in end view} = r_2\, d\theta_2$$

Now, as the small rectangular surface considered is in equilibrium, the upward pressure due to p must equal the downward pull of the tension σ. Hence, resolving vertically,

$$\text{upward force due to } p = \text{downward force due to } \sigma$$

or
$$p \times (r_1\, d\theta_1 \times r_2\, d\theta_2) = 2\sigma r_1\, d\theta_1 \frac{d\theta_2}{2} + 2\sigma r_2\, d\theta_2 \frac{d\theta_1}{2}$$

as $\sin \frac{d\theta}{2} = \frac{d\theta}{2}$ for small angles.

Then
$$p = \frac{\sigma(r_2 + r_1)}{r_1\, r_2}$$
$$\sigma\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \qquad (8)$$

If the surface is spherical of radius r,
$$r_1 = r_2 = r$$

Then
$$p = \frac{2\sigma}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

If the surface is cylindrical, r_2 is infinite.

Then
$$p = \frac{\sigma}{r_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

In a large vessel the surface of the liquid is curved near the perimeter only; in this case r_2 of eq. (8) is the radius of the vessel.

It will be seen from these equations that the value of the surface tension depends on the radius of the meniscus, and is found to vary with the nature of the fluids in contact and with the temperature. For a water surface in contact with the atmosphere the surface tension has a value of about 0·00042 Lb/in. at a temperature of 60°F.

1.15. Capillarity. If a tube of small bore be inserted in a liquid, the liquid is observed to rise, or fall, by a head h, as is shown in Fig. 17. This, in the case of (*a*), is caused by the fall in pressure p on the underside of the meniscus, due to the surface tension σ. At the point A the pressure is less than atmospheric as the meniscus is

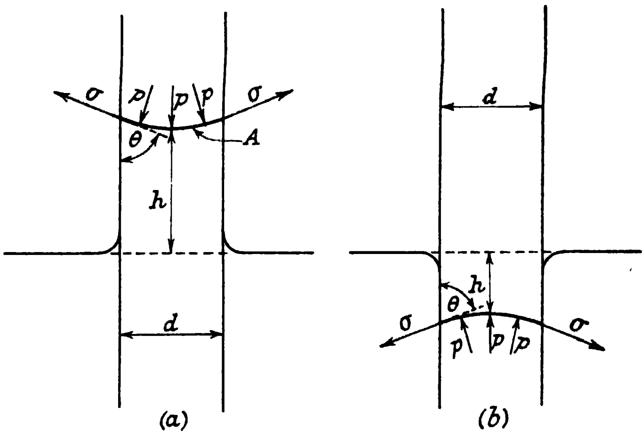


FIG. 17

sagging; this causes an elevation of the liquid in the tube. In the case of mercury (Fig. 17 (*b*)), the meniscus is reversed in curvature, thus causing a capillary depression in the tube.

For a circular-sectioned tube of diameter d inches, the weight of column of liquid elevated (Fig. 17 (*a*)) is supported by the surface tension σ acting around the perimeter of the tube. Then, resolving vertically,

weight of liquid raised = vertical component of σ acting on perimeter

or
$$w\left(\frac{\pi}{4}d^2h\right) \quad \sigma\pi d\cos\theta$$

where θ is the angle of the meniscus at the perimeter.

Hence
$$h = \frac{4\sigma\cos\theta}{wd} \quad . \quad (11)$$

For two parallel plates at distance d apart, resolving vertically and considering unit length,

$$whd = 2\sigma\cos\theta$$

hence
$$h = \frac{2\sigma\cos\theta}{wd} \quad (12)$$

This capillary elevation or depression will affect the readings of gauges of small bore, such as piezometer tubes, unless the diameter is of sufficient size to make the value of h negligible. If the diameter is extremely small, a correction must be made by the application of eq. (11).

If the liquid wets the walls of the tube, as in the case of water, the value of θ is zero, and eq. (11) becomes

$$h = \frac{4\sigma}{wd} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

This equation provides the best method of obtaining the value of the surface tension σ experimentally. By measuring the height h of the liquid in a capillary tube of diameter d and having wet walls, the value of σ can be calculated.

EXAMPLE 11

Find the height through which water is elevated by capillarity in a glass tube of $\frac{1}{4}$ in. bore if the surface tension at the existing temperature is $0{\cdot}343 \times 10^{-3}$ Lb per inch.

Using eq. (13),

$$\begin{aligned} h &= \frac{4\sigma}{wd} \\ &= \frac{4 \times 0{\cdot}343 \times 10^{-3} \times 12^3}{62{\cdot}4 \times 0{\cdot}25}\ \text{in.} \\ &= 0{\cdot}152\ \text{in.} \end{aligned}$$

1.16. Energy Stored in Liquid by Compression. The bulk modulus of a fluid (see § 13.9) is the ratio of the volumetric stress to the volumetric strain,

or
$$K = \frac{P}{\mathrm{d}V/V} \quad . \quad . \quad . \quad . \quad (14)$$

where P = pressure of fluid in pounds per square foot,
V = original volume of fluid in cubic feet,
and $\mathrm{d}V$ = change in volume due to pressure P.

When this equation is applied to a liquid the term $\mathrm{d}V/V$ is known as the volumetric strain.

The applied pressure P causes the volume to decrease by $\mathrm{d}V$ during its application, thus doing work on the liquid. The liquid receives an amount of energy of the same magnitude as the work done; this energy is stored in the liquid in the form of elastic energy, or strain energy. The pressure P is assumed to be applied gradually, so that whilst the liquid is being compressed the pressure commences at zero and gradually increases to P, the increase

following a straight-line law. Hence, the average pressure acting during the decrease of volume dV is $P/2$.

$$\begin{aligned}\text{Strain energy stored in liquid} &= \text{work done by } P\\ &= \text{average pressure} \times \text{change in volume}\\ &= \tfrac{1}{2}P \times \mathrm{d}V\end{aligned}$$

But, from eq. (14),

$$\mathrm{d}V = \frac{PV}{K}$$

Hence

$$\text{strain energy stored} = \frac{\tfrac{1}{2}P^2}{K} \times \text{volume} \qquad . \quad . \quad . \quad . \ (15)$$

where K is in foot units.

EXERCISES 1

1. A diver is working on the sea bottom at a depth of 74 ft. What is the pressure, above atmosphere, in pounds per square inch, at this depth? 1 ft^3 sea water weighs 64 Lb. *Ans.* 32·9 Lb/in.^{2}

2. The pressure of a gas is measured by a U-tube containing water, which has one limb open to the atmosphere, and is found to be 2·6 in. of water. The barometer reading is 76 cm of mercury. Express the pressure of this gas in pounds per square inch, (1) as gauge pressure, (2) as absolute pressure. The specific gravity of mercury is 13·6.
Ans. (1) 0·0939 Lb/in.^{2} (2) 14·7639 Lb/in.^{2}

3. A hydraulic press has a ram of 4 in. diameter and a piston of $\frac{3}{8}$ in. diameter. What load on the ram can be lifted by a force of 25 Lb on the piston?
Ans. 2,840 Lb.

4. A masonry dam, of rectangular section 20 ft high and 10 ft wide, has the water level with its top. Find (1) the total pressure on 1 ft length of the dam; (2) the height of the centre of pressure; (3) the point at which resultant cuts the base. The weight of 1 ft^3 of masonry is 110 Lb.
Ans. (1) 12,480 Lb. (2) 6·67 ft. (3) 3·78 ft from centre.

5. A hollow triangular box, the ends of which are equilateral triangles of 4 ft sides, is submerged in water so that one of its rectangular faces lies in the surface of the water. Find the net total pressure, and the position of the centre of pressure on one of the triangular ends, (*a*) when the inside of the box is at atmospheric pressure; (*b*) when the inside of the box is at a pressure of 1 Lb/in.^{2} above the atmospheric pressure. (*Lond. Univ.*)
Ans. (*a*) 496 Lb; 1·735 ft from surface. (*b*) 500 Lb; 0·585 ft from surface.

6. A pressure gauge, for use in a stokehold, is made of a glass U-tube with enlarged ends, one of which is exposed to the pressure in the stokehold and the other connected to the outside air. The gauge is filled with water on one side, and oil having a specific gravity of 0·95 on the other—the surface of separation being in the tube below the enlarged ends. If the area of the

enlarged end is 50 times that of the tube, how many inches of water pressure in the stokehold correspond to a displacement of 1 in. in the surface of separation ? (*Lond. Univ.*) *Ans.* 0·089 in.

7. A circular plate 5 ft in diameter is immersed in water, its greatest and least depths below the surface being 6 ft and 3 ft respectively; find (*a*) the total pressure on one face of the plate; (*b*) the position of the centre of pressure. (*Lond. Univ.*) *Ans.* (*a*) 5,520 Lb. (*b*) 4·63 ft below surface.

8. Each gate of a lock is 20 ft high and 6 ft wide, and is supported on pivots, situated 2 ft from the top and bottom. The angle between the gates when they are closed is 140°. If the depths of water on the two sides are 17 ft and 5 ft respectively, find the magnitude and position of the resultant water pressure on each gate, the magnitude of the reaction between the gates, and the magnitude and directions of the reactions at the pivots, on the assumption that the gate reaction acts in the same horizontal plane as the resultant water pressure. (*Lond. Univ.*)

Ans. 49,500 Lb; 6·04 ft from bottom; 72,550 Lb; 18,150 Lb (top); 54,400 Lb (bottom); 20° to gate.

9. A rectangular sluice-gate 6 ft square has its upper edge submerged to a depth of 6 ft. Determine the magnitude of the resultant pressure on one face, and the centre of pressure. (*A.M.I.Mech.E.*) *Ans.* 20,200 Lb; 9⅓ ft.

10. A hemispherical tank, 5 ft in diameter, is full of water. Determine (1) the resultant pressure on the wetted surface; (2) the total pressure on the wetted surface; (3) the centre of pressure on the wetted surface. (*A.M.I.Mech.E.*) *Ans.* (1) 2,040 Lb. (2) 3,060 Lb. (3) 2·5 ft.

11. A bulkhead closing one end of a floating dock is 30 ft wide at the bottom and 60 ft at the top, and is 30 ft deep. If submerged up to its upper edge, what is the pressure on the bulkhead, and what will be the depth of the centre of pressure? (*A.M.I.C.E.*) *Ans.* 1,122,000 Lb; 18·75 ft.

12. A 10 ft length of a semicircular culvert 6 ft in diameter has bulkheads at each end. If filled with water determine (*a*) the resultant force exerted by the water on the wetted surfaces; (*b*) the total pressure exerted on these surfaces. (*A.M.I.Mech.E.*) *Ans.* (*a*) 8,810 Lb; (*b*) 13,450 Lb.

13. Describe with sketches some form of differential gauge capable of enabling very small differences of head in a pipe to be measured. Explain the theory of its action. (*A.M.I.C.E.*)

14. A circular drum, 4 ft in diameter and 10 ft long, rests with its axis horizontal on the bottom of a dock in which the depth of water is 10 ft. Determine (*a*) the total pressure on the surface of the drum; (*b*) the resultant pressure on the surface of the drum; (*c*) the depth of the centre of pressure on each of the flat ends. (*A.M.I.Mech.E.*)

Ans. (*a*) 74,060 Lb. (*b*) 7,820 Lb. (*c*) 8·125 ft.

15. Water is 60 ft deep at the face of a dam, which is vertical to 30 ft from the water level, below which it slopes at 30° to the vertical. Specify completely the resultant pressure which acts on the face of the dam per foot run. (*Lond. Univ.*) *Ans.* P = 123,000 Lb at 23½° to the horizontal.

$\bar{h}$ = 41·3 ft.

16. Show that a vertical surface, subject to water pressure on one side, requires, for equilibrium, a balancing moment, about a horizontal axis through the centroid of area which is independent of the depth of submergence.

An aperture in a vertical wall of a water-tank is closed by a circular plate of 24 in. diameter. This is held in position by four stops, one at each end of the horizontal diameter and one at each lower end of the diameters at 60° to the horizontal. Determine the stop reactions when the water surface is 18 in. above the plate centre. (*Lond. Univ.*) *Ans.* 118·5 Lb, 28·5 Lb.

17. A capillary tube of $\frac{1}{8}$ in. bore is placed vertically in water. Calculate the elevation of the water in the tube due to capillarity. The surface tension is $0{\cdot}42 \times 10^{-3}$ Lb/in. *Ans.* 0·372 in.

CHAPTER 2

THE BUOYANCY OF A FLUID

2.1. Buoyancy. If a body is floating in a fluid and is at rest, it will be in equilibrium in a vertical plane; then the total upward force must equal the total downward force. This is true whether the body be immersed in a liquid or a gas. The downward force on the body will be due to gravity, whilst the upward force will be the resultant upward pressure of the fluid in which the body is floating. This resultant upward pressure is known as the buoyancy.

Consider a body immersed in a fluid, and let *oo* be the surface of the fluid (Fig. 18). Consider a vertical prism of the body of height H and end area a. Let p be the intensity of pressure of the fluid on the upper end of the prism. Then the intensity of pressure on the lower end of the prism will be $p + wH$, the additional amount wH being due to the additional depth H of the fluid.

Total downward pressure of fluid on prism

$$= pa$$

Total upward pressure of fluid on prism

$$= (p + wH)a$$

Resultant upward pressure of fluid on prism

$$= (p + wH)a - pa$$

$$= wHa$$

But volume of prism $= Ha$

Therefore

resultant upward pressure $= w \times$ volume of prism

= weight of fluid displaced by prism

If the whole body is imagined to be made up of similar vertical prisms, it follows that the total resultant upward pressure of the fluid will equal the weight of fluid displaced by the body. This is known as Archimedes' principle.

When a body is floating in a liquid, a normal pressure will be exerted by the liquid at all points on the surface of the body. The resultant of all these normal pressures will be vertically upwards and will act through the centre of gravity of the volume of liquid displaced by the body; this point is known as the centre of buoyancy. When dealing with a transverse section of a floating body, the centre of buoyancy is at the centre of area of the immersed section.

Referring to the transverse section of the ship in Fig. 22, page 34, let ac be the water line; then the immersed section will be the area $acde$. The centre of buoyancy will be at the centre of area of this immersed section and is denoted by the point B_1. If the ship rolls in a clockwise direction, as shown by the dotted position (Fig. 22), the

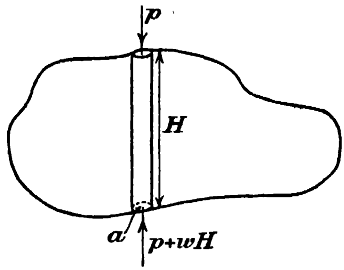


FIG. 18

immersed section will now be acd_1e_1, and the centre of buoyancy will be at the centre of area of this new immersed section.

EXAMPLE 1

The volume of the displacement of a ship in sea water is 4,000 ft³; find the weight of the ship if 1 ft³ sea water weighs 64 Lb. Find also the volume of the displacement in fresh water.

$$\text{Weight of ship} = \text{weight of sea water displaced}$$

$$= \frac{64 \times 4{,}000}{2{,}240} = 114{\cdot}2 \text{ tons}$$

Volume of fresh water displacement

$$= \frac{114{\cdot}2 \times 2{,}240}{62{\cdot}4}$$

$$= 4{,}100 \text{ ft}^3$$

2.2. Conditions of Equilibrium of a Floating Body. There are three conditions of equilibrium for a floating body: stable, unstable, and neutral. If the floating body is given a slight angular displacement, such as the rolling of a ship, after which it returns to its original position, the body is said to be stable. If, on being given a slight displacement, it heels farther over, it is said to be in unstable equilibrium. But if the body is given a small displacement into a new position and it remains at rest in that new position, the body is then said to be in neutral equilibrium.

Consider the cross-section of a ship, shown in Fig. 19, and let *oo* be the water-line, B the centre of buoyancy, and G the centre of gravity of the ship. If the ship is given a small angular displacement θ, it will rotate about some point on its vertical axis. As the rotation does not cause any alteration in the volume of water displaced, it

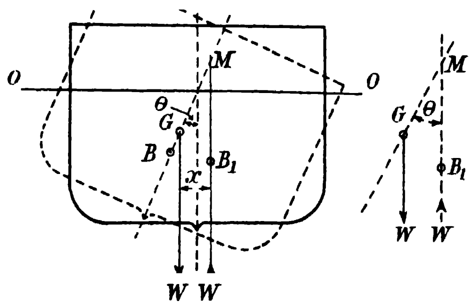


FIG. 19

follows that the ship must rotate about the point of intersection of the water-line with the vertical centre-line. The dotted lines show the position of the ship after it has heeled through the small angle θ.

Let B_1 = position of new centre of buoyancy after heeling,
G = position of centre of gravity after heeling,
W = weight of ship.

Draw a vertical line through B_1 to intersect the centre-line of the ship at M. The point M is called the *metacentre.*

The forces now acting on the ship are shown in the right-hand diagram of Fig. 19; the upward thrust of the water is equal to W and acts vertically through B_1, whilst the weight of the ship acts vertically downwards through G. Thus, there is a couple acting on the ship tending to restore it to its original position. This couple is known as the righting couple, or righting moment, its magnitude being $W \times$ horizontal distance between G and B_1. Or,

$$\text{righting couple} = W \times x$$
$$= W \times MG \times \tan \theta \text{ (as } \theta \text{ is small)}$$

The distance MG is known as the *metacentric height*, when the angle θ is infinitely small.

It will be seen from this that the ship behaves as a pendulum suspended at M, the point G corresponding to the bob, as shown in Fig. 20. Hence, the ship may be regarded as rotating about M, which point is considered by some authorities to be equivalent to the instantaneous centre of rotation. As M is always very close to the water-line, it does not materially affect the problem whether the ship

be assumed to rotate about the water-line or about its metacentre M. M can be regarded as a fixed point only for an extremely small angle of heel.

Now, referring to Fig. 20, the righting moment is $W \times GM \tan \theta$, so that if M is above G the ship will return to its original position and is, therefore, stable. If M is below G, the moment due to W would cause the ship to turn completely over. In this case the ship would be unstable. In the case when M coincides with G the ship is in neutral equilibrium, for there would then be no moment acting on the ship. In order to ensure that a ship is perfectly stable, M should be a certain distance above G. The distance varies with the size and type of the vessel; usually the metacentric height is between 1 ft and 4 ft.

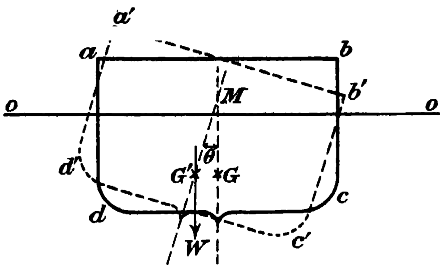


FIG. 20

The term "metacentre" was first defined by Bougier* in 1746. Bougier's definition was that the metacentre is the point at which the vertical through the centre of buoyancy intersects the vertical centre-line of the ship's section, after a small angle of heel.†

The metacentric height of a floating body can be determined both by calculation and by experiment.

2.3. The Experimental Determination of the Metacentric Height. The metacentric height of a ship or pontoon may be found experimentally whilst the vessel is floating, if the position of the centre of gravity is known.

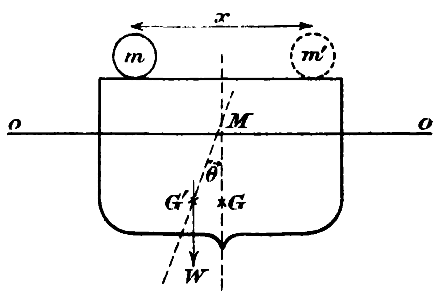


FIG. 21

Let W be the weight of the ship (Fig. 21), which is known, and let G be the centre of gravity. Let a known movable weight m be placed on one side of the ship.

A pendulum consisting of a weight suspended by a long cord is placed in the ship, and the position of the bob when at rest is marked. Let l be the length of the pendulum. The weight m is

* Bougier, *Traité du navire.*

† For a full discussion on the stability of ships see Sir William White's *Naval Architecture.*

then moved across the deck through the distance x, the new position of m being denoted by m'. This will cause the ship to swing through a *small* angle θ about its metacentre M. Then, as the pendulum inside the ship still remains vertical, the angle θ may be measured by the apparent deflection of the pendulum.

Let the apparent horizontal displacement of the pendulum weight $= y$. Then,

$$\tan\theta = \frac{y}{l}$$

Referring to Fig. 21, the moment caused by W about M equals the moment about M caused by moving m to m'. Or,

$$W \times GM \tan\theta = mx$$

from which

$$GM = \frac{mx}{W\tan\theta} \qquad . \quad . \quad . \quad (1)$$

and, as all the quantities on the right of this equation are known, the metacentric height can be calculated.

This experiment is often carried out on a ship in order to determine the exact position of G, which is difficult to estimate from the distribution of the ship's weight. The position of M is first calculated from the method given in § 2.4.

EXAMPLE 2

Define the term "metacentric height" in connexion with a floating body. Obtain an equation giving the metacentric height and apply it in the case of a ship which displaces 3,000 tons of sea water and which heels over 1/30 when a load of 15 tons is shifted across the deck a distance of 30 ft. (*Lond. Univ.*)

Taking moments about M (Fig. 21),

$$\text{moment due to } W = \text{moment due to } m$$

$$W \times GM \tan\theta = mx$$

$$3{,}000 \times GM \times \frac{1}{30} = 15 \times 30$$

$$GM = \frac{15 \times 30 \times 30}{3{,}000}$$

$$= 4{\cdot}5 \text{ ft}$$

2.4. Analytical Method for Metacentric Height. An equation for the metacentric height of a floating body may be obtained if the position of the centre of gravity G is known. Consider the transverse section of the ship of Fig. 22; let the ship heel in a clockwise direction through a small angle θ (radians). The immersed section has now changed from the area of *acde* to the dotted position acd_1e_1. The old centre of buoyancy, relative to the ship, is B, and the new centre

of buoyancy is B_1. Hence the centre of buoyancy has moved from B to B_1, relative to the ship. It will be noticed that the effect of the heeling is to move an immersed wedge from one side of the ship to the other; that is, the immersed wedge *aom* now occupies the position *con*. The apparent movement of this wedge across the ship

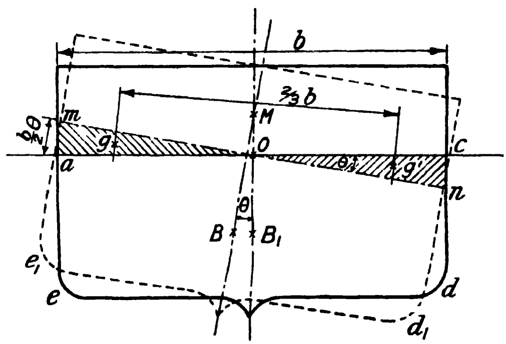


FIG. 22

causes the centre of buoyancy to move from B to B_1; these movements, of course, being relative to the ship. From the effect of these two movements the required equation may be obtained.

As the volume of water displaced remains constant, the shaded area *aom* must equal the shaded area *con*; hence the old water-line *mn* will pass through the point o. From this it follows that the ship is rotating about the point o; but for the extreme case of θ being infinitesimally small, the same result is obtained whether the ship be assumed to rotate about M or o.

Let l be the length of the ship and consider a thin transverse slice of length dl.

Let b = breadth of ship,
V = volume of water displaced by whole ship,
dV = volume of water displaced by slice considered,
I = 2nd moment of area of a horizontal section of ship at water-line about a longitudinal axis,
dI = 2nd moment of area of slice considered about a longitudinal axis,
g and g' = centres of gravity of triangular prisms *aom* and *con* respectively.

Then

$$\text{weight of ship} = wV$$
$$\text{weight of slice considered} = w\,dV$$
$$\text{distance between } g \text{ and } g' = \tfrac{2}{3}b$$

$$am = cn = \frac{b}{2}\theta$$

$$\text{volume of wedge of slice} = \tfrac{1}{2} \times \frac{b}{2} \times \frac{b}{2}\,\theta \times \mathrm{d}l$$

$$\text{weight of wedge of slice} = \frac{wb^2\theta\,\mathrm{d}l}{8}$$

Also
$$\mathrm{d}I = \frac{\text{breadth} \times (\text{depth})^3}{12}$$
$$= \frac{\mathrm{d}l \times b^3}{12}$$

Taking moments about M,

moment caused by moving triangular prism of water from g to g' = moment caused by moving upward thrust of water from B to B_1.

That is
$$\frac{wb^2\theta\,\mathrm{d}l}{8} \times \tfrac{2}{3}b = w\,\mathrm{d}V \times BB_1$$

or
$$w\left(\frac{\mathrm{d}l \times b^3}{12}\right)\theta = w\,\mathrm{d}V \times (BM \times \theta)$$

Hence
$$\mathrm{d}I = BM \times \mathrm{d}V$$

Integrating for whole length of ship,

$$I = BM \times V$$

or
$$BM = \frac{I}{V} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Then metacentric height $GM = BM - BG$

Hence, as BG is known, the metacentric height can be obtained.

The 2nd moment of area I is actually the 2nd moment of area of the horizontal section of the ship at the water-line. Usually the sides of a vessel are vertical at this section, so that I may be taken as the 2nd moment of area of the deck about a longitudinal axis. Referring to the plan of the vessel shown in Fig. 23, in order to find the 2nd moment of area of this figure about the axis oo it would be necessary to divide the section up into small horizontal rectangles and to add together their 2nd moments of area. Sometimes, the 2nd moment of area of the deck of a ship is given as a function of the 2nd moment of area of the circumscribing rectangle.

Let l = length of ship (Fig. 23). Then,

$$\text{2nd moment of area of circumscribing rectangle} = \frac{lb^3}{12}$$

and $$I = k\frac{lb^3}{12}$$

where k is a coefficient depending on the shape of the ship.

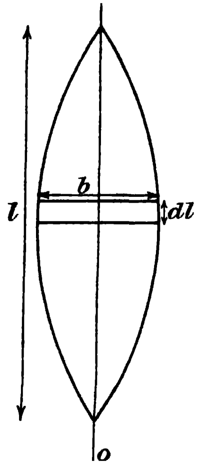


FIG. 23

In the case of a pontoon, the deck will be rectangular; then k will equal unity.

This method may be applied if the angle of heel is less than 10°. As the 2nd moment of area of the ship's water-plane is not constant, but increases with the angle of heel, the metacentric height will increase as the angle of heel increases.

The metacentric height of large ships varies between $1\frac{1}{2}$ ft and 4 ft.

EXAMPLE 3

A vessel has a length of 200 ft, a beam of 28 ft, and a displacement of 1,350 tons. A weight of 20 tons moved $22\frac{1}{2}$ ft across the deck inclines the vessel 5°. The second moment of the load water-plane about its fore and aft axis is 65 per cent of the second moment of the circumscribing rectangle, and the position of the centre of buoyancy is 5 ft below the water-line. Find the position of the metacentre and the centre of gravity of the vessel. The weight of 1 ft³ of sea water can be taken as 64 Lb. (*Lond. Univ.*)

From eq. (1),

$$GM = \frac{mx}{W\tan\theta}$$

$$= \frac{20 \times 22{\cdot}5}{1{,}350 \times 0{\cdot}0875}$$

$$= 3{\cdot}81 \text{ ft}$$

$$\text{Volume of displacement} = V = \frac{W}{w}$$

$$= \frac{1{,}350 \times 2{,}240}{64}$$

$$= 47{,}200 \text{ ft}^3$$

$$I = k\,.\frac{lb^3}{12}$$

$$= \frac{0{\cdot}65 \times 200 \times 28^3}{12} = 238{,}000 \text{ ft}^4$$

From eq. (2),

$$BM = \frac{I}{V}$$

$$= \frac{238{,}000}{47{,}200}$$

$$= 5{\cdot}05 \text{ ft}$$

Position of $M = 5{\cdot}05 - 5 = 0{\cdot}05$ ft above water-line

Position of $G = 3{\cdot}81 - 0{\cdot}05 = 3{\cdot}76$ ft below water-line

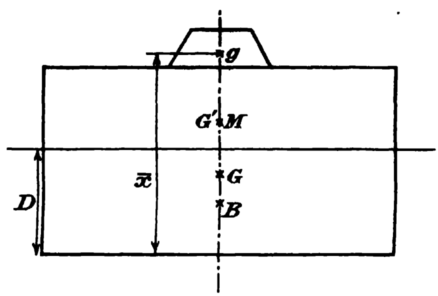


FIG. 24

EXAMPLE 4

State the condition for the stability of a floating body; and find an expression for the distance between the centre of buoyancy and the metacentre, in terms of the second moment of the water-plane area and the volume of displacement. A cylindrical buoy floats in salt water. It is 6 ft in diameter and 4 ft long, and weighs 2,500 Lb. The c.g. is 1·5 ft from the bottom. If a load of 500 Lb is placed on the top, find the maximum height of its c.g. above the bottom, so that the buoy may remain in stable equilibrium. (Weight of 1 ft^3 of salt water, 64 Lb.) (*Lond. Univ.*)

The floating buoy is shown in Fig. 24.

Let G = centre of gravity of buoy,

g = centre of gravity of weight on top,

G' = centre of gravity of buoy plus weight,

D = depth of buoy below water-line.

Then height of B from bottom $= \dfrac{D}{2}$

Let $\bar{x}$ = required height of centre of gravity of weight.

Then, $\bar{x}$ will be a maximum when the buoy reaches the state of neutral equilibrium. That is, when G' and M coincide.

Total weight of buoy plus load $= 2{,}500 + 500$

$= 3{,}000$ Lb

$$\text{Volume of water displaced} = V = \frac{3{,}000}{64} = 46{\cdot}9 \text{ ft}^3$$

$$D = \frac{V}{\text{area of base}}$$

$$= \frac{46{\cdot}9}{\frac{\pi}{4} \times 6^2} = 1{\cdot}66 \text{ ft}$$

Then $$\text{height of } B = \frac{1{\cdot}66}{2} = 0{\cdot}83 \text{ ft}$$

Using eq. (2),

$$BG' = BM = \frac{I}{V} = \frac{\pi \, (\text{diameter})^4}{64V}$$

$$= \frac{\pi \times 6^4}{64 \times 46{\cdot}9} = 1{\cdot}355 \text{ ft}$$

$$\text{Height of } G' \text{ above bottom} = 1{\cdot}355 + 0{\cdot}83 = 2{\cdot}185 \text{ ft}$$

In order to find $\bar{x}$ take moments about the bottom of cylinder.

$$500\,\bar{x} = (3{,}000 \times 2{\cdot}185) - (2{,}500 \times 1{\cdot}5)$$

$$\bar{x} = 5{\cdot}61 \text{ ft}$$

2.5. Floating Body Anchored at Base. Consider a floating body, such as a buoy, anchored by means of a chain from the centre of its base (Fig. 25). The tension in the anchor chain puts an additional downward force on the body causing it to displace a larger volume of water.

Let W = weight of body,

T = tension in anchor cable.

Then, applying Archimedes' principle,

total downward force = weight of water displaced

or $$W + T = wAd \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where A is the area of the body in the horizontal plane and d is the draught in feet.

Also $$BM = \frac{I}{V} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If the body is just to float with its axis vertical, the metacentric height M must coincide with the centre of the force system acting on the body; it is then in neutral equilibrium. Referring to Fig. 25, let R be the centre of the force system; that is, the point of application of the resultant of W and T when the body has a small angle of

heel. Let K be the point of application of the tension in the anchor chain. When the body has a small angle of heel, as shown to the right of the figure, the moments due to T and W about the point R will balance. Hence,

$$W \times RG = T \times RK \tag{5}$$

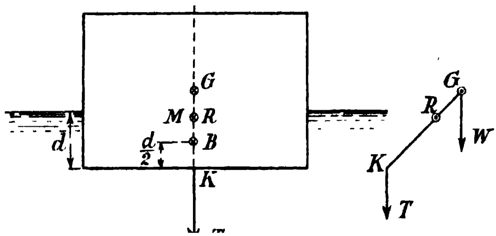


FIG. 25

If the system is in neutral equilibrium, the upward thrust of the water must pass through R, which coincides with M.

EXAMPLE 5

A cylindrical buoy is 5 ft in diameter and 6 ft high. It weighs 1,500 Lb, and its centre of gravity is 2·5 ft above the base and is on the axis. Show that the buoy will not float with its axis vertical in sea water.

If one end of a vertical chain is fastened to the centre of the base, find the pull on the chain in order that the buoy may just float with its axis vertical. Density of sea water, 64 Lb/ft³. (*Lond. Univ.*)

When there is no Anchor Chain

$$W = \frac{\pi}{4} D^2 \times d \times 64$$

that is $$1{,}500 = \frac{\pi}{4} \times 5^2 \times d \times 64$$

from which $$d = 1{\cdot}192 \text{ ft}$$

Then $$BG = 2{\cdot}5 - \frac{d}{2}$$

$$= 2{\cdot}5 - \frac{1{\cdot}192}{2} = 1{\cdot}904 \text{ ft}$$

Applying eq. (4),

$$BM = \frac{I}{V}$$

$$\frac{\frac{\pi}{64}5^4}{\frac{\pi}{4}5^2 \times 1{\cdot}192}$$

$$= 1{\cdot}31 \text{ ft}$$

$$MG = BM - BG$$
$$= 1{\cdot}31 - 1{\cdot}904$$
$$= -0{\cdot}594 \text{ ft}$$

Hence, M is below G; the body is, therefore, unstable.

When Anchor Chain is Fitted

Applying eq.(3),

$$1{,}500 + T = 64 \times \frac{\pi}{4}5^2 d$$

Applying eq. (4),

$$BM = \frac{I}{V}$$

and M now coincides with R, i.e.

$$2{\cdot}5 - RG - \frac{d}{2} = \frac{\frac{\pi}{64}5^4}{\frac{\pi}{4}5^2 d}$$

Applying eq. (5),

$$1{,}500 \times RG = T \times (2{\cdot}5 - RG)$$

Substituting in eq. (3) the value of T from eq. (5),

$$RG = -\frac{3}{d} + 2{\cdot}5$$

Substituting this value of RG in eq. (4),

$$d = 1{\cdot}7 \text{ ft}$$

Then, from eq. (3),

$$T = 640 \text{ Lb}$$

2.6. Floating Body with Bilge Water. The effect of water in the bottom of a floating body, if free to move, is to reduce the righting moment when the floating body has an angle of heel.

Consider the rectangular pontoon of Fig. 26, and let there be a quantity of free bilge water in the bottom, its surface being denoted by the line *aa*. Now imagine the pontoon to heel through the small angle θ radians; the surface of the bilge water will remain horizontal

and is now represented by the line a_1a_1. This has the effect of moving the shaded wedge of bilge water from one side of the pontoon to the other; the centre of gravity of the pontoon and its contents is thus moved by a corresponding amount. Eq. (2),

$$BM = \frac{I}{V}$$

is not affected by the relative movement of the bilge water, as this equation is based on the displacement of the external water only.

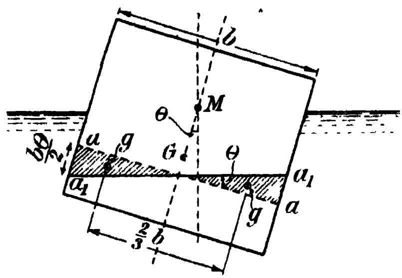


FIG. 26

Let G = original centre of gravity of pontoon and contents before heeling,

g = mass centre of wedge of bilge water,

m = weight of wedge of bilge water

$$= w\left(\frac{1}{2} \times \frac{b}{2} \times \frac{b\theta}{2}\right)$$

$$= \frac{wlb^2\theta}{8},$$

x = horizontal distance moved by mass centre of wedge

$= \frac{2}{3}b$ (approximately, if θ is small),

I = second moment of bilge water surface about longitudinal axis

$$= \frac{lb^3}{12}.$$

Then righting moment $= W\,\overline{MG}\,\theta - mx$

$$= W\,\overline{MG}\,\theta - \left(\frac{wlb^2\theta}{8} \times \frac{2}{3}b\right)$$

$$= W\,\overline{MG}\,\theta - w\theta\left(\frac{lb^3}{12}\right)$$

$$= W\,\overline{MG}\,\theta - wI\theta \quad . \quad . \quad . \quad (6)$$

It will be noticed from eq. (6) that the effect of the free bilge water is to reduce the righting moment on a floating body.

2.7. Transverse Oscillation of a Floating Body. In § 2.2 it was shown that a floating body, when given a lateral heel, may be regarded as oscillating instantaneously about the metacentre M (Fig. 19) in the same manner as a pendulum oscillates about its point of suspension.

Let W = weight of floating body,
m = metacentric height in feet,
θ = angle of displacement (radians) in t sec,
α = angular acceleration in radians/sec^2,
$= \dfrac{d^2\theta}{dt^2}$,
T = time of complete oscillation in sec,
I = moment of inertia of body about its centre of gravity G,
k = radius of gyration about G.

Then
$$I = \frac{W}{g} k^2$$

It is assumed that the axis of oscillation passes through G; this is approximately correct if m is small.

Assuming θ to be small,
$$\text{righting moment} = Wm\theta \qquad (\S\ 2.2)$$
and
$$\text{inertia torque} = -I\alpha$$
Then
$$Wm\theta = -I\alpha$$
Substituting for I and α,
$$Wm\theta = -\frac{W}{g} k^2 \times \frac{d^2\theta}{dt^2}$$
from which
$$\frac{d^2\theta}{dt^2} + \frac{mg}{k^2}\theta = 0$$

The solution of this differential equation is
$$\theta = A \sin\left(\sqrt{\frac{mg}{k^2}} \times t\right) + B \cos\left(\sqrt{\frac{mg}{k^2}} \times t\right)$$
where A and B are the constants of integration.

When $t = 0$, $\theta = 0$, hence $B = 0$.

Then
$$\theta = A \sin\left(\sqrt{\frac{mg}{k^2}} \times t\right)$$
Also, when
$$t = \frac{T}{2},\ \theta = 0.$$
Then
$$0 = A \sin\left(\sqrt{\frac{mg}{k^2}} \times \frac{T}{2}\right)$$

As A cannot be zero, then,

$$\sin\left(\sqrt{\frac{mg}{k^2}} \times \frac{T}{2}\right) = 0$$

Hence

$$\sqrt{\frac{mg}{k^2}} \times \frac{T}{2} = \pi$$

from which

$$T = 2\pi\sqrt{\frac{k^2}{mg}} \quad . \quad . \quad . \quad (7)$$

From this equation the time of oscillation of the floating body can be calculated if m and k are known.

It will be noticed from eq. (7) that if m is increased the time of oscillation is shorter. This is noticeable with a ship travelling in ballast, as the effect of the ballast is to lower the centre of gravity G and thus increase m.

EXAMPLE 6

Define the metacentric height of a floating body. Derive the formula for the period of rolling about the horizontal longitudinal axis through the centre of gravity, and state the assumptions made in deriving the formula.

The metacentric height of a ship is 2 ft and the period of rolling is 20 sec. What is the value of the relevant radius of gyration? (*Lond. Univ.*)

Assumptions made are (1) the angle θ is small; (2) the metacentric height is small so that the axis of oscillation is approximately through G.

Using eq. (7),

$$T = 2\pi\sqrt{\frac{k^2}{mg}}$$

that is

$$20 = 2\pi\sqrt{\frac{k^2}{2 \times 32{\cdot}2}}$$

from which

$$k = 25{\cdot}4 \text{ ft}$$

2.8. Buoyancy of Balloon. The *lift* of a balloon is mainly due to the weight of atmospheric air it displaces. Archimedes' principle applies to bodies immersed in a gas in the same way as it applies to bodies immersed in liquids.

A balloon consists of a light fabric container filled with a gas of less density than the atmosphere; its gross lift is equal to the weight of atmosphere it displaces. Its actual lift is the gross lift less the weight of the container and the gas inside the container. This would have its maximum value if the inside of the container was a vacuum, but this condition is impossible as the container would collapse

inwards due to the atmospheric pressure on the outside. To prevent this, the container is filled with the lightest gas possible at a pressure approximately equal to that of the atmosphere. The container is thus prevented from collapsing inwards, but its actual lift is reduced by the weight of the gas used. It should be noted that the balloon gets no lift from this gas, which merely acts as an internal structure inserted to resist the external pressure of the atmosphere. The gas used is hydrogen or helium.

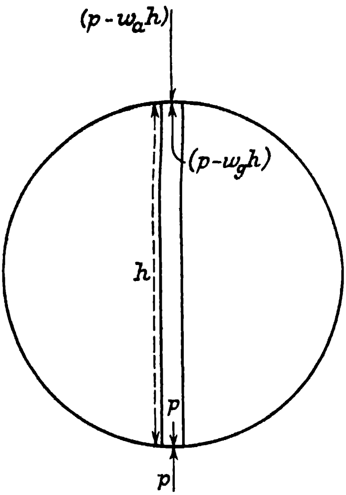


FIG. 27

Consider the spherical balloon of Fig. 27, and consider a vertical column of gas of height h and cross-sectional area a as shown.

Let p = pressure of atmosphere at base of container,

w_a = average weight of 1 ft^3 of air surrounding container,

w_g = average weight of 1 ft^3 of gas inside container.

Assume the pressure of gas at base of container to be the same as that of the atmosphere at the base. Now, the pressure of gas at the top of the column will be less than that at the base by an amount equal to the weight of the column of gas. In the same way, the pressure of the atmosphere at the top of the container is less than that at the base by an amount equal to the weight of a corresponding column of atmosphere. Or,

$$\text{pressure of gas at top} = p - w_g h \text{ Lb/ft}^2$$
$$\text{pressure of atmosphere at top} = p - w_a h \text{ Lb/ft}^2$$

As w_a is greater than w_g it follows that the gas pressure at the top is greater than that of the atmosphere at the top; hence, there is an upward force on the container.

Considering the column of gas and ignoring the weight of the container,

$$\text{upward force on column} = a(p - w_g h)$$
$$\text{downward force on column} = a(p - w_a h)$$

Subtracting,

$$\begin{aligned}\text{net upward force on column} &= -aw_g h + aw_a h \\ &= (w_a - w_g)ah \\ &= (w_a - w_g) \times \text{volume of column} \qquad (8)\end{aligned}$$

If the whole container is imagined to be made up of similar columns, it follows that

$$\text{lift on container} = (w_a - w_g) \times \text{volume of container}$$

The weight of the container must be subtracted from this amount for the actual lift of the balloon.

It will be seen from eq. (8) that the apparent lift of the gas is equal to $(w_a - w_g)$ Lb/ft³.

For hydrogen at N.T.P., $w_a = 0{\cdot}0756$ Lb and $w_g = 0{\cdot}0053$ Lb

Then apparent lift of hydrogen $= 0{\cdot}0756 - 0{\cdot}0053$

$= 0{\cdot}0703$ Lb/ft³

As the hydrogen can be obtained only at about 97 per cent pure, this figure is reduced in practice to about 0·068 Lb/ft³.

For helium at N.T.P., $w_g = 0{\cdot}0116$ Lb/ft³

Then apparent lift of helium $= 0{\cdot}0756 - 0{\cdot}0116$

$= 0{\cdot}064$ Lb/ft³

This is reduced to 0·062 Lb/ft³ if a purity of 97 per cent is assumed.

It will be noticed from Fig. 27 that the buoyancy of the balloon is due to the upward pressure on the lower half being larger than the downward pressure on the upper half; but the lift on the container itself is due to the pressure of the gas on the upper portions. In the case of a rigid airship, the containers press on to a rigid structure, being in direct contact with a network of wires fixed circumferentially around the structure. The manner in which the apparent lift of the gas is distributed around the circumference of the structure will depend on the tautness and on the method of fixing of these wires.

As a balloon rises to a higher altitude, the atmosphere is less dense; consequently, the weight of air displaced by the balloon is less: this reduces the lift. As the balloon rises the inward pressure on the outside becomes less. This produces an excess of internal gas pressure which is liable to burst the container. In order to prevent this, an automatic valve is fitted at the base so that gas may escape as the altitude increases. The containers of an airship are filled to 95 per cent of their volume when on the ground; they will then be just full when the ship has risen to its cruising height; consequently, no gas is wasted.

It should be noted that the lift of a balloon or airship will depend on the barometer reading, the temperature of the atmosphere, and on the heating of its gas by the sun's rays; the latter process is termed *superheating*.

EXERCISES 2

1. A ship has a displacement of 2,200 tons in sea water. Find the volume of the ship below the water-line. 1 ft^3 of sea water weighs 64 lb.

Ans. 77,000 ft^3.

2. A solid cube of wood of specific gravity 0·9 floats in water with a face parallel to the water-plane. If the length of one edge is 4 in., find the metacentric height. *Ans.* 0·17 in.

3. A pontoon of 1,500 tons displacement floats in fresh water. A weight of 18 tons is moved 24 ft across the deck; this causes a pendulum 10 ft long to move $4\frac{1}{2}$ in. horizontally. Find the metacentric height of the pontoon.

Ans. 7·68 ft.

4. A rectangular pontoon weighing 240 tons has a length of 60 ft. The centre of gravity is 1 ft above the centre of the cross-section, and the metacentric height is to be 4 ft when the angle of heel is 10°. The freeboard must not be less than 2 ft when the pontoon is vertical. Find the breadth and height of the pontoon, if floating in fresh water. *Ans.* 21·8 ft and 8·6 ft.

5. State the conditions which govern the stability or instability of a floating vessel.

A buoy carrying a beacon light has the upper portion cylindrical, 7 ft in diameter and 4 ft deep. The lower portion, which is curved, displaces a volume of 14 ft^3, and its centre of buoyancy is situated 4 ft 3 in. below the top of the cylinder. The centre of gravity of the whole buoy and beacon is situated 3 ft below the top of the cylinder, and the total displacement is 2·6 tons. Find the metacentric height. (Weight of sea water, 64 Lb/ft^3.) (*Lond. Univ.*)

Ans. 1·101 ft.

6. A rectangular pontoon, 35 ft long, 24 ft broad, 8 ft deep, weighs 70 tons. It carries on its upper deck a boiler 16 ft in diameter weighing 50 tons. The centres of gravity of the boiler and pontoon may be assumed to be at their centres of figure and in the same vertical line. Find the metacentric height. (Weight of sea water, 64 Lb/ft^3.) (*Lond. Univ.*) *Ans.* 3·1 ft.

7. A cylinder has a diameter of 12 in. and a relative density of 0·8. What is the maximum permissible length in order that it may float with its axis vertical? (*Lond. Univ.*) *Ans.* 10·6 in.

8. A cylindrical buoy is 6 ft in diameter and 8 ft high and weights 1·8 tons. Show that it will not float with its axis vertical in sea water. If one end of a vertical chain is fastened to the centre of the base, find the pull on the chain, in order that the buoy may just float with its axis vertical. (*Lond. Univ.*)

Ans. $MG = -1{\cdot}87$ ft; 1·15 tons.

9. If a floating body is assumed to roll about a fixed horizontal axis through its centre of gravity, prove that the period of rolling in seconds is given by

$$T = 2\pi\sqrt{(k^2/gh)}$$

in which h is the metacentric height, and k is the relevant radius of gyration.

Find the value of k for a ship which has a period of rolling of 20 sec. The displacement is 10,000 tons; the second moment of the load-water-plane about its fore and aft axis is $3{\cdot}5 \times 10^6$ ft^4; and the centre of buoyancy is 8 ft below the centre of gravity. (Sea water, 64 Lb/ft^3.) (*Lond. Univ.*)

Ans. 25·57 ft.

10. A log of wood of square section, 14 in. $\times$ 14 in., weighing 50 Lb/ft^3, floats in water. One edge is depressed and released, causing the log to roll. Estimate the period of a roll. (*Lond. Univ.*) *Ans.* 7·0 sec.

11. Obtain a formula for the metacentric height of a floating body. A solid right cone of wood weighing 44 Lb/ft^3 is required to float in water with the axis vertical. Determine the minimum apex angle which will enable the cone to float in stable equilibrium. (*Lond. Univ.*) *Ans.* 39°.

12. An airship filled with hydrogen has a gas capacity of 7,000,000 ft^3. If the weight of air at N.T.P. is 0·0756 Lb/ft^3 and the weight of hydrogen at N.T.P. is 0·0053 Lb/ft^3, what is the total lift of the airship at N.T.P.?

Calculate the loss of lift if the airship is filled with helium instead of hydrogen. 1 ft^3 of helium weighs 0·0116 Lb at N.T.P.

Ans. 219·5 tons; 19·5 tons.

CHAPTER 3

THE FLOW OF A FLUID

3.1. Flow of Fluids. When a fluid is flowing along a passage, such as a pipe, it will be subjected to a resistance due to viscosity, or friction. If the velocity of flow is very small, the fluid will flow in bands parallel to the sides of the passage; such a flow is called a viscous or laminar flow. If the velocity is large, cross-currents or eddies will be formed causing greater resistance to flow; such a flow is known as a turbulent or eddy flow. Also, the velocity of the fluid is not uniform over the cross-section, being slower towards the sides of the passage. In engineering problems, however, it is usual to assume the velocity to be uniform over the cross-section and equal to the mean velocity.*

Any obstruction in the passage or any change of section or direction will interfere with the steady flow. This will cause eddies or

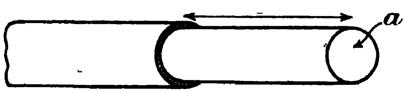


FIG. 28

transverse motions of the particles, and, consequently, there will be an additional loss of energy due to the friction caused by these transverse currents.

Consider a pipe of a cross-sectional area of a ft² containing a fluid which is flowing with a velocity of v ft/sec (Fig. 28); the pipe is running full. Consider any section of the pipe; a quantity of fluid in the shape of a cylinder of length v and area a will pass by this section in 1 sec. Or,

$$\text{quantity of fluid flowing} = \text{volume of cylinder}$$

or

$$Q = av \text{ ft}^3/\text{sec}$$

3.2. Equation of Continuity of Flow. If a fluid is flowing through any channel or pipe, the weight of fluid passing any transverse section in a given interval of time must be equal at all such sections. This is known as the equation of continuity of flow.

Let Fig. 29 represent a tapering pipe through which a fluid is flowing. Let the pipe be running full.

* For viscous flow of a fluid see § 3.16.

Let a_a = area at section aa,
a_b = area at section bb,
W = weight of fluid flowing per second.

1. **Flow of Gases.** In dealing with gases, which are compressible, the density w varies with the temperature and pressure; hence

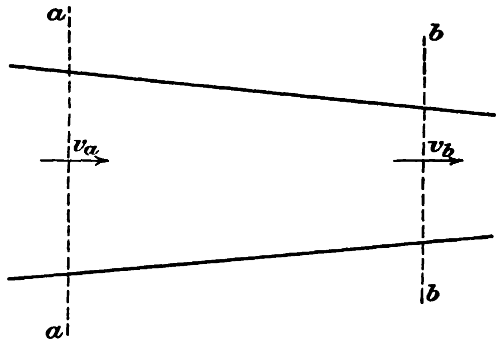


Fig. 29

the quantity of flow passing any section per second is not constant unless the temperature and pressure remain constant. But the weight of flow per second passing any section must remain constant. Hence, the equation of continuity of flow for a gas may be written—

$$W = wav = \text{constant}$$

Applying this equation to sections aa and bb,

$$W = w_a a_a v_a = w_b a_b v_b \qquad (1)$$

Then $$v_a = \frac{W}{w_a a_a} \text{ ft/sec}$$

and $$v_b = \frac{W}{w_b a_b} \text{ ft/sec}$$

2. **Flow of Liquids.** In the case of a liquid, the density is assumed to remain constant; hence,

$$w_a = w_b$$

In which case the quantity of flow per second passing any section is constant. The equation of continuity of flow may now be written—

$$\text{Quantity} = Q = av = \text{constant}$$

Hence $$Q = a_a v_a = a_b v_b \qquad (2)$$

If Q is known $$v_a = \frac{Q}{a_a} \text{ ft/sec}$$

and $$v_b = \frac{Q}{a_b} \text{ ft/sec}$$

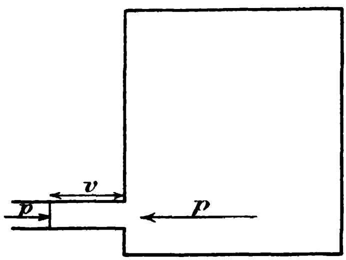


FIG. 30

3.3. Work Done in Overcoming Pressure. Let a large tank be full of water under a constant pressure of p Lb/ft^2, and let water be forced into the tank through a small pipe of cross-sectional area of a ft^2 (Fig. 30). Let v be the velocity in feet per second with which the water is forced through the pipe.

Then, work done per second in forcing water through pipe

$$= \text{force} \times \text{distance moved per second}$$
$$= pa \times v$$

But $$av = \text{volume of water forced into tank per second}$$

Therefore
$$\text{work done} = p \times \text{volume of flow per second}$$
or $$\text{work done} = wH \times \text{volume}$$
$$= WH$$

where W = weight of water injected per second and H = equivalent static head in feet of water.

EXAMPLE 1

300 gal of water are pumped into a tank per minute under a pressure of 20 Lb/in.^{2} Find the horse-power required. 1 gal water weighs 10 Lb.

$$\text{Weight of water per second} = \frac{300 \times 10}{60} = 50 \text{ Lb}$$

$$\text{Volume of water per second} = \frac{50}{62 \cdot 4} = 0 \cdot 802 \text{ ft}^3$$

$$\text{Work done per second} = p \times \text{volume}$$
$$= 20 \times 144 \times 0 \cdot 802$$

$$\text{Horse-power required} \quad \frac{20 \times 144 \times 0 \cdot 802}{550}$$
$$= 4 \cdot 2$$

Alternative Method. Convert the pressure to pressure head in feet of water.

Then

$$\text{Horse-power} = \frac{WH}{550}$$

where W = weight of water per second.

$$\text{Static head} = H = \frac{p}{w} = \frac{20 \times 144}{62\cdot4} = 46\cdot2 \text{ ft of water}$$

$$\text{Horse-power} = \frac{50 \times 46\cdot2}{550} = 4\cdot2$$

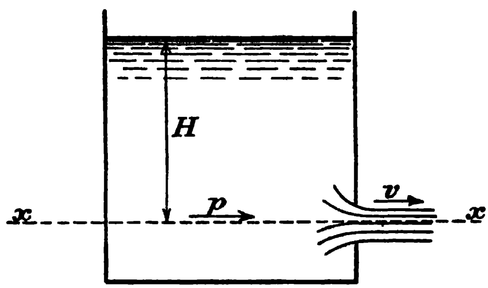


FIG. 31

3.4. Velocity Head of Liquid. Consider a liquid flowing from a tank under a constant head H (Fig. 31). Let v be the velocity of the liquid in feet per second. Consider a small quantity of liquid of weight W on the surface at the top of the tank. This quantity will have a potential energy of WH. This same quantity of liquid, when issuing through the orifice, may be looked upon as having fallen through the height H and converted its potential energy to kinetic energy. Then, ignoring frictional losses,

loss of potential energy = gain of kinetic energy

or
$$WH = \frac{Wv^2}{2g}$$

Therefore
$$H = \frac{v^2}{2g} \text{ ft of liquid}$$

or
$$v = \sqrt{2gH} \text{ ft/sec}$$

By making use of these equations, the energy of moving liquid may be given as a static head in feet of liquid; this static head is known as the velocity head.

Alternative Proof

Let p = intensity of pressure of liquid on line xx. Then,

$$p = wH$$

Consider the liquid as being forced out of the orifice by pressure p.

Let a = area of jet,

W = weight of liquid issuing per second.

Work done on liquid $= p \times$ volume of liquid moved per second

$= p \times av$

Work done on liquid = gain of kinetic energy

Or $$p \times av \quad \frac{Wv^2}{2g}$$

Hence $$wH \times av = \frac{Wv^2}{2g}$$

But $$W = wav \text{ Lb/sec}$$

Therefore $$H = \frac{v^2}{2g} \text{ ft of liquid}$$

3.5. The Total Head of Liquid. The total head of a particle of liquid at any instant is the sum of its datum head, its velocity head, and its pressure head. The datum head is reckoned above a convenient datum level. Let Z be the height of the particle considered above the chosen datum level.

Datum head per pound of liquid $= Z$ ft

Velocity head per pound of liquid $= \frac{v^2}{2g}$ ft of liquid

Pressure head per pound of liquid $= \frac{p}{w}$ ft of liquid

Then $$\text{total head per pound} = Z + \frac{p}{w} + \frac{v^2}{2g}$$

For any mass of flowing liquid in which there is a continuous connexion between all the particles, the total head of each particle is the same. This is known as Bernoulli's theorem and is very important in solving problems dealing with the flow of liquids.

Consider the vessel in Fig. 32; let the liquid flow through an orifice in the side of the vessel with a velocity v under the static head H. Apply Bernoulli's theorem to the points A, B, and C.

Total head at A = total head at B = total head at C

$$Z_A + \frac{p_A}{w} + \frac{v_A{}^2}{2g} = Z_B + \frac{p_B}{w} + \frac{v_B{}^2}{2g} = Z_C + \frac{p_C}{w} + \frac{v_C{}^2}{2g}$$

Assuming the line xx to be the datum level, and ignoring the atmospheric pressure, which is constant throughout, the equations become

$$H + 0 + 0 = 0 + \frac{p_B}{w} + 0 = 0 + 0 + \frac{v^2}{2g}$$

or
$$H = \frac{p_B}{w} = \frac{v^2}{2g} \text{ ft of liquid}$$

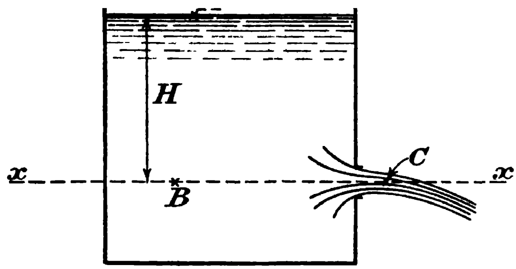


FIG. 32

It should be noticed that no account has been taken of any frictional losses which may occur between the points chosen. Any such losses should be added or subtracted from one side of the equation.

As an example, supposing there is a loss of head between B and C equal to h ft of liquid. Then,

$$Z_B + \frac{p_B}{w} + \frac{v_B^2}{2g} = Z_C + \frac{p_C}{w} + \frac{v_C^2}{2g} + h$$

PROOF OF BERNOULLI'S THEOREM. Consider a liquid flowing through the non-uniform pipe of Fig. 33. The pipe is running full and under pressure. Consider the volume of liquid between the two sections AA and BB.

Let Z, p, v, and a be the height above datum, pressure, velocity, and area of pipe respectively at section AA. Let Z_1, p_1, v_1, and a_1 be the corresponding values at BB. Let the whole quantity of liquid between AA and BB move to the position $A'A'$, $B'B'$, the movement being small.

Let distance between AA and $A'A' = \mathrm{d}l$

BB and $B'B' = \mathrm{d}l_1$

Then $$a\,\mathrm{d}l = a_1\,\mathrm{d}l_1 \quad . \quad . \quad . \quad (3)$$

This movement of the whole quantity of liquid between AA and BB is equivalent to moving the quantity between AA and $A'A'$ to

the position between BB and $B'B'$; the remainder of the liquid between $A'A'$ and BB is unaffected.

Let W = weight of liquid between AA and $A'A'$

$= wa\,dl$ . . . . (4)

$= wa_1\,dl_1$ . . . . (5)

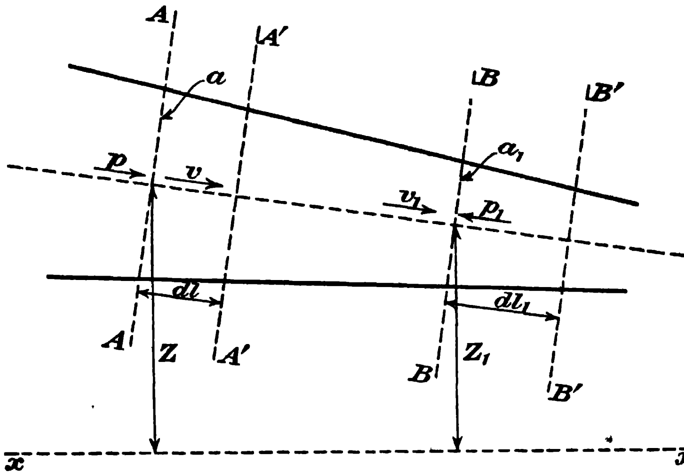


FIG. 33

Work done by pressure at AA $pa\,dl$

Work done by pressure at BB $= -p_1a_1\,dl_1$

Net work done by pressure $= pa\,dl - p_1a_1\,dl_1$

$= (p - p_1)a\,dl$ (from eq. (3))

$= W\dfrac{(p - p_1)}{w}$ (from eq. (4))

Loss of potential energy $= W(Z - Z_1)$

Gain of kinetic energy $\dfrac{W}{2g}(v_1^2 - v^2)$

Then, loss of potential energy + work done by pressure
= gain of kinetic energy

i.e. $$W(Z - Z_1) + W\frac{(p - p_1)}{w} = \frac{W}{2g}(v_1^2 - v^2)$$

Therefore $$Z + \frac{p}{w} + \frac{v^2}{2g} = Z_1 + \frac{p_1}{w} + \frac{v_1^2}{2g}$$

EXAMPLE 2

Water is flowing down a vertical tapering pipe 6 ft long. The top of the pipe has a diameter of 4 in., and the diameter of the bottom of the pipe is 2 in. If the quantity of water flowing is 300 gal/min, find the difference of pressure between the top and bottom ends of the pipe.

Let v_1, p_1, Z_1, and a_1 refer to lower end of pipe,
v_2, p_2, Z_2, and a_2 refer to top end of pipe.

$$\text{Quantity of water flowing per second} = \frac{300 \times 10}{60 \times 62{\cdot}4} = 0{\cdot}802 \text{ ft}^3$$

$$\text{Area of lower end of pipe} = \frac{\pi}{4} \times 2^2 = 3{\cdot}14 \text{ in.}^2$$

$$\text{Area of top end of pipe} = \frac{\pi}{4} \times 4^2 = 12{\cdot}56 \text{ in.}^2$$

$$v_1 = \frac{\text{quantity}}{\text{area in ft}^2} = \frac{0{\cdot}802 \times 144}{3{\cdot}14} = 36{\cdot}8 \text{ ft/sec}$$

$$v_2 = \frac{0{\cdot}802 \times 144}{12{\cdot}56} = 9{\cdot}2 \text{ ft/sec}$$

Applying Bernoulli's equation to both ends of pipe, and taking the datum level through the lower end,

$$Z_1 + \frac{p_1}{w} + \frac{v_1^2}{2g} = Z_2 + \frac{p_2}{w} + \frac{v_2^2}{2g}$$

$$0 + \frac{p_1}{w} + \frac{36{\cdot}8^2}{64{\cdot}4} = 6 + \frac{p_2}{w} + \frac{9{\cdot}2^2}{64{\cdot}4}$$

$$\frac{p_2 - p_1}{w} = 21{\cdot}01 - 1{\cdot}31 - 6$$

$$= 13{\cdot}7 \text{ ft of water}$$

or

$$p_2 - p_1 = \frac{13{\cdot}7 \times 62{\cdot}4}{144}$$

$$= 5{\cdot}93 \text{ Lb/in.}^2$$

3.6. The Venturi Meter. A practical application of Bernoulli's theorem is found in the Venturi meter,* an instrument for measuring the quantity of liquid flowing through a pipe. The meter, in its simplest form, consists of a short length of pipe, tapering to a narrow throat in the middle (Fig. 34). Tubes enter the pipe at the enlarged end and at the throat, by means of which the pressure of

* For a description of an actual Venturi meter see § 20.3. For flow of gas through a Venturi meter see § 14.5.

the liquid at these sections may be measured. Piezometer tubes may be used, or the tubes may be connected to a U-tube. As the liquid flows through the meter the velocity will increase at the throat owing to the reduction of area; consequently the pressure will be reduced. This reduction of pressure is measured by means of the

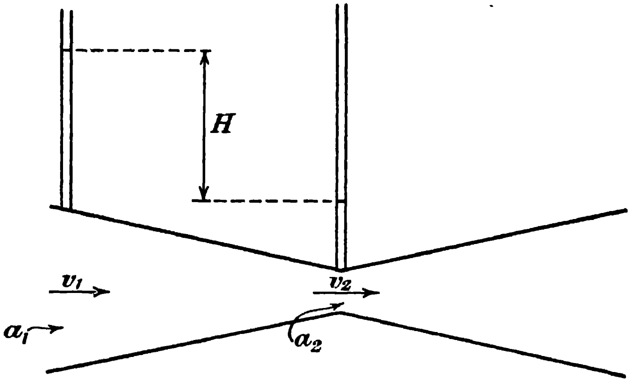


FIG. 34

piezometer tubes. Then, by applying Bernoulli's equation to the enlarged end and to the throat, the quantity of liquid flowing may be calculated.

Let H = difference of pressure head in feet of liquid in the piezometer tubes,

a_1 = area of enlarged end in square feet,

a_2 = area of throat in square feet,

Q = quantity of liquid flowing in cubic feet per second,

v_1 = velocity of liquid at enlarged end,

v_2 = velocity of liquid at throat.

Then
$$Q = a_1 v_1 = a_2 v_2$$

Therefore
$$v_1 = v_2 \frac{a_2}{a_1} \tag{6}$$

Applying Bernoulli's equation, and assuming the meter to be horizontal,

$$\frac{p_1}{w} + \frac{v_1^2}{2g} = \frac{p_2}{w} + \frac{v_2^2}{2g}$$

or
$$\frac{p_1}{w} - \frac{p_2}{w} = \frac{v_2^2}{2g} - \frac{v_1^2}{2g}$$

But
$$\frac{p_1}{w} - \frac{p_2}{w} = H$$

Therefore $$H = \frac{v_2^2 - v_1^2}{2g}$$

Substituting for v_1 from eq. (6),

$$H = \frac{v_2^2}{2g}\left(1 - \frac{a_2^2}{a_1^2}\right)$$

Therefore $$v_2 = \frac{a_1}{\sqrt{a_1^2 - a_2^2}} \sqrt{2gH}$$

$$Q = a_2 v_2$$

$$= \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \sqrt{2g} \sqrt{H}$$

But, $\frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \sqrt{2g}$ is a constant for any one meter; let this constant equal c. Then,

$$Q = c\sqrt{H} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

It will be noticed that, if the Venturi meter discharges into the atmosphere, the pressure at the throat must be less than atmospheric; hence, a vacuum pressure must occur at the throat. This is known as a *Venturi vacuum* and it is used as a suction pump, for small quantities of flow, by connecting a suction tube to the throat.

In this case H is the theoretical head, as no frictional losses have been taken into account. In practice it is found that there is a loss of head in the meter between the enlarged end and the throat; consequently the liquid will not rise so high in the pressure tube at the throat. This means that a larger difference of head will be measured. In order to allow for this, a coefficient k is introduced into the equation, the magnitude of k being found experimentally.

Let h = difference of head, in feet of water, actually measured.

Then $$Q = kc\sqrt{h} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

But $$Q = c\sqrt{H}$$

Therefore $$kc\sqrt{h} = c\sqrt{H}$$

from which $$k = \sqrt{\frac{H}{h}} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The actual head measured, h, is known as the Venturi head.

In the converging cone of the meter, h will be larger than H; then k will be less than unity. An average value of k is 0·97.*

The loss of head in the meter will be partly due to friction and partly due to shock caused by a change of section. Consequently,

* A curve showing the values of k for the Venturi meter is shown in Fig. 360, Appendix 1.

k will not be truly a constant for all velocities; but the variation is slight.

The Venturi meter is not accurate for very low velocities on account of the variation of k.

It will be noticed that there is a limit to the ratio of the diameters of the throat and enlarged end. The larger this ratio is, the smaller

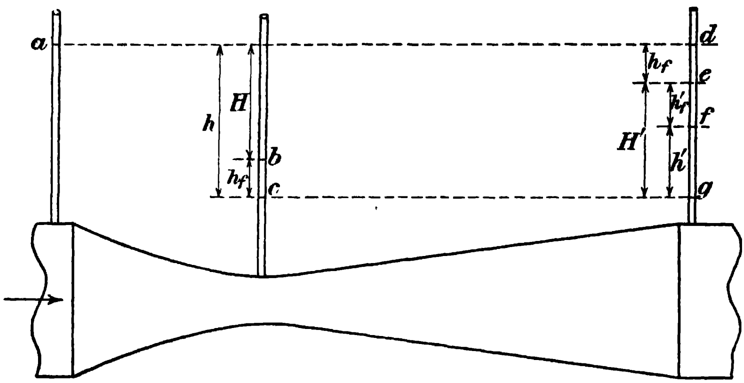


FIG. 35

will be the pressure in the throat; if the fluid is water and if the pressure in the throat falls below 8 ft of water absolute, dissolved gases and vapour will be given off from the water; this will interfere with the flow (§ 1.2). Hence, the limiting ratio of the diameters is reached when the throat pressure is approximately 8 ft of water absolute.

The coefficient k will have a different value for the converging and diverging cones of the meter. In the converging cone the theoretical head is less than the actual head, whilst in the diverging cone the theoretical head is greater than the actual head. Consider the Venturi meter shown in Fig. 35; let pressure tubes be fitted at both enlarged ends and throat. Assume the liquid is flowing from left to right.

Let h_f = head lost in converging cone,

h_f' = head lost in diverging cone,

H' = theoretical difference of head in diverging cone,

h' = actual difference of head in diverging cone.

Let the liquid level at the left enlarged end be at a. Then, if there were no losses in the meter, the liquid level at the right enlarged end would be at the same level d. The friction loss in the converging cone reduces this water level to e; whilst the frictional loss in the diverging cone further reduces the level to f.

Referring to the converging cone only, from eq. (9),

$$H = k^2h$$

But $$h_f = h - H$$

Therefore $$h_f = h(1 - k^2) \quad . \quad . \quad . \quad . \quad (10)$$

Referring to the diverging cone, let k' be the coefficient of the diverging cone. Then, if there were no frictional loss in this cone, the liquid level at the enlarged end would rise to e; the theoretical difference of head between this section and the throat would then be the height eg. But due to the frictional loss the liquid level only reaches f. Then the difference of head actually measured is fg.

Then $$\text{quantity flowing} = c\sqrt{H'} = k'c\sqrt{h'}$$

Therefore $$H' = k'^2h' \quad . \quad . \quad . \quad . \quad (11)$$

But $$h_f' = H' - h'$$

Therefore $$h_f' = h'(k'^2 - 1) \quad . \quad . \quad . \quad (12)$$

Also, from Fig. 35,

$$fg = dg - df$$

or $$h' = h - h_f - h_f' \quad . \quad . \quad . \quad (13)$$

It will be noticed that k' is greater than unity.

EXAMPLE 3

State and prove Bernoulli's theorem. The difference of head registered in the two limbs of a mercury gauge, with water above the mercury, connected to a Venturi meter was 7 in. The diameters of the pipe and the throat of the meter are 6 in. and 3 in. respectively. The coefficient of the meter is 0·97. Find the discharge through the meter. (*Lond. Univ.*)

$$\text{Difference of head in feet of water} = \frac{7(13{\cdot}6 - 1)}{12} \qquad (\S\ 1.7)$$

$$= 7{\cdot}35 \text{ ft}$$

$$a_1 = \frac{\pi}{4}(0{\cdot}5)^2$$

$$a_2 = \frac{\pi}{4}(0{\cdot}25)^2$$

$$c = \frac{a_1a_2\sqrt{2g}}{\sqrt{a_1^2 - a_2^2}}$$

$$= \frac{\pi}{4} \times \frac{0{\cdot}25 \times 0{\cdot}0625\sqrt{64{\cdot}4}}{\sqrt{0{\cdot}0625 - 0{\cdot}0039}}$$

$$= 0{\cdot}407$$

$$\text{Quantity} = kc\sqrt{h} \qquad (\text{eq. } (8))$$

$$= 0{\cdot}97 \times 0{\cdot}407\sqrt{7{\cdot}35}$$

$$= 1{\cdot}07 \text{ ft}^3/\text{sec}$$

EXAMPLE 4

Show that in a Venturi meter the quantity of water passing through the meter will only be proportional to the root of the "Venturi head" if the head lost in friction is proportional to the head lost due to increased velocity.

A Venturi meter placed in a 3 in. diameter pipe has a throat diameter of 1 in. The constant of the meter is 0·97. Determine the number of cubic feet passing per minute when the Venturi head is 16·2 in. of water.

If the frictional loss in the diverging cone is double that in the converging cone, find the total head lost in the meter due to friction when the water is passing at the above rate. (*Lond. Univ.*)

This question assumes that the whole of the head lost in the meter is due to friction.

The coefficient k can only be a constant if $h_f \propto H$; because

$$k = \sqrt{\frac{H}{h}} \qquad \text{(from eq. (9))}$$

Also $$h = H + h_f$$

Let $h_f = mH$ where m is a constant. Then,

$$k = \sqrt{\frac{H}{H + mH}} = \sqrt{\frac{1}{1 + m}} = \text{a constant}$$

$$\text{Quantity per second} = k\,\frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}}\,\sqrt{2g}\,\sqrt{h}$$

$$= 0{\cdot}97\;\frac{\frac{\pi}{4}\left(\frac{1}{144} \times \frac{1}{16}\right)}{\sqrt{\frac{1}{256} - \frac{1}{20{,}700}}}\;\sqrt{\frac{16{\cdot}2}{12}}$$

$$= 0{\cdot}97 \times 0{\cdot}0055 \times 8{\cdot}02 \times 1{\cdot}16 = 0{\cdot}0496 \text{ ft}^3$$

$$\text{Quantity per minute} = 0{\cdot}0496 \times 60 = 2{\cdot}98 \text{ ft}^3$$

From eq. (10), $$h_f = h(1 - k^2)$$

$$= \frac{16{\cdot}2}{12}(1 - 0{\cdot}97^2) = 0{\cdot}08 \text{ ft}$$

$$\text{Head lost in diverging cone} = 2 \times 0{\cdot}08 = 0{\cdot}16 \text{ ft}$$

$$\text{Total head lost} = 0{\cdot}08 + 0{\cdot}16 = 0{\cdot}24 \text{ ft}$$

3.7. Horse-power of Jet of Fluid. The horse-power of a jet of fluid may be obtained by dividing the kinetic energy of the jet per second by 550.

Let a = area of cross-section of jet in square feet,

v = velocity of jet in feet per second,

W = weight of fluid flowing per second
$= wav.$

Then kinetic energy of jet $= \dfrac{Wv^2}{2g}$ ft-Lb/sec

$$\text{Horse-power} = \frac{Wv^2}{2g \times 550}$$
$$= \frac{wav^3}{2g \times 550}.$$

EXAMPLE 5

A jet of water has a velocity of 20 ft/sec. If the diameter of the jet is 2 in., find the horse-power.

$$\text{Area of jet} = \frac{\pi}{4} \times \frac{2^2}{144}$$
$$= 0{\cdot}0218 \text{ ft}^2$$
$$\text{Horse-power} = \frac{wav^3}{2g \times 550}$$
$$= \frac{62{\cdot}4 \times 0{\cdot}0218 \times 20^3}{64{\cdot}4 \times 550}$$
$$= 0{\cdot}308$$

3.8. The Radial Flow of a Liquid. Consider a liquid flowing radially between two horizontal circular flat plates placed parallel with a small distance between them (Fig. 36). The space between the plates is full of liquid. Let the liquid flow up a central pipe and then flow radially outwards between the plates. The outside of the plates is open to the atmosphere, so that the liquid will be discharged at atmospheric pressure.

Let v_0 = velocity of liquid in pipe,
p_0 = absolute pressure of liquid in pipe,
a_0 = area of cross-section of pipe,
p_a = pressure of atmosphere,
v_a = velocity of liquid when leaving plates.

As the liquid flows between the plates radially outwards, the area of flow will increase; therefore, the velocity will decrease. This will cause an increase in pressure.

Consider the total energy of the liquid at A, just inside the pipe, and at B which is at the outer edge of the plates.

Let R = radius of plates at B
and t = distance between the plates.

Then,

$$\text{energy at } A = \text{energy at } B$$

$$\frac{p_0}{w} + \frac{v_0^2}{2g} = \frac{p_a}{w} + \frac{v_a^2}{2g} = H$$

where H is a constant.

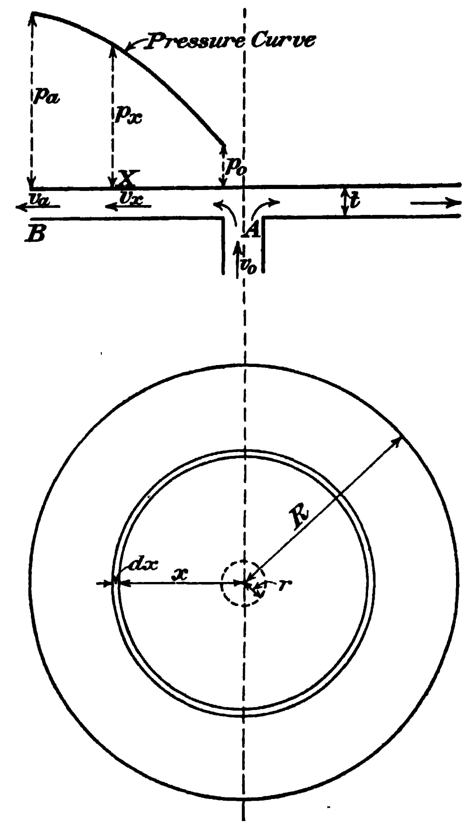


Fig. 36

As p_0, v_0, and p_a are known, this equation will give v_a.

Consider any point X at a radius of x from the centre of the plates. Let v_x be velocity of the liquid at this point and p_x the pressure. Then, as quantity of liquid flowing is a constant at all sections,

$$v_a \times 2\pi Rt = v_x \times 2\pi xt$$

or

$$v_x = v_a \frac{R}{x} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Also $\qquad$ total energy at $X = H$

$$= \frac{p_x}{w} + \frac{v_x^2}{2g}$$

Substituting from eq. (14),

$$\frac{p_x}{w} = H - \frac{v_a^2 R^2}{2gx^2} \quad . \quad . \quad . \quad (15)$$

Thus, the pressure at any point varies with the square of the radius at that point and will increase towards the outer edge, the increase following a parabolic law. If the pressure at any radius is plotted as shown in Fig. 36, the parabolic curve thus obtained is known as Barlow's curve.

Having found the intensity of pressure at any radius, the total static pressure on the plate may be obtained by finding an equation for the pressure on a thin ring and integrating between the required limits.

Let r = radius of pipe.

Consider the pressure at X on a thin ring of thickness $\mathrm{d}x$. From eq. (15),

$$p_x = w\left(H - \frac{k}{x^2}\right)$$

where the constant $k = v_a^2 R^2/2g$.

$$\text{Area of ring} = 2\pi x\,\mathrm{d}x$$

$$\text{Total pressure on ring} = p_x 2\pi x\,\mathrm{d}x$$

$$= 2\pi w\left(H - \frac{k}{x^2}\right) x\,\mathrm{d}x$$

Total static pressure due to liquid pressure on upper plate

$$= 2\pi w \int_r^R \left[Hx\,\mathrm{d}x - \frac{k}{x}\,\mathrm{d}x\right] + p_0\pi r^2$$

$$= 2\pi w \left[\frac{Hx^2}{2} - k\log_e x\right]_r^R + p_0\pi r^2$$

$$= 2\pi w \left\{\frac{H}{2}(R^2 - r^2) - k\log_e\frac{R}{r}\right\} + p_0\pi r^2$$

$$. \quad . \quad . \quad . \quad (16)$$

This is the total upward absolute static pressure. If the atmosphere is pressing on the outside of the plate, the net static pressure will be the total atmospheric pressure on the plate minus the above liquid pressure.

$$\text{Total atmospheric pressure on plate} = p_a\pi R^2$$

It will be noticed that the dynamic force due to the entering liquid has not been included.

The principle is made use of in the nozzles of fire hydrants in order to produce an even distribution of flow.

The same reasoning will hold when the liquid is flowing radially inwards, passing away down the centre pipe.

Example 6

Water flows radially outwards between two horizontal discs which are $\frac{1}{2}$ in. apart and 12 in. in diameter. The water enters at the centre of the lower disc through a 2 in. diameter pipe, with a velocity of 20 ft/sec. Find the pressure of the water in this pipe if the pressure at the outer edge of the discs is atmospheric. Find also the resultant static pressure on the upper disc. Neglect the dynamic force of the entering water.

Using the notation of Fig. 36,

$$v_a = \frac{v_0 r^2 \pi}{2\pi R t}$$

$$= \frac{20 \times 1}{2 \times 6 \times 0{\cdot}5} = 3{\cdot}33 \text{ ft/sec}$$

Applying Bernoulli's equation,

$$\frac{p_0}{w} + \frac{v_0^2}{2g} = \frac{p_a}{w} + \frac{v_a^2}{2g}$$

$$\frac{p_0}{w} = 34 + \frac{3{\cdot}33^2}{64{\cdot}4} - \frac{20^2}{64{\cdot}4}$$

$$= 28 \text{ ft of water (absolute)}$$

Let $$H = \frac{p_a}{w} + \frac{v_a^2}{2g} = 34 + 0{\cdot}173 = 34{\cdot}173$$

Also $$k = \frac{v_a^2 R^2}{2g} = \frac{11{\cdot}11 \times 0{\cdot}5^2}{64{\cdot}4} = 0{\cdot}0432$$

Using eq. (16),

total upward pressure on upper plate

$$= 2\pi w \left[\frac{34{\cdot}173}{2}\left(0{\cdot}5^2 - \frac{1}{12^2}\right) - 0{\cdot}0432 \log_e 6\right] + p_0 \pi r^2$$

$$= 2\pi \times 62{\cdot}4\,\{(17{\cdot}086 \times 0{\cdot}243) - (0{\cdot}0432 \times 2{\cdot}303 \times 0{\cdot}778)\} + 12{\cdot}1\pi$$

$$= 1{,}598 + 38$$

$$= 1{,}636 \text{ Lb}$$

Downward pressure of atmosphere

$$= p_a \pi R^2$$

$$= 14{\cdot}7 \times \pi \times 6^2 = 1{,}660 \text{ Lb}$$

$$\text{Net pressure on plate} = 1{,}660 - 1{,}636$$

$$= 24 \text{ Lb}$$

3.9. Centrifugal Head Impressed on Revolving Liquid. A rotating fluid is called a vortex. If the fluid is rotating freely without any external forces being impressed upon it, it is called a free vortex. An example of a free vortex is the whirlpool formed in the emptying of a wash basin having a central drain. If the fluid is rotated by an external force the vortex is termed a forced vortex. A forced vortex will have a centrifugal head impressed on the liquid, caused by its rotation.

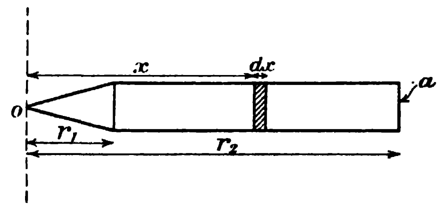


Fig. 37

Referring to Fig. 37, imagine a hollow cylinder containing liquid to be rotating in a horizontal plane about the centre O and with an angular velocity of ω. Let the hollow cylinder be full of liquid between a radius of r_1 and r_2 and let the cross-sectional area of the hollow cylinder be a.

Consider a small section of the liquid of thickness dx and at a radius of x. Then,

volume of small section of liquid $= a\,\mathrm{d}x$

weight of small section of liquid $= wa\,\mathrm{d}x$

Centrifugal force acting on liquid considered

$$= \frac{wa\,\mathrm{d}x}{g}\omega^2 x$$

Total centrifugal force impressed on whole of rotating liquid

$$= \int_{r_1}^{r_2} \frac{wa\,\mathrm{d}x}{g}\,\omega^2 x$$

$$= \frac{wa\omega^2}{2g}\Big[x^2\Big]_{r_1}^{r_2}$$

$$= \frac{wa\omega^2}{2g}\,(r_2^2 - r_1^2)$$

Let v_1 = tangential velocity at radius of r_1

and v_2 = tangential velocity at radius of r_2.

Then, total centrifugal force impressed

$$= \frac{wa}{2g}\,(v_2^2 - v_1^2)$$

since $v_1 = \omega r_1$ and $v_2 = \omega r_2$.

Intensity of pressure at end of cylinder due to centrifugal force

$$= \frac{w}{2g}\,(v_2^2 - v_1^2)$$

$$\text{Centrifugal head impressed} = \frac{p}{w} = \frac{v_2^2}{2g} - \frac{v_1^2}{2g}$$

Thus, the centrifugal head impressed on a revolving fluid is the difference between the tangential velocity heads.

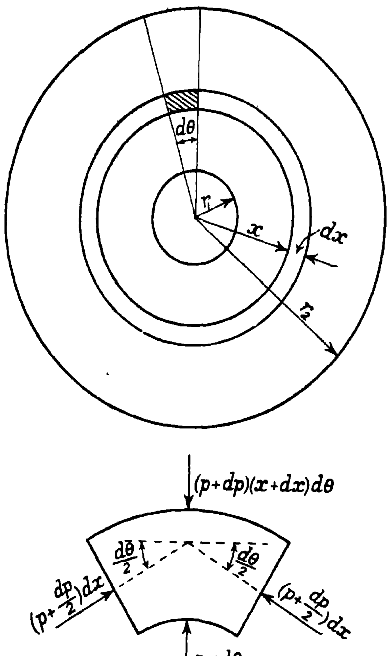


FIG. 38

This is the principle of the centrifugal pump, which obtains its lifting power from this head.

Alternative Proof. A more general proof for the centrifugal head impressed on revolving liquid may be obtained by considering the annular ring of liquid revolving with an angular velocity ω (Fig. 38). Let r_1 and r_2 be the internal and external radii, and consider a thin ring of the liquid of radius x and thickness dx. Consider a portion of this thin ring subtending a small angle dθ at the centre and let p be the intensity of pressure on the inside of the element, due to the centrifugal force. Then the centrifugal pressure will increase by dp over the thickness of the ring dx. Consider the whole annular ring to be of unit thickness in the plane of the paper. Then—

$$\text{area of inside of element} = x\,\mathrm{d}\theta$$

$$\text{area of outside of element} = (x + \mathrm{d}x)\mathrm{d}\theta$$

$$\text{area of sides of element} = \mathrm{d}x$$

$$\text{intensity of pressure on outside of element} = p + \mathrm{d}p$$

$$\text{intensity of pressure on sides of element} = p + \frac{\mathrm{d}p}{2}$$

$$\text{weight of element} = w(x\,\mathrm{d}\theta)\mathrm{d}x$$

$$\text{centrifugal force on element} = \frac{(wx\,\mathrm{d}\theta\,\mathrm{d}x)\omega^2 x}{g}$$

Consider the enlarged view of the element (Fig. 38); the normal forces due to the pressure of the liquid are shown in the figure. These, together with the centrifugal force, keep the element in equilibrium. Hence, by resolving radially, the required equation may be obtained.

Resolving radially, and assuming the sine of a small angle to be equal to the angle in radians,

$$-px\,\mathrm{d}\theta - 2\left(p + \frac{\mathrm{d}p}{2}\right)\mathrm{d}x\frac{\mathrm{d}\theta}{2} + (p + \mathrm{d}p)(x + \mathrm{d}x)\mathrm{d}\theta = \frac{wx\,\mathrm{d}\theta\,\omega^2 x\,\mathrm{d}x}{g}$$

Dividing throughout by $\mathrm{d}\theta$, and ignoring all small quantities of the second order,

$$\mathrm{d}p = \frac{w\omega^2 x\,\mathrm{d}x}{g}$$

Integrating between r_1 and r_2,

$$\text{centrifugal intensity of pressure} = \int \mathrm{d}p = \int_{r_1}^{r_2} \frac{w\omega^2 x\,\mathrm{d}x}{g}$$

$$= \frac{w\omega^2}{2g}(r_2^2 - r_1^2)$$

Then $$\text{centrifugal head} = \frac{p}{w} = \frac{\omega^2}{2g}(r_2^2 - r_1^2)$$

or, as $v_1 = \omega r_1$ and $v_2 = \omega r_2$,

$$\text{centrifugal head} = \frac{v_2^2}{2g} - \frac{v_1^2}{2g}$$

EXAMPLE 7

Water enters a revolving turbine wheel at the centre and flows through the wheel in a radial direction. The wheel is running full. If the inlet radius of the wheel is 2 ft and the outlet radius 3·5 ft, find the centrifugal head impressed on the water when the wheel is running at 300 r.p.m.

$$\text{Velocity of wheel at inlet} = v_1 = 2\pi \times 2 \times \frac{300}{60}$$

$$= 62{\cdot}8 \text{ ft/sec}$$

$$\text{Velocity of wheel at outlet} = v_2 = 62{\cdot}8 \times \frac{3{\cdot}5}{2}$$

$$= 110 \text{ ft/sec}$$

$$\text{Centrifugal head} = \frac{v_2^2}{2g} - \frac{v_1^2}{2g}$$

$$= \frac{110^2 - 62{\cdot}8^2}{64{\cdot}4}$$

$$= 126{\cdot}7 \text{ ft of water}$$

3.10. Revolving Cylinder of Liquid. Consider a cylinder containing a liquid to be revolved about a vertical axis OC (Fig. 39). The surface of the liquid will take the shape of a paraboloid as shown. This is another example of a forced vortex.

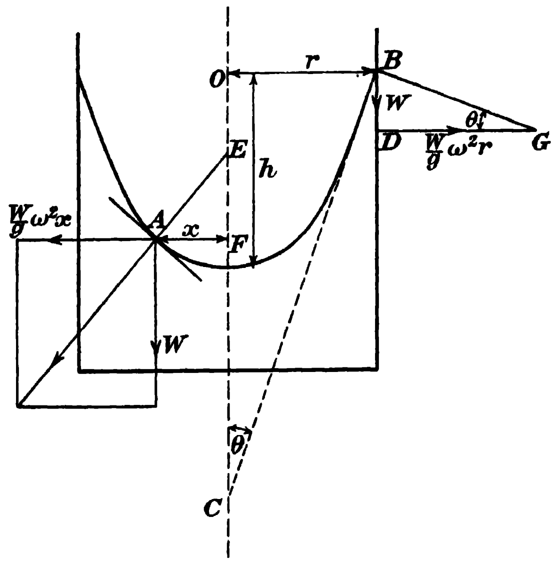


FIG. 39

Consider a small particle of the liquid at the point A on the surface. Let W be the weight of the particle. It will be in equilibrium under the action of three forces: the weight, the centrifugal force, and the pressure.

Let ω = angular velocity of cylinder,
x = radius of particle.

$$\text{Centrifugal force on particle} = \frac{W}{g}\,\omega^2 x$$

The centrifugal force will act horizontally outwards, and the weight vertically downwards. The resultant of these two will be opposed by the pressure of the fluid. As the latter must act normal to the surface, it follows that a tangent to the surface at A will be at right angles to the resultant of the centrifugal force and the weight.

It follows from Fig. 39 that

$$\frac{EF}{x} = \frac{W}{W\omega^2 x} \text{ (from similar triangles)}$$

Therefore $$EF = \frac{g}{\omega^2}$$

and is, therefore, a constant.

As EF is the subnormal of the liquid surface, the shape of the surface is a paraboloid.

Consider the liquid at B.

Let r = radius at B,

θ = angle of inclination of surface at B to vertical,

h = height of paraboloid.

Consider the similar triangles BDG and BOC,

$$\frac{BD}{DG} = \frac{BO}{OC}$$

or $$\frac{W}{\frac{W}{g}\omega^2 r} = \frac{r}{2h}$$

as OC will be twice the height of the paraboloid.

Therefore,

$$h = \frac{\omega^2 r^2}{2g} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$= \frac{v^2}{2g}$$

where v is the tangential velocity at the point B.

It will be noticed that h is a direct function of the square of the speed of rotation. This is made use of in a type of speedometer used in engine testing. A vertical, cylindrical glass vessel containing a liquid is rotated about its vertical axis by the engine, the speed of which can be estimated by the height h of the paraboloid formed in the vessel.

EXAMPLE 8

In order to measure the speed of a steam engine during a test, a glass cylinder containing oil is rotated about a vertical axis by the engine and is geared at twice the speed. If the paraboloid formed by the rotating liquid is 3 in. high, with a maximum radius of $1\frac{1}{2}$ in., find the number of revolutions per minute made by the engine at that instant.

Using eq. (17),

$$h = \frac{\omega^2 r^2}{2g}$$

Then $$\frac{1}{4} = \frac{\omega^2}{64{\cdot}4}\left(\frac{1}{8}\right)^2$$

Therefore $\omega = 32{\cdot}12$ rad/sec

$$\text{Speed of cylinder} = \frac{\omega}{2\pi}$$

$$= \frac{32{\cdot}12}{2\pi} = 5{\cdot}11 \text{ rev/sec}$$

$$\text{Speed of engine} = \frac{5{\cdot}11 \times 60}{2} = 153{\cdot}3 \text{ r.p.m.}$$

EXAMPLE 9

A closed cylinder, 12 in. in diameter and 0·1 in. deep, is completely filled with water. It is rotated about its axis, which is vertical, at 240 r.p.m. Calculate the total pressure of the water on each end. (*A.M.I.C.E.*)

Consider a vertical thin hollow cylinder of water of radius x and thickness dx.

From eq. (17),

$$\text{centrifugal head on thin cylinder} = \frac{\omega^2 x^2}{2g}$$

Hence, intensity of pressure at radius $x = p_x = wh$

$$= \frac{w\omega^2 x^2}{2g}$$

Although this centrifugal pressure is horizontal it will also act vertically on top and bottom of the cylinder, as the pressure of water is transmitted in all directions.

Let r = radius of cylinder in question.

Then, total vertical pressure on top or bottom of cylinder due to centrifugal pressure

$$= \int_0^r p_x \times 2\pi x \, dx$$

$$= \int_0^r \frac{w\omega^2 x^2}{2g} \times 2\pi x \, dx$$

$$= \frac{w\omega^2 \pi}{g} \int_0^r x^3 dx$$

$$= \frac{\pi\omega^2 r^4 w}{4g}$$

$$= \frac{\pi(2\pi 4)^2(\tfrac{1}{2})^4}{4 \times 32{\cdot}2} \times 62{\cdot}4 = 60{\cdot}10 \text{ Lb}$$

Hence, total pressure on top of cylinder $= 60{\cdot}10$ Lb

Total pressure on bottom of cylinder

$$\begin{aligned} &= \text{centrifugal pressure} + \text{weight of water} \\ &= 60{\cdot}10 + (\pi r^2 \times \text{depth} \times w) \\ &= 60{\cdot}10 + \left(62{\cdot}4 \times \pi(\tfrac{1}{2})^2 \times \frac{0{\cdot}1}{12}\right) \\ &= 60{\cdot}10 + 0{\cdot}408 \\ &= 60{\cdot}508 \text{ Lb} \end{aligned}$$

3.11. Energy Variation Across Streamlines. Let AB and CD be two adjacent streamlines in a vertical plane (Fig. 40) in a liquid having motion in two perpendicular planes. Consider a short length of the streamlines subtending an angle $d\theta$ at the centre of curvature of AB, and let the radius of this short length of AB be constant and equal to r.

Let p = pressure on streamline AB,

$p + dp$ = pressure on streamline CD,

dr = distance between AB and CD,

v = velocity of AB,

$v + dv$ = velocity of CD,

Z = height above datum.

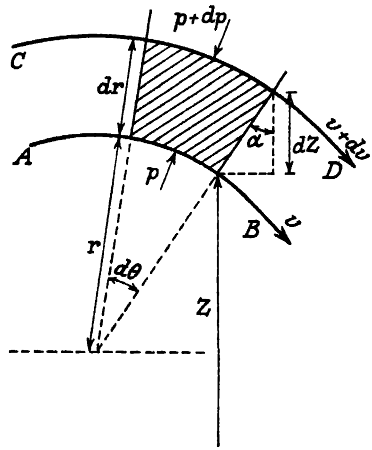


FIG. 40

Let α be the inclination to the vertical of section of streamlines considered, and dZ be the vertical height of CD above AB. Then, from Fig. 40,

$$\cos\alpha = \frac{dZ}{dr} \quad . \quad . \quad . \quad . \quad (18)$$

Also $$w = \rho g$$

where ρ = absolute density.

Consider the section of liquid between the streamlines, shown shaded, and consider it to have unit thickness. Then,

$$\text{weight of section considered} = wr\,d\theta\,dr$$

and $$\text{centrifugal force on section} = \frac{(wr\,d\theta\,dr)v^2}{gr}$$

In Fig. 41 is shown an enlarged view of this section of liquid, the dimensions and forces acting being inserted in the figure. These

forces, plus the centrifugal force and its weight, must balance, as the element is in equilibrium.

Hence, resolving radially and writing $\frac{d\theta}{2}$ for $\sin\frac{d\theta}{2}$,

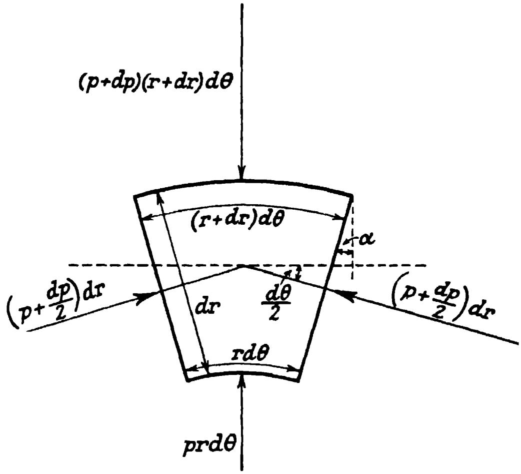


FIG. 41

$$pr\,d\theta + 2\left(p + \frac{dp}{2}\right)dr\,\frac{d\theta}{2} - (p + dp)(r + dr)d\theta - (wr\,d\theta\,dr)\cos\alpha + \frac{(wr\,d\theta\,dr)v^2}{gr} = 0$$

Substituting for $\cos\alpha$ from eq. (18) and ignoring small quantities of the second order,

$$dp = -\,w\,dZ + \frac{w\,dr\,v^2}{gr}$$

from which

$$\frac{dp}{w} = -\,dZ + \frac{v^2\,dr}{gr} \qquad . \quad (19)$$

Now, applying Bernoulli's equation to any streamline,

$$E = Z + \frac{p}{w} + \frac{v^2}{2g}$$

where E is the total energy of the streamline considered.

Differentiating each term for small changes dZ, dp and dv over the width dr,

$$dE = dZ + \frac{dp}{w} + \frac{v\,dv}{g}$$

Substituting for $\mathrm{d}p/w$ from eq. (19),

$$\mathrm{d}E = \mathrm{d}Z + \left(-\mathrm{d}Z + \frac{v^2\,\mathrm{d}r}{gr}\right) + \frac{v\,\mathrm{d}v}{g}$$

Hence $$\mathrm{d}E = \frac{v^2\,\mathrm{d}r}{gr} + \frac{v\,\mathrm{d}v}{g}$$

or $$\frac{\mathrm{d}E}{\mathrm{d}r} = \frac{v}{g}\left(\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{v}{r}\right). \qquad (20)$$

This equation represents the change of energy across any streamline flow in two perpendicular directions. It will be noticed that the Z term does not appear in the final equation. Eq. (20) will, therefore, apply to streamlines moving in any plane.

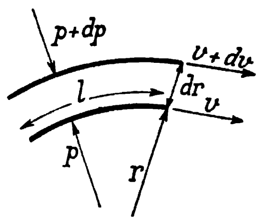


Fig. 42

3.12. Energy Variation Across Horizontal Streamlines. If the streamlines of a moving fluid are in the horizontal plane only, the effect of gravity on the fluid need not be taken into account, in which case eq. (20) can be obtained in a simplified manner.

Consider a short length l of the two adjacent streamlines of Fig. 42, having unit depth. Let the fluid at this section be moving in the horizontal plane only with a velocity v at a radius of r. Assume the two streamlines are at a distance of $\mathrm{d}r$ apart and over this distance let the velocity v increase by $\mathrm{d}v$ and let the pressure p increase by $\mathrm{d}p$; this increase of pressure is due to the centrifugal force on the element considered. Then,

$$\text{weight of element considered} = wl\,\mathrm{d}r$$

As difference of radial force on element = centrifugal force,

then $$\mathrm{d}p \times l = \frac{(wl\,\mathrm{d}r)v^2}{gr}$$

from which $$\frac{\mathrm{d}p}{w} \quad \frac{v^2\,\mathrm{d}r}{gr} \qquad (21)$$

$$\text{Total energy of fluid} = E = \frac{p}{w} + \frac{v^2}{2g}$$

Differentiating for small changes of energy, pressure and velocity across the thickness of the element $\mathrm{d}r$,

$$\mathrm{d}E = \frac{\mathrm{d}p}{w} + \frac{v\,\mathrm{d}v}{g}$$

Substituting for dp/w from eq. (21),

$$dE = \frac{v^2\,dr}{gr} + \frac{v\,dv}{g}$$

from which

$$\frac{dE}{dr} = \frac{v}{g}\left(\frac{v}{r} + \frac{dv}{dr}\right) \quad . \quad . \quad . \quad (22)$$

which is the same equation as eq. (20). It will be noticed that this proof is much simpler than the more general proof given in § 3.11, but is based on a horizontally moving fluid only.

3.13. Two-dimensional Flow of a Liquid. The motion of a liquid in a two-dimensional plane may be in the form of a free cylindrical vortex, a free spiral vortex, a forced vortex, a radial flow, or a straight-line motion. The last type is a simple direct flow and need not be dealt with further; the other four types will be treated separately.

1. A Free Cylindrical Vortex. In this type of flow the streamlines are moving freely in horizontal concentric circles and there is no variation of the total energy E across the streamlines. Then,

$$dE = 0$$

Hence, applying this to eq. (20),

$$\frac{v}{g}\left(\frac{dv}{dr} + \frac{v}{r}\right) = 0$$

then

$$\frac{dv}{v} + \frac{dr}{r} = 0$$

Integrating, $\log_e v + \log_e r = \text{constant}$

Then $vr = \text{constant}$

or

$$v = \frac{C}{r} \quad . \quad . \quad . \quad . \quad (23)$$

where C is the value of the constant for the liquid considered and is known as the *strength* of the vortex.

Applying Bernoulli's equation to any two concentric horizontal streamlines of radius r_1 and r_2, and letting p_1 and p_2 be the pressures at the two streamlines considered,

$$\frac{p_1}{w} + \frac{v_1^2}{2g} = \frac{p_2}{w} + \frac{v_2^2}{2g}$$

Then

$$\frac{p_1 - p_2}{w} = \frac{v_2^2 - v_1^2}{2g}$$

But, from eq. (23),

$$v_2 = \frac{C}{r_2}$$

and

$$v_1 = \frac{C}{r_1}$$

Hence

$$\frac{p_1 - p_2}{w} = \frac{C^2}{2g}\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right) \quad . \quad . \quad . \ (24)$$

If the fluid is a gas, on account of its compressibility, eq. (24) should be written

$$\frac{p_1}{w_1} - \frac{p_2}{w_2} = \frac{C^2}{2g}\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right)$$

This equation gives the difference of pressure head between the two streamlines considered. It will be noticed that, if a curve representing the pressure variation is plotted on a base representing the radius of the vortex, a parabola is obtained having a maximum value at the outer circumference. If the upper surface of the liquid be free, it will assume this parabolic shape.

As the pressure near the outer edge of a free cylindrical vortex is greater than that near the centre, the fluid is caused to flow radially inwards towards its central core, through which it drains away by passing along the core of the cylinder longitudinally, thus causing a suction at the opposite end of the core. The combination of the circumferential flow and of this inward radial flow results in a spiral motion and converts the free cylindrical vortex into a free spiral vortex (Case 4).

It will be seen from this that a free cylindrical vortex cannot be maintained in nature, as it tends to develop into a free spiral vortex. This latter type of vortex occurs in such phenomena as a tornado, a waterspout, and in the emptying of a vessel of liquid by means of a drain at the base.

2. A Forced Vortex. A forced vortex is the name given to a circular stream of liquid the whirl of which is caused by power from an external source. An example of a forced vortex is the stream of water in the casing of a centrifugal pump.

In a forced vortex the liquid has a constant angular velocity. Let ω be the angular velocity of the liquid; then

$$\omega = \frac{v}{r} = \frac{dv}{dr} = \text{constant}$$

From eq. (20),

$$\frac{dE}{dr} = \frac{v}{g}\left(\frac{dv}{dr} + \frac{v}{r}\right) \quad . \quad . \quad . \ (25)$$

Substituting the above value of ω for v/r and dv/dr,

$$dE = \frac{2v}{g}\,\omega\, dr$$

Substituting for $v = \omega r$,

$$dE = \frac{2\omega^2 r\, dr}{g}$$

Using the suffix 1 for the inside radius of the vortex and suffix 2 for the outside, and integrating between these limits,

$$E_2 - E_1 = \frac{\omega^2}{g}\Big[r^2\Big]_{r_1}^{r_2}$$

$$= \frac{\omega^2}{g}(r_2^2 - r_1^2)$$

$$\frac{v_2^2 - v_1^2}{g} \quad . \quad . \quad . \quad (26)$$

But, applying Bernoulli's equation to the limiting streamlines,

$$E_1 = \frac{p_1}{w} + \frac{v_1^2}{2g}$$

and

$$E_2 = \frac{p_2}{w} + \frac{v_2^2}{2g}$$

Hence

$$E_2 - E_1 = \frac{p_2 - p_1}{w} + \frac{v_2^2 - v_1^2}{2g} \quad . \quad . \quad (27)$$

From eqs. (26) and (27),

$$\frac{p_2 - p_1}{w} + \frac{v_2^2 - v_1^2}{2g} = \frac{v_2^2 - v_1^2}{g}$$

from which

$$\frac{p_2 - p_1}{w} = \frac{v_2^2 - v_1^2}{2g} \quad . \quad . \quad . \quad (28)$$

It will be noticed that eq. (28) is the centrifugal head impressed on the liquid and is the same result as that obtained in § 3.9.

If the fluid is a gas, eq. (28) should be written

$$\frac{p_2}{w_2} - \frac{p_1}{w_1} = \frac{v_2^2 - v_1^2}{2g}$$

3. Radial Flow of a Liquid. In this case it is assumed that the liquid is flowing radially outwards between two horizontal flat discs, placed parallel a fixed distance apart. The liquid is assumed to enter by a hole at the centre and leave at the circumference, as is shown in Fig. 36 (page 62). As the path of the liquid is a straight

line, the radius r of the streamline is infinity. Then, applying this to eq. (25), and putting r and dr equal to infinity,

$$\frac{dE}{\infty} = \frac{v}{g}\left(\frac{dv}{\infty} + \frac{v}{\infty}\right) = 0$$

Hence, the change of energy across the streamlines is zero. Applying Bernoulli's equation,

$$E = \frac{p}{w} + \frac{v^2}{2g} = \text{constant for all streamlines}$$

Let Q = discharge in cubic feet per second,
and t = distance between discs in feet.

Then $$v = \frac{Q}{2\pi rt} \text{ at any radius } r$$

Using the suffix 1 for the inner radius and suffix 2 for the outer radius,

$$v_2 = \frac{Q}{2\pi r_2 t}$$

and $$v_1 = \frac{Q}{2\pi r_1 t}$$

As E is a constant for all streamlines,

$$E_2 = E_1$$

that is $$\frac{p_2}{w} + \frac{v_2^2}{2g} = \frac{p_1}{w} + \frac{v_1^2}{2g}$$

from which $$\frac{p_2 - p_1}{w} = \frac{v_1^2 - v_2^2}{2g}$$

$$= \frac{Q^2}{8\pi^2 t^2 g}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) \quad . \quad . \quad . \quad (29)$$

This equation gives the pressure distribution shown in Fig. 36, which is known as Barlow's curve. This type of flow is the same as that dealt with in § 3.8.

4. Free Spiral Vortex. This type of flow is a combination of radial flow (Case 3) and free cylindrical vortex (Case 1). It will be noticed by comparing eqs. (24) and (29) that the pressure variations of these two types of flow are similar.

In a free spiral vortex the liquid is rotating and flowing radially at the same time, thus moving in the form of a horizontal spiral.

Let v_r = radial velocity of liquid

$$= \frac{Q}{2\pi rt} \text{ (Case 3)}$$

and $$v_c = \text{circumferential velocity of liquid}$$
$$= \frac{C}{r} \text{ (Case 1).}$$

Then, at any radius r,

$$\frac{v_r}{v_c} = \frac{Q}{2\pi rt} \times \frac{r}{C}$$
$$= \frac{Q}{2\pi tC} = \text{a constant for all radii}$$

Hence, the resultant velocity of the fluid flows at a constant angle to the tangent at all radii. Let this angle be represented by a; then

$$\tan a = \frac{v_r}{v_c}$$
$$= \frac{Q}{2\pi tC} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

It will be noticed that the above four types of flow have been applied to liquids only, and the density has been assumed constant. The equations obtained may also be applied to gases if the pressure variation is small, in which case the alteration in density is negligible.

EXAMPLE 10

Calculate the difference of pressure between radii of 6 in. and 3 in. of a forced vortex of water which is rotated at 1,450 r.p.m.

Using eq. (28),

$$\frac{p_2 - p_1}{w} = \frac{v_2^2 - v_1^2}{2g}$$
$$= \frac{\omega^2(r_2^2 - r_1^2)}{2g} \text{ (as } v = \omega r)$$
$$= \frac{(2\pi \times 1{,}450)^2(0{\cdot}5^2 - 0{\cdot}25^2)}{(60)^2 \times 2 \times 32{\cdot}2}$$
$$= 67{\cdot}3 \text{ ft of water}$$
$$p_2 - p_1 = \frac{67{\cdot}3 \times 62{\cdot}4}{144}$$
$$= 29{\cdot}2 \text{ Lb/in.}^2$$

EXAMPLE 11

In a free cylindrical vortex of water it is found that at a radius of 3 in. the tangential velocity of the water is 20 ft/sec and its pressure 30 Lb/in.^{2} Calculate the pressure at a radius of 6 in.

For a free cylindrical vortex,

$$v = \frac{C}{r}$$

Hence
$$C = v_1 \times r_1$$
$$= 20 \times \tfrac{3}{12}$$
$$= 5.$$

Using eq. (24),
$$\frac{p_1 - p_2}{w} = \frac{C^2}{2g}\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right)$$

that is
$$\frac{144(30 - p_2)}{62{\cdot}4} = \frac{5^2}{64{\cdot}4}\left(\frac{1}{0{\cdot}5^2} - \frac{1}{0{\cdot}25^2}\right)$$

Then
$$30 - p_2 = \frac{25 \times (-12) \times 62{\cdot}4}{144 \times 64{\cdot}4}$$

Hence
$$p_2 = 30 + 2{\cdot}02$$
$$= 32{\cdot}02 \text{ Lb/in.}^2$$

3.14. Flow of Gases under Constant Head. The velocity with which a gas will flow from one chamber to another may be obtained in the same manner as for a liquid, providing the density remains constant. This will hold only if the pressure difference is small.

Let a gas flow from a chamber A through an orifice or pipe into a chamber B. Let the pressure in A remain constant and equal p_1 Lb/in.² Let the pressure in B also remain constant and equal p_2 Lb/in.² Then p_1 must be greater than p_2. Assume there is no change of temperature. Let w_1 be the density of the gas in A in pounds per cubic foot.

The head causing flow will be due to the difference of pressure in A and B. This head may be expressed as an equivalent static head in feet of gas under the same condition as the gas in A.

$$\text{Equivalent static head} = H_1 = \frac{(p_1 - p_2)144}{w_1} \text{ ft of gas}$$

Velocity of gas

This method of solution is approximate only and will apply only for small velocities.

If the gas being dealt with is atmospheric air, the barometer reading and temperature must be known in order to convert the standard density to the density under the required conditions. The density of air at 0°C and 14·7 Lb/in.² may be taken as 0·081 Lb/ft³. This should be converted to the required density by the law of gases

$$\frac{pV}{T} = \text{a constant}$$

where T is the absolute temperature.

3.15. The Pitot Tube. The Pitot tube is an instrument by which the velocity head of a flowing liquid may be measured. In its simplest form, it consists of a glass tube with the lower end bent through 90° (Fig. 43). It is placed in the moving liquid with the

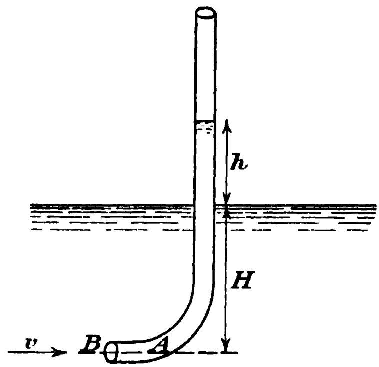


FIG. 43

lower opening facing the direction of motion. The liquid flows up the tube until all its kinetic energy is converted to potential energy; the velocity of the liquid may then be estimated by the height of the liquid in the tube.

This instrument is often used for measuring the velocity of rivers.

Let h = height of liquid in tube above surface,
H = depth of tube in liquid,
v = velocity of liquid.

Applying Bernoulli's equation to the points A and B, which are just outside and inside the mouth of the tube respectively,

$$\text{total head at } A = \text{total head at } B$$

$$H + h = H + \frac{v^2}{2g}$$

Therefore
$$h = \frac{v^2}{2g}$$

In practice, this is usually multiplied by a coefficient k; then

$$h = \frac{kv^2}{2g}$$

In well-formed instruments, k is equal to unity.

Attempts have been made to deduce the value of h by considering the total force at B as equal to the rate of change of momentum;

but this method gives results twice too high. This method obviously cannot be used, as there is a cone of still liquid in front of B which deviates the moving liquid from its sloping sides. This reduces the pressure on the tube, just as the windward side of a structure or building does not get the full force of the wind.

One type of Pitot tube consists of two tubes, one bent at the base, as in Fig. 43, and facing towards the motion of the water, and one straight tube open at the top end with a hole in the lower end parallel to the direction of motion. The velocity head is the difference of water level in the two tubes. The object of this is to eliminate any losses due to the tube.

If the Pitot tube is faced downstream, the water level in the tube is depressed by the amount h.

A view of an actual Pitot tube is shown in Fig. 44; this is known as the Amsler Hydrometrical Tube. It consists of two vertical tubes each having the lower end bent at right angles, one to point upstream against the current, the other to point downstream with the current; both lower ends are tapered to a fine nozzle. The water level in the tube facing downstream is depressed by the amount h. In order to read the height of the water columns in the tubes a small hand pump is fitted at the top of the instrument, by means of which the water columns can be sucked up to any convenient height. The upper parts of the tubes are of glass and are fitted with a sliding graduated scale.

The difference of water level in the two tubes is proportional to the velocity head of the current. Let h_1 be the reading of the upstream tube and h_2 be the reading of the downstream tube. Then,

$$v = c\sqrt{h_1 - h_2}$$

where c is the constant of the instrument.

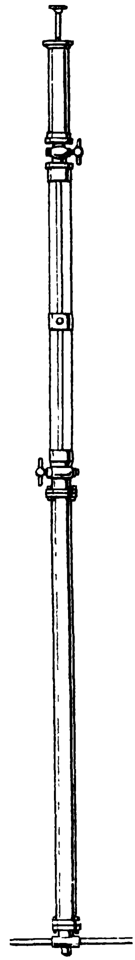


FIG. 44. AMSLER HYDROMETRICAL TUBE

(Courtesy of Amsler and Co., Ltd.)

EXAMPLE 12

The following observations were made for the purpose of calibrating a Pitot tube—

v = velocity of fluid (ft/sec)	1·86	2·96	4·20	6·47	7·97
H = head (in. of water)	0·756	1·72	3·50	9·12	14·40

Plot v against $\sqrt{H}$ and determine the mean value of the constant for the tube. (*Lond. Univ.*)

The values of v and $\sqrt{H}$ are shown plotted in Fig. 45, and a straight line is drawn a mean through the points. This line will pass through the origin as $v = 0$ when $H = 0$.

Let c = constant for the meter. Then,

$$v = c\sqrt{H}$$

Therefore
$$c = \frac{v}{\sqrt{H}}$$

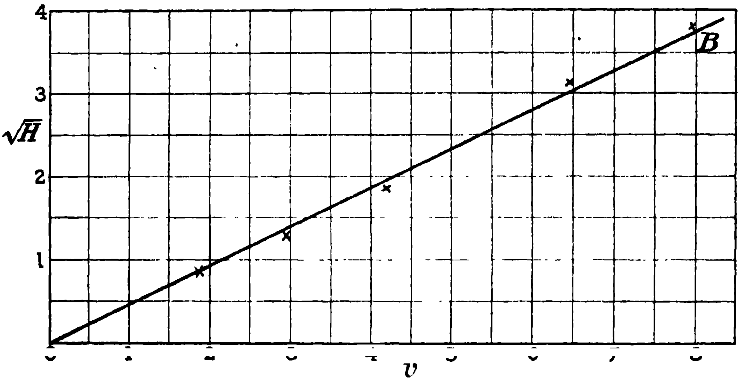


FIG. 45

Using the values of v and $\sqrt{H}$ at the point B (Fig. 45),

$$c = \frac{8}{3 \cdot 7} = 2 \cdot 162$$

Then
$$v = 2 \cdot 162 \sqrt{H}$$

where H is the measured head in inches.

EXAMPLE 13

The velocity of water in a pipe was measured with a Pitot tube consisting of one tube with orifice facing the direction of flow and the other orifice perpendicular to the first orifice. The difference of head at the centre of the pipe was 3·5 in. of water. If the mean velocity of the water is two-thirds the velocity at the centre, find the quantity of water flowing per minute. The diameter of the pipe is 10 in. Take the coefficient of the Pitot tube as unity.

$$\text{Area of pipe} = \frac{\pi}{4} \times \frac{10^2}{144} = 0 \cdot 545 \text{ ft}^2$$

$$\text{Velocity at centre of pipe} = k\sqrt{2gh}$$
$$= \sqrt{2g \frac{3 \cdot 5}{12}}$$
$$= 4 \cdot 33 \text{ ft/sec}$$

$$\text{Mean velocity in pipe} = \tfrac{2}{3} \times 4{\cdot}33$$
$$= 2{\cdot}885 \text{ ft/sec}$$
$$\text{Quantity flowing per minute} = 2{\cdot}885 \times 60 \times 0{\cdot}545$$
$$= 94{\cdot}3 \text{ ft}^3$$

3.16. Viscous Flow. A fluid with a low velocity flowing past a surface has a velocity which varies with the distance from the surface, within a limiting distance from the surface. As the velocity

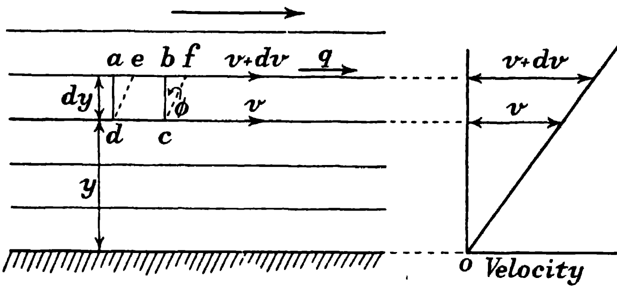


FIG. 46

is very small the fluid may be regarded as flowing in thin parallel layers known as streambands; the flow is then said to be *laminar* flow (Fig. 46).

Referring to Fig. 46, the layer adjacent to the surface is at rest relative to the surface. The next adjacent layer moves with a small velocity. The velocity of each successive layer uniformly increases, as shown by the graph to the right of the figure. Thus, the velocity in each layer increases with its distance from the surface. The relative movement of any layer causes a stress on the adjacent layer, known as a viscous stress. This stress is transmitted from layer to layer until the surface is reached. It is then resisted by what is known as the frictional drag of the surface.

The resistance between the adjacent layers of the fluid is known as viscosity. The viscous resistance of a fluid depends on its viscosity, which is a function of its temperature, on its density, on its relative velocity to the surface, and on the dimensions of the surface.

Consider the laminated layers of fluid shown in Fig. 46. Consider the layer of thickness dy at y from the surface.

Let v = velocity of base of layer,

$v + \mathrm{d}v$ = velocity of upper edge of layer,

q = viscous stress on layer due to drag of adjacent layer

= viscous resistance per unit area.

Consider a rectangular section of the layer *abcd*; an enlarged view of this section is shown in Fig. 47. After an interval of one second the section *abcd* will distort to the shape *efcd* where

$$ae = bf = \mathrm{d}v$$

Let ϕ = angle of distortion

$= \frac{\mathrm{d}v}{\mathrm{d}y}$ as the angle ϕ is extremely small.

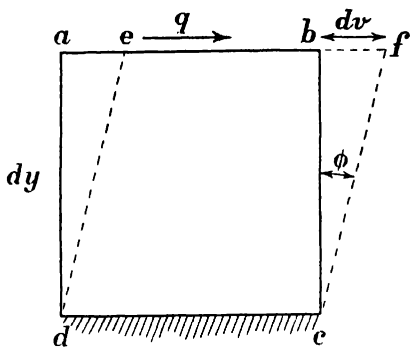


FIG. 47

Newton found that ϕ was a function of the viscous stress q and was a measure of the viscosity of the fluid. The relation between q and ϕ he termed the *coefficient of viscosity.** The coefficient of viscosity is an important physical property of the fluid and varies with its temperature. For gases, it is independent of the pressure.

Let η = coefficient of viscosity

$$= \frac{q}{\phi} \text{ (from definition)}$$

$$= \frac{q}{\frac{\mathrm{d}v}{\mathrm{d}y}} \quad . \quad . \quad (31)$$

from which
$$q = \eta \frac{\mathrm{d}v}{\mathrm{d}y} \quad . \quad . \quad (32)$$

Another convenient method is to incorporate the fluid's density with its coefficient of viscosity. The ratio of these values is known as the *kinematic viscosity*. Then

* For experimental methods of measuring the coefficient of viscosity see 3.8. For values of η see Appendix 2.

$$\text{kinematic viscosity} = \nu = \frac{\eta}{\rho} \quad . \quad . \quad . \quad (33)$$

where ρ = absolute density of fluid
$= w/g$.

It is interesting to notice the analogy between this viscous distortion of a fluid and the shear distortion of a solid. Imagine the area *abcd* (Fig. 47) to represent a solid under a shear stress q; the solid will distort to the shape *efcd*, ϕ being the angle of shear distortion. If G is the shear modulus or modulus of rigidity, then

$$G = q/\phi$$

This is analogous to $\eta = q/\phi$, the coefficient of viscosity η corresponding to the modulus of rigidity G, and the viscous stress corresponding to the shear stress q.

If M, L, and T represent the fundamental dimensional units of mass, space, and time, the units of η may be found by substituting these in eq. (31).

$$\begin{aligned}
\eta &= q \div \frac{dv}{dy} \\
&= \frac{\text{force}}{\text{area}} \div \frac{\text{velocity}}{\text{distance}} \\
&= \frac{\text{mass} \times \text{acceleration}}{\text{area}} \div \frac{\text{velocity}}{\text{distance}} \\
&= \left(\frac{\mathrm{M}}{\mathrm{L}^2} \times \frac{\mathrm{L}}{\mathrm{T}^2}\right) \div \left(\frac{\mathrm{L}}{\mathrm{TL}}\right) \\
&= \frac{\mathrm{M}}{\mathrm{TL}}
\end{aligned}$$

Poiseuille investigated the viscous resistance of water flowing through capillary tubes* and found that the resistance to flow varied inversely with the temperature. This is, of course, well known in the case of oils, which flow more easily when warm.

Let η = coefficient of viscosity at any temperature t in degrees centigrade,

η_0 = coefficient of viscosity at 0°C.

Then, from Poiseuille's experiments,

$$\eta = \eta_0 \left(\frac{1}{1 + at + bt^2}\right)$$

where a and b are constants.

* *Comptes Rendus*, 1840–41.

For water, Poiseuille found that

$$\eta = \eta_0 \left(\frac{1}{1 + 0{\cdot}03368t + 0{\cdot}000221t^2}\right)$$

$$= \frac{0{\cdot}0179}{(1 + 0{\cdot}03368t + 0{\cdot}000221t^2)} \text{ c.g.s. absolute units*}$$

$$= \frac{0{\cdot}0179}{1 + 0{\cdot}03368t + 0{\cdot}000221t^2} \times \frac{30{\cdot}5}{453{\cdot}6 \times 32{\cdot}2} \text{ engineers' units } (\text{slug-ft}^{-1}\text{-sec}^{-1})$$

$$= \frac{0{\cdot}00003738}{(1 + 0{\cdot}03368t + 0{\cdot}000221t^2)} \text{ slug-ft}^{-1}\text{-sec}^{-1} \quad . \quad . \quad . \ (34)$$

Now as $\nu = \eta/\rho$, substituting for η from eq. (34), for water,

$$\nu = \left(\frac{0{\cdot}00003738}{1 + 0{\cdot}03368t + 0{\cdot}000221t^2}\right) \times \frac{32{\cdot}2}{62{\cdot}4}$$

$$= \frac{0{\cdot}00001929}{(1 + 0{\cdot}03368t + 0{\cdot}000221t^2)} \text{ ft}^2\text{/sec}\dagger \quad . \quad . \ (35)$$

It will be noticed that the units of the kinematic viscosity ν are L^2/T; for,

$$\nu = \eta \div \rho$$

$$= \frac{M}{LT} \div \frac{M}{L^3}$$

$$= \frac{L^2}{T}$$

3.17. Non-dimensional Constants. The resistance of a body moving in a fluid, or of a fluid flowing past a body or surface, or flowing through a pipe or channel, may be due to one, or two, of the following types.

1. Viscous or Frictional Resistance. This is due to the frictional drag of the surface (§ 3.16), and the physical property governing this resistance is its coefficient of viscosity η.

2. Surface Wave Formation Resistance. This occurs with ships and other surface craft, and is due to the formation of the bow, stern, and similar surface waves. It may also occur in the flow of water through channels and tidal rivers. The resistance is due to

* This absolute unit is known as a *poise*. A smaller unit, known as a *centipoise*, is also used; a centipose is $\frac{1}{100}$ of a poise.

† This unit is known as a *Stokes* when in centimetres.

the work absorbed in raising the liquid to the height of the waves. This is work done against gravity and is consequently governed by the value of g.

3. RESISTANCE DUE TO THE FORMATION OF COMPRESSION WAVES. This occurs with projectiles and aeroplanes moving in the atmosphere, and with submarines moving when submerged. It also occurs with vapours and gases flowing at high speeds through pipes and nozzles. The nose of the body, and changes in its surfaces such as projections and abrupt changes of slope, will cause pressure waves to be propagated in the fluid. It will be shown in § 13.10 that the governing factor in the formation of pressure waves is the bulk elastic modulus of the fluid.

Each of the above three types of resistance is governed by what is known as a non-dimensional factor, and it will be shown in Chapter 11 that the resistance of each type is a function of this non-dimensional factor.

1. THE REYNOLDS NUMBER. The non-dimensional factor governing viscous or frictional resistance is known as the *Reynolds number*.

Let R_e = Reynolds number of the flow past a surface or object,

ρ = absolute density of fluid = w/g,

v = relative velocity between body, or surface, and fluid (feet per second),

l = linear dimension of body or surface (feet),

η = coefficient of viscosity of fluid (engineers' units).

Then
$$R_e = \frac{\rho v l}{\eta} \quad . \quad . \quad . \quad . \quad . \quad (36)$$

or
$$R_e = \frac{vl}{\nu} \left(\text{as } \nu = \frac{\eta}{\rho}\right)$$

When applying these equations to pipe flow, l is taken as the diameter of pipe, d.

2. THE FROUDE NUMBER. The non-dimensional factor governing surface wave formation resistance is known as the *Froude number*.

Let F_r = Froude number of the fluid flow,

l = linear dimension of body, or depth of channel in channel flow,

v = relative velocity of fluid to surface.

Then
$$F_r = \frac{v}{\sqrt{gl}} \quad . \quad . \quad . \quad . \quad . \quad (37)$$

3. THE MACH NUMBER. The non-dimensional factor governing compression wave resistance is known as the *Mach number.*

Let M_a = Mach number of flow,

v = relative velocity between fluid and body or surface,

v_s = velocity of sound in fluid at its given temperature (§ 17.2).

Then $$M_a = \frac{v}{v_s} \quad . \quad . \quad . \quad . \quad . \quad (38)$$

If the fundamental dimensions M, L, and T are substituted in eqs. (36), (37), and (38), it will be found that each case cancels to unity; thus showing the R_e, F_r, and M_a are non-dimensional.

EXERCISES 3

1. Find the work done in forcing 50 gal of water into a boiler in which the pressure is 180 Lb/in.^{2} gauge. If this work is done in 5 min, what is the horse-power expended? *Ans.* 208,000 ft-Lb. 1·26 h.p.

2. Water is flowing along a pipe with a velocity of 24 ft/sec. Express this as velocity head in feet of water. What is the corresponding pressure in pounds per square inch? *Ans.* 8·95 ft. 3·88 Lb.

3. Water at an altitude of 120 ft above sea level has a velocity of 16 ft/sec and a pressure of 60 Lb/in.^{2} Give the total head of 1 Lb of this water reckoned above sea level. *Ans.* 262·58 ft.

4. A pipe 1,000 ft long has a slope of 1 in 100 and tapers from 4 ft diameter at the high end to 2 ft diameter at the low. The quantity of water flowing is 1,200 gal/min. If the pressure at the high end is 10 Lb/in.^{2}, find the pressure at the low end. Neglect friction. *Ans.* 14·25 Lb/in.^{2}

5. Water flows from a supply tank into a chamber in which the pressure is 10 Lb/in.^{2} vacuum. If the level of the water in the supply tank is 20 ft above the vacuum chamber, find the velocity of the entering water. *Ans.* 52·6 ft/sec.

6. A Venturi meter has an enlarged end of 2 ft^2 area and a throat area of 0·25 ft^2. The coefficient of the meter is 0·97. If the Venturi head is 9 in. of water, find the quantity of water flowing. *Ans.* 1·695 ft^3/sec.

7. A jet of water 1 in. in diameter has a velocity of 60 ft/sec. Find the horse-power of the jet. *Ans.* 2·07.

8. Two horizontal circular discs of 8 in. diameter are 1 in. apart. Water flows between the discs radially towards the centre and leaves by a vertical pipe of 1 in. diameter situated at the centre of the lower disc. If the pressure of the entering water is 14·7 Lb/in.^{2}, find the pressure inside the vertical pipe when the water is flowing at the rate of 40 gal/min. Find, also, the intensity of pressure of the water between the discs at a radius of 2 in. *Ans.* 12·12 Lb/in.^{2} 14·6924 Lb/in.^{2}

9. A cylindrical arm full of water is rotated in a horizontal plane at 100 r.p.m. about one end. The arm is 2 ft long and its diameter is 2 in. Find the centrifugal head impressed on the water and the total pressure on the outer end of the arm. *Ans.* 6·81 ft of water. 9·28 Lb.

10. The air supply to a gas engine is measured by drawing the air into a large chamber through a small orifice. If the difference of pressure between the outside air and the air in the chamber is 16 in. of water, find the velocity with which the air flows through the orifice. Temperature of atmosphere is 18° C, reading of barometer is 29 in. of mercury. Weight of 1 ft^3 of air at 0° C and a pressure of 30 in. of mercury is 0·081 Lb. *Ans.* 270 ft/sec.

11. A Pitot tube was placed in the centre of a pipe 8 in. diameter with one orifice facing the stream and the other perpendicular to it. The difference of pressure on the two orifices as measured by an air gauge was $1\frac{1}{2}$ in. of water. The coefficient of the tube was unity. Taking the mean velocity of the water in the pipe to be 0·83 of the maximum velocity, find the discharge through the pipe. (*Lond. Univ.*) *Ans.* 0·822 ft^3/sec.

12. State Bernoulli's theorem for streamline flow of a liquid and give an elementary proof of the theorem.

A portion of a pipe for conveying water is vertical and the diameter of the upper part of the pipe is 2 in., and the section is gradually reduced to 1 in. diameter at the lower part. A pressure gauge is inserted where the diameter is 2 in., and a second gauge is placed 6 ft below the first and where the pipe is 1 in. in diameter. When the quantity of water flowing up through the pipe is 6·85 ft^3/min, the gauges show a pressure difference of 4·5 Lb/$in.^2$ Assuming that the frictional losses vary as the square of the velocity, determine the quantity of water passing through the pipe when the two gauges show no pressure difference and the water is flowing downwards. (*Lond. Univ.*)
Ans. 4·05 ft^3/min.

13. Find, from Bernoulli's theorem, an expression for the theoretical discharge of a horizontal Venturi meter. State how the actual discharge compares with the theoretical. A Venturi meter tapers from 12 in. diameter at the entrance to 4 in. diameter at the throat, and the discharge coefficient is 0·98. The difference of pressure between entrance and throat is 2·2 in. of mercury. Calculate the discharge in gallons per minute. (*Lond. Univ.*) *Ans.* 409 gal/min.

14. A vertical pipe of radius r_1 in. is fitted at the outlet end with a flange of radius r_2 in. A disc of the same diameter r_2 is placed above the flange, and separated from it by a narrow gap. Water from the pipe flows radially between them and is discharged into the atmosphere. Neglecting friction, find general expressions for the pressure between the surfaces at any radius, and for the resultant inward force on the disc. Sketch the curve of pressure distribution. (*Lond. Univ.*)

15. A Venturi has an entrance diameter of 6 in. and a throat diameter of 2 in. Pipes from the entrance and throat lead water to the limbs of a U-tube containing mercury, and the difference of pressure at these two places in the meter is thus recorded by a difference of mercury level. If the coefficient of the meter is 0·96 draw a curve showing a relation between gallons of water passing through the meter per minute and the difference of mercury level over a range 0 to 15 in. (*Lond. Univ.*)

16. Give a proof of Bernoulli's theorem and show how this is used to determine the discharge from a Venturi meter. (*A.M.I.Mech.E.*)

17. A conical tube is fixed vertically with its smaller end upwards, and forms part of a pipe line. The velocity at the smaller end is 15 ft/sec, and at the larger end 5 ft/sec, and the tube is 5 ft long; the pressure at the upper end is equivalent to a head of 10 ft; the loss in the tube expressed in feet head is given by

$$\frac{0{\cdot}3(v_1 - v_2)^2}{2g}$$

where $v_1 = 15$ and $v_2 = 5$.

Determine the pressure at the lower end of the tube. (*A.M.I.Mech.E.*).
Ans. 17·64 ft of water.

18. Explain the theory of the Pitot tube and obtain an expression for the velocity in terms of the observed difference of level of the liquid, of specific gravity *s*, in the U-tube connected to the up- and down-stream orifices immersed in flowing water.

If the difference of level is 1·2 ft, the specific gravity of the liquid 1·25, and the calibration coefficient for the orifices 0·865, what is the velocity in feet per second? (*A.M.I.C.E.*) *Ans.* 6·08.

19. A Venturi contraction is introduced in a 30 in. diameter horizontal pipe. The area of the pipe is six times that of the throat. The upper end of a vertical cylinder 12 in. in diameter is connected by a pipe to the throat and the lower end to the beginning of the convergence. Neglecting friction losses, and the thickness of the piston in the cylinder, determine the flow through the pipe in cusecs at which the piston begins to rise when the gross effective load—piston, piston rod, and external weight—on the piston rod is 450 Lb. The piston rod is $1\frac{1}{2}$ in. in diameter, and passes through both ends of the cylinder. (*A.M.I.C.E.*) *Ans.* 20·4.

20. State Bernouilli's Theorem. The diameter of a pipe changes gradually from 6 in. at a point *A*, 20 ft above datum, to 3 in. at *B*, 10 ft above datum. The pressure at *A* is 15 Lb/in.^{2}, and the velocity of flow 12 ft/sec. Neglecting losses between *A* and *B*, determine the pressure at *B*. (*A.M.I.Mech.E.*)
Ans. 4·82 Lb/in.^{2}

21. A closed vertical cylinder of 3 ft internal diameter is filled with water and rotates about its axis at 950 r.p.m. Neglecting the effect of the shaft, find the total pressure of the water against the top of the cylinder. (*Lond. Univ.*)
Ans. 76,500 Lb.

22. A Pitot tube used to measure the air flow in a duct 2 ft in diameter gave the following readings—

Distance across duct (inches)	0	0·25	0·50	1·00	2·00	3·6	6·0	9·0	12·0
Pitot head (inches of water)	0	0·24	0·40	0·59	0·82	1·05	1·25	1·44	1·58

Find the air flow in pounds per second. Weight of air = 0·078 Lb/ft.^{3} (*I. Mech. E.*) *Ans.* 1·035 Lb/ft.^{3}

23. Water is rotating in the form of a free cylindrical vortex. At a radius of 1 in. its tangential velocity is 5 ft/sec, and its pressure 20 Lb/in.^{2} Calculate the pressure at a point in the vortex at a radius of 2·5 in. and in the same horizontal plane. If this vortex has a free surface, find the difference in height of the water surface above these two points. *Ans.* 20·141 Lb/in.^{2}; 3·92 in.

24. A cylindrical vessel contains water and rotates about its vertical axis. Show that the surface of the forced vortex thus formed is a paraboloid.

If the cylinder is 3 in. in diameter and 4 in. deep and is exactly half full of water, find the speed of rotation at which water will just begin to spill over the top edge. (*Lond. Univ.*) *Ans.* 354 r.p.m.

25. Deduce an expression for the velocity at a given radius in a free vortex. A point in the free surface at 6 in. radius is 3 in. below the free surface at the boundary of the vessel whose radius is large. What will be the surface level at a radius of 12 in.? (*I. Mech. E.*) *Ans.* 0·75 in.

26. Distinguish between a free and forced vortex. Derive formulae for the relation between radius, velocity and pressure for both types, assuming a perfect fluid.

A U-tube has a horizontal part 2 ft long with vertical end limbs. If the whole tube is rotated about a vertical axis 18 in. from one end and 6 in. from the other, calculate the speed when the difference of level in the tubes is 10 in. (*Lond. Univ.*) *Ans.* 49·7 r.p.m.

CHAPTER 4

ORIFICES AND MOUTHPIECES

4.1. Flow Through Orifices. Supposing a tank containing a liquid were to have a hole made in the side or base through which the liquid would flow; such a hole is termed an orifice, and the quantity of liquid which would flow through this orifice in a given time would depend partly on the shape, size and form of the orifice. There would be a certain amount of frictional resistance at the sides of the orifice; this may be reduced by making them sharp-edged. The jet of liquid, in passing through the orifice, will contract in area, which will further reduce the rate of discharge. This contraction of area is caused by the liquid in the tank around the sides of the orifice, which, in flowing to the orifice, will have a motion parallel to it and perpendicular to that of the jet (Fig. 48). The velocity in this direction is destroyed on reaching the orifice; this causes a lateral force on the jet and a consequent reduction of area. The contraction of area will depend on the shape and size of the orifice and on the head causing flow.

The section of the jet at which the streamlines first become parallel is known as the *vena contracta*. This section is the line *cc* in Fig. 48. The velocity at the vena contracta has reached its maximum and there will be no further contraction of the jet beyond this section.*

4.2. The Coefficient of Contraction. The ratio between the area of the jet at the vena contracta and the area of the orifice is known as the coefficient of contraction.

Let C_c = coefficient of contraction.

Then $$C_c = \frac{\text{area of jet at vena contracta}}{\text{area of orifice}}$$

This coefficient varies slightly with the head and with the size and shape of the orifice. An average value for small, sharp-edged orifices is 0·64.

The coefficient of contraction may be found experimentally by direct measurement of the area of jet at the vena contracta. This may be done with the instrument shown in Fig. 49. It consists of a small collar or ring having four radial screws, equally spaced. The ring is held at the vena contracta so that the jet passes through

* For the flow of gas through an orifico see § 14.6.

its centre. The screws are then adjusted until all their points are in contact with the surface of the jet. The instrument is then removed and the space between the screw points measured. Micrometer screws may be used.

This method is not very satisfactory in practice as the section of the jet is not absolutely regular; also, it is difficult to adjust the

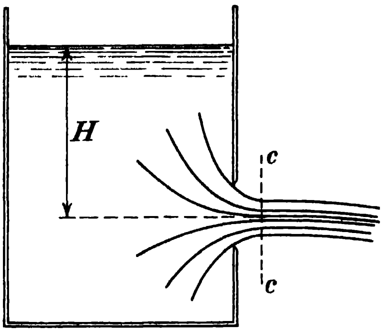


FIG. 48

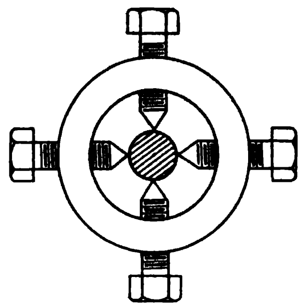


FIG. 49

instrument so that all four screws are just in contact with the surface simultaneously.

A more accurate method of finding C_c is given in § 4.4.

4.3. The Coefficient of Velocity. The ratio between the actual velocity of the jet at the vena contracta and the theoretical velocity is known as the coefficient of velocity.

Let C_v = coefficient of velocity.

Then $$C_v = \frac{\text{actual velocity at vena contracta}}{\text{theoretical velocity}}$$

Let H = head causing flow,

v = actual velocity.

Then $$C_v = \frac{}{\sqrt{2gH}}$$

or $$v = C_v\sqrt{2gH}$$

The difference between the theoretical and actual velocities is due to friction at the orifice and is very small for sharp-edged orifices. The coefficient of velocity will vary slightly for different orifices, depending on the shape and size of the orifice and on the head. An average value for C_v is about 0·97.

The coefficient C_v may be found experimentally for a vertical orifice by measuring the horizontal and vertical co-ordinates of the issuing jet.

Consider the tank in Fig. 50.

Let H = height of liquid in feet above centre of orifice,

cc = vena contracta.

The jet of liquid has a horizontal velocity of v but is acted upon by gravity with a downward acceleration of g. Consider a particle

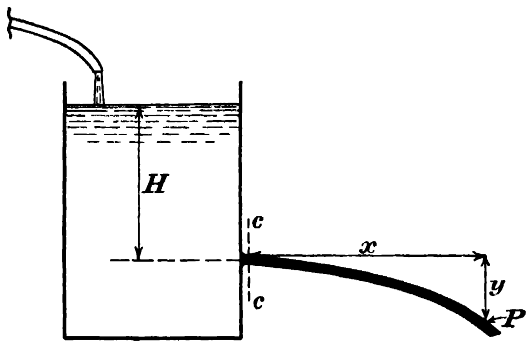


FIG. 50

of liquid in the jet at P and let the time taken for this particle to move from cc to P be t sec.

Let x = horizontal co-ordinate of P from cc in feet,

y = vertical co-ordinate of P from cc in feet.

Then $$x = vt$$

and $$y = \tfrac{1}{2}gt^2$$

Equating the values of t^2 from these two equations,

$$\frac{x^2}{v^2} = \frac{2y}{g}$$

or $$v = \sqrt{\frac{gx^2}{2y}}$$

But $$C_v = \frac{v}{\sqrt{2gH}}$$

Substituting for v,

$$C_v = \sqrt{\frac{x^2}{4yH}}$$

The value of C_v can be found from this equation by measuring the distances x and y for a certain point on the jet and for a known value of H.

The coefficient of velocity may also be found by measuring the actual mean velocity of the jet with a Pitot tube (§ 3.15).

EXAMPLE 1

In order to determine the coefficient of velocity of a small circular sharp-edged orifice under low heads, the horizontal and vertical co-ordinates of the jet were measured when the head was 8 in. The horizontal co-ordinate of a certain point of jet, from the vena contracta, was found to be 32·5 in., whilst the vertical co-ordinate for the same point was 33·7 in. Find the coefficient of velocity.

$$C_v = \sqrt{\frac{x^2}{4yH}}$$

where $H = 8$ in.,
$x = 32{\cdot}5$ in.
and $y = 33{\cdot}7$ in.

Then
$$C_v = \sqrt{\frac{(32{\cdot}5)^2}{4 \times 33{\cdot}7 \times 8}} = 0{\cdot}988$$

4.4. The Coefficient of Discharge. Owing to the reduction in velocity and to the contraction of the jet, the actual discharge will be much less than the theoretical; the relation between them being known as the coefficient of discharge.

Let C_d = coefficient of discharge,
a = area of orifice.

Then
$$C_d = \frac{\text{actual discharge}}{\text{theoretical discharge}} = \frac{\text{measured discharge}}{\sqrt{2gH} \times a}$$

But
$$\text{actual discharge} = \text{actual velocity of jet} \times \text{actual area of jet} = C_v\sqrt{2gH} \times C_c a$$

Therefore
$$\text{actual discharge} = C_v C_c \sqrt{2gH} \times a$$

But
$$a\sqrt{2gH} = \text{theoretical discharge}$$

Therefore
$$C_d = C_v \times C_c$$

The coefficient of discharge of an orifice may therefore be found by first determining its C_v and C_c and then multiplying these together.

The coefficient of discharge will also vary with the head and type of orifice.* Usually, its value is between 0·61 and 0·64.

The simplest method of determining the coefficient of discharge is by actually measuring the quantity of liquid discharged through the orifice in a given time under a known constant head, and by dividing this quantity by the theoretical discharge.

* For the non-dimensional factor for orifices see § 11.6.

Let Q be the measured volume of liquid in cubic feet discharged in a time t sec. Then,

$$C_d = \frac{Q}{a\sqrt{2gH}\,t}$$

A good method of finding the coefficient of contraction is to find the value of C_d by the above method, then $C_c = C_d/C_v$.

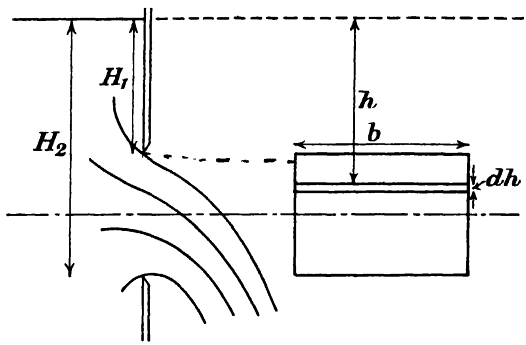


FIG. 51

4.5. Large Vertical Orifices. If a vertical orifice is large compared with the head, the velocity of the liquid may no longer be regarded as constant, as the variation in head at different heights of the orifice will be considerable.

Consider the large orifice in Fig. 51. Let the height of the liquid level be H_1 above the top of the orifice and H_2 above the lower edge. Let b be the breadth of the orifice.

Consider a horizontal strip of the orifice of depth h and thickness dh, and assume velocity at strip to be proportional to $\sqrt{(2gh)}$.

$$\text{Area of strip} = b\,dh$$

$$\text{Velocity of liquid through strip} = k\sqrt{2gh}$$

where k is an unknown coefficient which is included in the value of C_d.

$$\begin{aligned}\text{Discharge through strip} &= C_d \times \text{area} \times \text{velocity} \\ &= C_d b\,dh\,\sqrt{2gh} \\ \text{Total discharge} &= C_d b\sqrt{2g}\int_{H_1}^{H_2} h^{1/2}\,dh \\ &= \frac{2}{3}\,C_d b\sqrt{2g}\left[h^{3/2}\right]_{H_1}^{H_2} \\ &= \frac{2}{3}\,C_d b\sqrt{2g}(H_2{}^{3/2} - H_1{}^{3/2})\end{aligned}$$

EXAMPLE 2

A rectangular orifice in the side of a large tank is 4 ft broad and 2 ft deep. The level of the water in the tank is 2 ft above the top edge of the orifice. Find the quantity of water flowing through the orifice per second if the coefficient of discharge is 0·62.

$$\text{Discharge} = \frac{2}{3} C_d \sqrt{2g} b (H_2^{3/2} - H_1^{3/2})$$

$$= \frac{2}{3} \times 0{\cdot}62 \times \sqrt{64{\cdot}4} \times 4(4^{3/2} - 2^{3/2})$$

$$= 68{\cdot}8 \text{ ft}^3/\text{sec}$$

4.6. Drowned Orifices. If an orifice does not discharge into the atmosphere, but discharges into more liquid, the whole of the outlet side of the orifice being under liquid, it is known as a drowned or submerged orifice. If the outlet side of the orifice is only partly under the surface of the liquid it is known as a partially submerged or drowned orifice.

In a drowned orifice the discharge of the jet is interfered with by the liquid on the outlet side. This has the effect of slightly reducing the coefficient of discharge; the discharge will, therefore, be less for a drowned orifice than for a free, assuming the net head causing flow to be the same.

The discharge through a drowned orifice may be obtained from the same equations as for an orifice running free, excepting that the head causing flow will be the difference between the heads on either side of the orifice.

The discharge through a partially drowned orifice may be found by treating the lower portion as a drowned orifice and the upper portion as an orifice running free and by adding together the two discharges thus found.

EXAMPLE 3

An orifice in the side of a large tank is rectangular in shape, 4 ft broad and 2 ft deep. The water level on one side of the orifice is 4 ft above the top edge; the water level on the other side of the orifice is 1 ft below the top edge. Find the discharge per second if $C_d = 0{\cdot}62$.

The orifice in the question is partially drowned; the lower half may be treated as a drowned orifice and the upper half as a free orifice.

Considering upper half of orifice,

$$\text{discharge} = \frac{2}{3} C_d b \sqrt{2g} (H_2^{3/2} - H_1^{3/2})$$

$$= \frac{2}{3} \times 0{\cdot}62 \times \sqrt{64{\cdot}4} \times 4(5^{3/2} - 4^{3/2})$$

$$= 42{\cdot}2 \text{ ft}^3/\text{sec}$$

Considering lower half of orifice,

head causing flow = 5 ft, and is constant at all depths

$$\text{Discharge} = C_d\sqrt{2g} \times \text{area} \times \sqrt{H}$$
$$= 0{\cdot}62\sqrt{64{\cdot}4} \times 4 \times \sqrt{5}$$
$$= 44{\cdot}5 \text{ ft}^3/\text{sec}$$
$$\text{Total discharge} = 42{\cdot}2 + 44{\cdot}5$$
$$= 86{\cdot}7 \text{ ft}^3/\text{sec}$$

4.7. Time of Emptying Tank. Consider a tank of uniform cross-sectional area A (Fig. 52) to contain a liquid and let the liquid be discharged through an orifice in the base of the tank so that the liquid level falls from a height H_1 to a height H_2 in T sec. The rate of discharge will decrease as the liquid level falls.

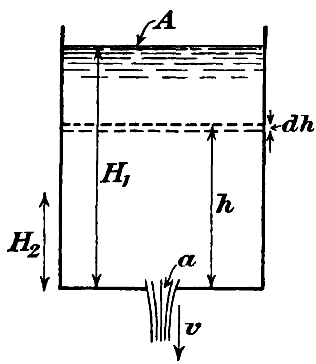


FIG. 52

Let a = area of orifice,

v = velocity of liquid passing through orifice at any particular instant.

At any particular instant let the liquid level be at a height h above the orifice and let the level fall by a small amount dh in the time dt. Let the corresponding quantity of liquid passing through the orifice due to this small change of liquid level be dq. Then, as volume displaced by liquid level equals quantity flowing through orifice, and as dh, being measured downwards, is negative,

$$\mathrm{d}q = -A\,\mathrm{d}h = C_d a v\,\mathrm{d}t$$

But
$$v = \sqrt{2gh}$$

Therefore
$$-A\,\mathrm{d}h = C_d a\sqrt{2gh}\,\mathrm{d}t$$

or
$$\mathrm{d}t = -\frac{A\,\mathrm{d}h}{C_d a\sqrt{2gh}}$$
$$= -\frac{Ah^{-1/2}\,\mathrm{d}h}{C_d a\sqrt{2g}}$$

Then, total time taken
$$= \int_0^T \mathrm{d}t = -\frac{A}{C_d a\sqrt{2g}}\int_{H_1}^{H_2} h^{-1/2}\,\mathrm{d}h$$

from which
$$T = -\frac{2A}{C_d a\sqrt{2g}}\Big[h^{1/2}\Big]_{H_1}^{H_2}$$

$$= \frac{2A}{C_d a\sqrt{2g}}(H_1^{1/2} - H_2^{1/2}) \quad . \quad . \quad (1)$$

If the tank is completely emptied, $H_2 = 0$. Then,

$$T = \frac{2A}{C_d a\sqrt{2g}}\sqrt{H_1} \quad . \quad . \quad . \quad (2)$$

4.8. Time of Emptying Hemispherical Vessel. The time taken to lower the liquid level in a hemispherical vessel may be found in the same manner as in § 4.7; but in this case the cross-sectional area of the vessel is not constant (Fig. 53).

Let R be the radius of vessel and let the liquid level fall from H_1 to H_2 in the time T.

Consider the instant when the liquid level is at a height h, and let the radius of the vessel's cross-section at this level be x.

Let a small quantity dq flow through the orifice at the base in a time dt, and let the corresponding fall of liquid level due to this be dh.

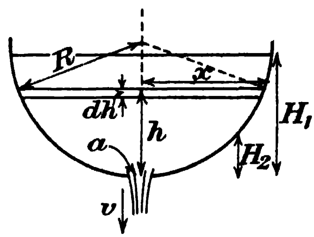


FIG. 53

Let a = area of orifice,

and v = theoretical velocity of liquid passing through orifice at time considered

$= \sqrt{2gh}$.

As volume displaced by liquid level equals volume flowing through orifice, and as dh is negative,

$$dq = -\pi x^2\,dh = C_d av\,dt$$

$$= C_d a\sqrt{2gh}\,dt$$

But, from Fig. 53,

$$x^2 = R^2 - (R - h)^2$$

$$= 2Rh - h^2$$

Therefore

$$dt = -\frac{\pi(2Rh - h^2)dh}{C_d a\sqrt{2gh}}$$

$$\frac{\pi(2Rh - h^2)h^{-1/2}\,dh}{C_d a\sqrt{2g}}$$

Total time required to lower liquid level

$$= -\frac{\pi}{C_d a\sqrt{2g}}\int_{H_1}^{H_2}(2Rh^{1/2} - h^{3/2})\mathrm{d}h$$

$$= -\frac{\pi}{C_d a\sqrt{2g}}\left[\frac{4}{3}Rh^{3/2} - \frac{2}{5}h^{5/2}\right]_{H_1}^{H_2}$$

$$= \frac{2\pi}{C_d a\sqrt{2g}}\left[\frac{2}{3}R(H_1^{3/2} - H_2^{3/2}) - \frac{1}{5}(H_1^{5/2} - H_2^{5/2})\right] \quad . \quad (3)$$

If the vessel were full at the commencement and is completely emptied, then

$$H_1 = R$$

and

$$H_2 = 0$$

Eq. (3) then becomes

$$T = \frac{2\pi}{C_d a\sqrt{2g}}\left(\frac{2}{3}R^{5/2} - \frac{1}{5}R^{5/2}\right)$$

$$\frac{14\pi R^{5/2}}{15C_d a\sqrt{2g}} \quad (4)$$

Example 4

A hemispherical tank 12 ft in diameter is emptied through a hole, 8 in. in diameter, at the bottom. Assuming that the coefficient of discharge is 0·6, find the time required to lower the level of the water surface from 6 ft to 4 ft, and deduce the formula you use. (*Lond. Univ.*)

Using eq. (3),

$$T = \frac{2\pi}{C_d a\sqrt{2g}}\left[\frac{2}{3}R(H_1^{3/2} - H_2^{3/2}) - \frac{1}{5}(H_1^{5/2} - H_2^{5/2})\right]$$

$$= \frac{2\pi}{0{\cdot}6 \times \frac{\pi}{4} \times \frac{4}{9}\sqrt{64{\cdot}4}}\left[\frac{2}{3}\times 6(6^{3/2} - 4^{3/2}) - \frac{1}{5}(6^{5/2} - 4^{5/2})\right]$$

$$= 58{\cdot}2 \text{ sec}$$

4.9. Time of Flow from One Vessel to Another. Suppose liquid is flowing from one vessel into another (Fig. 54), so that as the liquid level falls in one vessel it will rise by a corresponding amount in the other. In this case, the orifice will be drowned, and the head causing flow at any instant will be the difference between the two liquid levels at that instant.

Let the liquid flow from a vessel of area A_1 into a vessel of area A_2, and let a be the area of the orifice between the vessels. Let the difference of head between the two vessels be H_1 at the beginning;

it is required to find the time taken for the difference of head to reach H_2.

Let v = theoretical velocity of flow through orifice.

At a certain instant let the difference of head between the two vessels be h, and let a small quantity dq flow through the orifice in the time dt. This will cause the liquid level in A_1 to fall by the

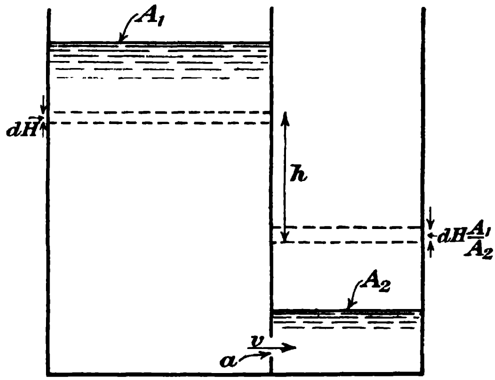


FIG. 54

small amount dH; the liquid level in A_2 will rise, therefore, by the amount d$H(A_1/A_2)$, because the quantity flowing from one tank equals the quantity entering the other.

$$\text{New difference of head} = h - \mathrm{d}H - \mathrm{d}H\,\frac{A_1}{A_2}$$

$$= h - \mathrm{d}H\left(1 + \frac{A_1}{A_2}\right)$$

Therefore, change of head causing flow

$$= \mathrm{d}h = \mathrm{d}H\left(1 + \frac{A_1}{A_2}\right)$$

or

$$\mathrm{d}H = \frac{\mathrm{d}h}{\left(1 + \dfrac{A_1}{A_2}\right)} \quad . \tag{5}$$

As quantity flowing from A_1 equals quantity flowing through orifice, and as dH is negative,

$$\mathrm{d}q = -A_1\,\mathrm{d}H = C_d a v\,\mathrm{d}t$$

But

$$v = \sqrt{2gh}$$

Therefore

$$\mathrm{d}t = \frac{A_1\,\mathrm{d}H}{C_d a\sqrt{2gh}}$$

Substituting from eq. (5),

$$dt = \frac{A_1\,dh}{C_d a\left(1 + \frac{A_1}{A_2}\right)\sqrt{2gh}}$$

or

$$dt = -\frac{A_1 h^{-1/2}\,dh}{C_d a\left(1 + \frac{A_1}{A_2}\right)\sqrt{2g}}$$

$$\text{Total time taken} = \int_0^T dt = -\frac{A_1}{C_d a\left(1 + \frac{A_1}{A_2}\right)\sqrt{2g}}\int_{H_1}^{H_2} h^{-1/2}\,dh$$

Hence

$$T = -\frac{2A_1}{C_d a\left(1 + \frac{A_1}{A_2}\right)\sqrt{2g}}\Big[h^{1/2}\Big]_{H_1}^{H_2}$$

$$= \frac{2A_1(H_1^{1/2} - H_2^{1/2})}{C_d a\left(1 + \frac{A_1}{A_2}\right)\sqrt{2g}} \quad . \quad . \quad (6)$$

If both the vessels have the same area,

$$A_1 = A_2$$

Then

$$T = \frac{A_1(H_1^{1/2} - H_2^{1/2})}{C_d a\sqrt{2g}} \quad . \quad . \quad . \quad (7)$$

It will be noticed that the time taken to reduce the difference of liquid level between two vessels of different areas is the same whether the liquid flows from the larger to the smaller or from the smaller to the larger, providing the reduction in liquid level is the same in each case.

Example 5

A tank 10 ft long and 5 ft wide is divided into two parts by a partition so that the area of one part is three times the area of the other. The partition contains a square orifice of 3 in. sides through which the water may flow from one part to the other. If the water level in the smaller division is 10 ft above that of the larger, find the time taken to reduce the difference of water level to 2 ft. $C_d = 0{\cdot}62$.

$$A_1 = 5 \times 2\tfrac{1}{2} = 12\tfrac{1}{2}\ \text{ft}^2$$

$$A_2 = 5 \times 7\tfrac{1}{2} = 37\tfrac{1}{2}\ \text{ft}^2$$

$$H_1 = 10\ \text{ft}$$

$$H_2 = 2\ \text{ft}$$

$$a = \frac{3 \times 3}{144} = \frac{1}{16}\ \text{ft}^2$$

Using eq. (6),

$$T = \frac{2A_1(H_1^{1/2} - H_2^{1/2})}{C_d a\left(1 + \frac{A_1}{A_2}\right)\sqrt{2g}}$$

$$= \frac{2 \times 12{\cdot}5(\sqrt{10} - \sqrt{2})}{0{\cdot}62 \times \frac{1}{16}\left(1 + \frac{1}{3}\right)\sqrt{64{\cdot}4}}$$

$$= 105{\cdot}5 \text{ sec}$$

4.10. Discharge From Tank with Inflow. The problem of finding the time required to lower the liquid surface in a tank by means of a small orifice was dealt with in § 4.7. This problem is more complex if there is an inflow of liquid whilst the discharge occurs. Consider the tank of Fig. 55 and let its area, in plan, be A ft^2. Let there be a constant inflow of liquid of Q ft^3/sec, whilst, at the same time, the tank is discharging through a small orifice at the base.

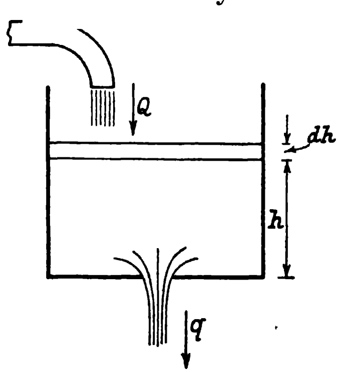


FIG. 55

Let a = area of orifice in square feet,

q = discharge from orifice in cubic feet per second.

Consider the tank at the instant when the liquid surface is at a height h ft above the orifice; let this height increase by dh in a small interval of time dt. Then,

$$\text{amount of inflow} = Q\,dt$$

$$\text{amount of discharge} = q\,dt$$

$$= C_d a\sqrt{2gh}\,dt$$

$$= k\sqrt{h}\,dt$$

where $$k = C_d a\sqrt{2g}$$

Then, increase of liquid in tank $= A\,dh = Q\,dt - kh^{1/2}\,dt$

$$= (Q - kh^{1/2})\,dt$$

from which $$dt = \frac{A\,dh}{Q - kh^{1/2}} \quad . \quad . \quad . \quad (8)$$

The time required to raise or lower the liquid surface between the heights H_1 and H_2 can be obtained by integrating this equation.

This is not a simple expression to integrate, but the solution can be obtained by letting the denominator be represented by the symbol z. Then,

$$z = Q - kh^{1/2} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Then $$h^{1/2} = \frac{Q - z}{k}$$

or $$h = \frac{(Q - z)^2}{k^2}$$

Differentiating with respect to z,

$$dh = -\frac{2(Q - z)dz}{k^2}$$

Substituting this value of dh in eq. (8) and substituting z for the denominator,

$$dt = \frac{A2(Q - z)dz}{zk^2}$$

that is $$dt = -\frac{2A}{k^2}\left(\frac{Q}{z} - 1\right)dz$$

Integrating for a total time of T sec,

$$\int_0^T dt = -\frac{2A}{k^2}(Q \log_e z - z)$$

Substituting for z from eq. (9),

$$T = -\frac{2A}{k^2}\left[Q \log_e (Q - kh^{1/2}) - (Q - kh^{1/2})\right]_{H_1}^{H_2}$$

$$\text{Hence, } T = -\frac{2A}{k^2}\Big[\{Q \log_e (Q - k\sqrt{H_2}) - (Q - k\sqrt{H_2})\}$$

$$- \{Q \log_e (Q - k\sqrt{H_1}) - (Q - k\sqrt{H_1})\}\Big]$$

$$= -\frac{2A}{k^2}\left[Q \log_e \left(\frac{Q - k\sqrt{H_2}}{Q - k\sqrt{H_1}}\right) + k(\sqrt{H_2} - \sqrt{H_1})\right] \quad (10)$$

The problem may also be approached from a graphical conception. Suppose the rise in head h is measured at various intervals of time t, commencing with the tank empty. The head h may then be plotted on a time base, t; the curve obtained is shown in Fig. 56.

Now, as shown above, $A\,dh = (Q - kh^{1/2})dt$

Hence $$\frac{dh}{dt} = \frac{Q}{A} - \frac{k}{A}h^{1/2} \quad . \quad . \quad . \quad (11)$$

It will be seen from this equation that if the values of dh/dt are plotted on a base representing $h^{1/2}$ a straight line is obtained; this is shown plotted in Fig. 56. The values of dh/dt for various values

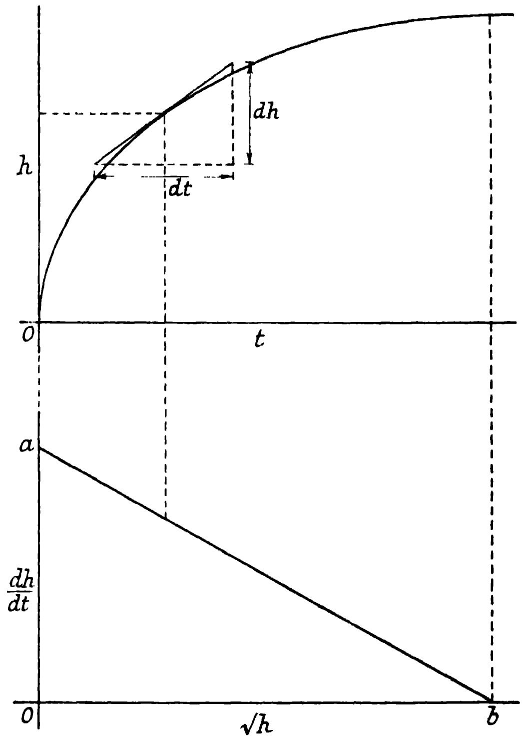


FIG. 56

of h were obtained by measuring the slopes of the h/t curve; this is done by drawing tangents for several values of h.

The values of the two constants of eq. (11), Q/A and k/A, can be obtained for any two known sets of values of dh/dt and h.

It will be noticed from eq. (11) that, when $\sqrt{h} = 0$,

$$\frac{dh}{dt} = \frac{Q}{A}$$

$$= \text{height } Oa \text{ on graph}$$

Also, when dh/dt = 0,

$$\frac{Q}{A} = \frac{k}{A} h^{1/2}$$

Hence $$h^{1/2} = \frac{Q}{k}$$

$$= \text{ordinate } Ob \text{ on graph}$$

This latter condition occurs when the liquid surface in the tank has reached its maximum height. At this level the discharge from the orifice is at the same rate as the inflow; consequently, no further increase in head will occur.

EXAMPLE 6

A concrete tank is 50 ft long and 30 ft wide, and its sides are vertical. Water enters the tank at the rate of 6 ft³/sec and is discharged from the sluices the centre line of which is 1 ft above the bottom of the tank.

When the depth of water in the tank was 17 ft, the instantaneous rate of discharge was observed to be 12 ft³/sec. How long will it take for the level in the tank to fall 10 ft? (*Lond. Univ.*)

Now, $$q = kh^{1/2}$$

that is $$12 = k\sqrt{16}$$

Hence $$k = 3$$

Applying eq. (10), and putting $Q = 6$, $A = 1{,}500$, $H_2 = 6$, and $H_1 = 16$,

$$T = -\frac{2 \times 1{,}500}{3^2}\left[6\log_e\left(\frac{6 - 3\sqrt{6}}{6 - 3\sqrt{16}}\right) + 3(\sqrt{6} - \sqrt{16})\right]$$

$$= -333{\cdot}3(6\log_e 0{\cdot}225 - 4{\cdot}65)$$

$$= 4{,}540 \text{ sec}$$

$$= 75{\cdot}7 \text{ min}$$

EXAMPLE 7

A cylindrical tank is placed with its axis vertical and is provided with a circular orifice, 1½ in. in diameter, in the bottom. Water flows into the tank at a uniform rate, and is discharged through the orifice. It is found that it takes 107 sec for the head in the tank to rise from 2 ft to 2 ft 6 in. and 120 sec for it to rise from 4 ft to 4 ft 3 in. Find the rate of inflow in cusecs and the cross-section of the tank, assuming a coefficient of discharge of 0·62 for the orifice. (*Lond. Univ.*)

This problem is demonstrated by the curves of Fig. 56. Two values of dh/dt are given in the question, namely,

for an average head of 2 ft 3 in., $dh = 6$ in. and $dt = 107$ sec

for an average head of 4 ft 1½ in., $dh = 3$ in. and $dt = 120$ sec

Hence, when $h = 2{\cdot}25$, $$\frac{dh}{dt} = \frac{0{\cdot}5}{107}$$

$$= 0{\cdot}00468$$

When $h = 4{\cdot}125$, $$\frac{dh}{dt} = \frac{0{\cdot}25}{120}$$

$$= 0{\cdot}00208$$

Also,
$$k = 0{\cdot}62 \times \frac{\pi}{4}\left(\frac{1}{8}\right)^2\sqrt{2g}$$
$$= 0{\cdot}0609$$

Substituting for each value of dh/dt in eq. (11),
$$\frac{dh}{dt} = \frac{Q}{A} - \frac{k}{A}\sqrt{h}$$

When $h = 2{\cdot}25$,
$$0{\cdot}00468A = Q - 0{\cdot}0609\sqrt{2{\cdot}25}$$
$$= Q - 0{\cdot}0913 \qquad (12)$$

When $h = 4{\cdot}125$,
$$0{\cdot}00208A = Q - 0{\cdot}0609\sqrt{4{\cdot}125}$$
$$= Q - 0{\cdot}1233 \qquad (13)$$

Solving the simultaneous eqs. (12) and (13),
$$A = 12{\cdot}3 \text{ ft}^2$$
$$Q = 0{\cdot}1489 \text{ ft}^3/\text{sec}$$

4.11. Losses of Head of Flowing Fluid. Fluid flowing along a straight uniform passage with perfectly smooth walls would suffer no loss of energy except that due to viscous resistance. It is not possible in practice to obtain this condition, on account of the roughness of the sides of the passage. The loss of energy due to such a resistance is usually expressed as a head in feet of fluid. Flowing fluid will also be subjected to losses of head due to changes of section, changes of direction, and obstructions. All such losses are expressed in terms of the velocity head.

1. Loss of Head due to Friction of Sides of Passage. This loss is expressed as a function of $v^2/2g$, and will depend on the Reynolds number of the flow (§ 3.17) which, in turn, depends on the length and diameter of the pipe, the velocity, the coefficient of viscosity, the density of the fluid and the roughness of the surface. This loss is dealt with fully in a subsequent chapter.

2. Loss of Head due to Change of Direction. This loss is due to the resistance of sharp bends and elbows, and is expressed as a function of $v^2/2g$. Or,
$$\text{loss of head} = k\frac{v^2}{2g}$$
where k is a coefficient found by experiment and depends on the radius of the bend and the angle of deviation. For 90° elbows, k is found to be approximately unity. The loss of energy due to a sudden change of direction is ultimately lost in the friction of the eddies formed.

3. Loss of Head due to Change of Section of Passage. Losses of head under this heading are due to a sudden enlargement of section, a sudden contraction, and the loss at entrance of a pipe.

There are also losses due to a gradual enlargement or contraction of the section.

4. Loss of Head due to Obstruction in Passage. Any obstruction in the passage, such as a diaphragm or a projection from the passage walls, will interfere with the steady flow of the fluid and form eddies, the energy of which will be ultimately lost in friction.

An obstruction will cause a contraction of the area of flow which will be followed by an enlargement when the obstruction is passed. The loss of head will be due to this sudden enlargement.

Summary of Losses of Head

1. Loss of head due to friction

$$= \frac{4fl}{d}\frac{v^2}{2g} \qquad (\S\,7.6)$$

2. Loss of head due to bends and elbows

$$= k\frac{v^2}{2g} \qquad (\S\,4.11)$$

3. Loss of head due to sudden enlargement

$$= \frac{(v_1 - v_2)^2}{2g} \qquad (\S\,4.12)$$

Loss of head due to sudden contraction

$$= 0{\cdot}5\frac{v^2}{2g} \qquad (\S\,4.13)$$

Loss of head at entrance to pipe

$$= 0{\cdot}5\frac{v^2}{2g} \qquad (\S\,4.14)$$

4. Loss of head due to obstruction

$$= \left[\frac{A}{0{\cdot}66(A - a)} - 1\right]^2 \frac{v^2}{2g} \qquad (\S\,4.15)$$

4.12. Loss of Head Due to a Sudden Enlargement. Consider a fluid flowing along a pipe of area a_1, with a velocity v_1 and a pressure p_1; let the pipe be suddenly enlarged to an area a_2, and let the velocity of the fluid in the enlarged section be v_2 and the pressure p_2 (Fig. 57). The fluid will flow by the enlargement as shown in the figure, and a backwash of eddies will be formed in the corner. It is the formation of these eddies which causes the loss of head.

The eddies press on the annular ring of area $a_2 - a_1$ with a pressure of p_0 Lb/in.^{2} It is found by experiment that p_0 is approximately equal to p_1 and it is on this assumption that the solution is obtained.

Consider the quantity of fluid between *aa* and *bb*. The resultant force acting on this mass of fluid is

$$p_2a_2 - p_1a_1 - p_0(a_2 - a_1)$$

Assuming $p_0 = p_1$,

$$\text{total force} = a_2(p_2 - p_1)$$

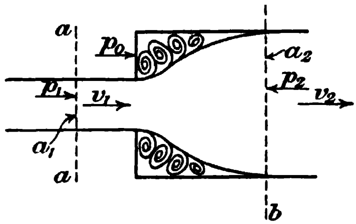


FIG. 57

The change of momentum per second of this mass of fluid is

$$\frac{wa_1v_1^2}{g} - \frac{wa_2v_2^2}{g}$$

But $$a_1v_1 = a_2v_2$$

Therefore, change of momentum per second

$$= \frac{wa_2v_2v_1}{g} - \frac{wa_2v_2^2}{g}$$

Then, as force equals change of momentum per second,

$$a_2(p_2 - p_1) = wa_2\left(\frac{v_2v_1}{g} - \frac{v_2^2}{g}\right)$$

or $$\frac{p_2}{w} - \frac{p_1}{w} = \frac{v_2v_1}{g} - \frac{v_2^2}{g} \quad . \quad . \quad . \quad . \quad (14)$$

Let h_L = loss of head due to enlargement. Applying Bernoulli's equation to sections *aa* and *bb*,

$$\frac{p_1}{w} + \frac{v_1^2}{2g} = \frac{p_2}{w} + \frac{v_2^2}{2g} + h_L$$

from which $$\frac{p_2}{w} - \frac{p_1}{w} = \frac{v_1^2}{2g} - \frac{v_2^2}{2g} - h_L \quad . \quad . \quad . \quad (15)$$

Equating eqs. (14) and (15),

$$\frac{v_2v_1}{g} - \frac{v_2^2}{g} = \frac{v_1^2}{2g} - \frac{v_2^2}{2g} - h_L$$

from which $$h_L = \frac{v_1^2}{2g} - \frac{2v_2v_1}{2g} + \frac{v_2^2}{2g}$$

$$= \frac{(v_1 - v_2)^2}{2g} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The loss given by eq. (16) occurs at any sudden enlargement of the cross-section of a passage containing a moving fluid.

EXAMPLE 8

A pipe of diameter 6 in. is suddenly enlarged to a diameter of 1 ft. Find the loss of head due to this enlargement when the quantity of water flowing is 4 ft^3/sec.

$$\text{Velocity in 6 in. pipe} = \frac{Q}{\text{area}} = \frac{4}{\frac{\pi}{4} \times (\frac{1}{2})^2} = 20{\cdot}4 \text{ ft/sec}$$

$$\text{Velocity in 12 in. pipe} = \frac{4}{\frac{\pi}{4}} = 5{\cdot}1 \text{ ft/sec}$$

$$\text{Loss of head} = \frac{(v_1 - v_2)^2}{2g}$$

$$= \frac{(20{\cdot}4 - 5{\cdot}1)^2}{64{\cdot}4}$$

$$= 3{\cdot}64 \text{ ft of water}$$

4.13. Loss of Head Due to a Sudden Contraction. The loss of head due to a sudden contraction is not due to the contraction itself but to the sudden enlargement which follows the contraction.

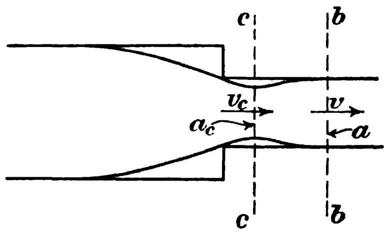


FIG. 58

Consider the pipe in Fig. 58. Let the pipe change section from an area of a_1 to an area of a. The fluid, in flowing into the narrow section, will be further contracted at the section cc, forming a vena contracta in the same way as a jet issuing from an orifice. Let the velocity at cc be v_c and let the contracted area be a_c. Then,

$$a_c = C_c a$$

where C_c is the coefficient of contraction.

Let v = velocity of fluid at the section bb.

At the section bb the jet of fluid will have expanded and filled the pipe; consequently, there will be a loss of head between cc and bb due to this expansion.

$$\text{Loss of head} = \frac{(v_c - v)^2}{2g} \qquad (\S\ 4.12)$$

But
$$av = a_c v_c$$
$$= C_c a v_c$$

Therefore
$$v_c = \frac{v}{C_c}$$

Then,
$$\text{loss of head} = v^2 \frac{\left(\frac{1}{C_c} - 1\right)^2}{2g}$$

Assuming C_c to be 0·62 for a circular orifice,

$$\text{loss of head} = \left(\frac{1}{0{\cdot}62} - 1\right)^2 \frac{v^2}{2g}$$
$$= 0{\cdot}375 \frac{v^2}{2g}$$

It is found by experiment that the actual value of the constant is nearer 0·5 than 0·375; this higher value is generally used.

Then, loss of head due to sudden contraction $= 0{\cdot}5(v^2/2g)$.

4.14. Loss of Head at Entrance to Pipe. The loss of head due to the fluid entering a pipe from a large container is actually a loss due to a sudden contraction.

Let v = velocity in pipe. Then,

$$\text{loss of head at entrance} = 0{\cdot}5 \frac{v^2}{2g}$$

In cases of fluid flowing along long pipes, this loss of head is very small compared with the frictional loss and may be neglected.

4.15. Loss of Head Due to Obstruction. The loss of head due to an obstruction in a pipe may be looked upon as due to the sudden enlargement beyond the obstruction.

Consider a pipe of cross-sectional area A (Fig. 59), and let an obstruction of area a be placed in the pipe. The fluid will flow in streamlines by the obstruction, the vena contracta occurring just beyond at the section cc.

Let v = velocity of fluid in free section of pipe,

v_c = velocity at vena contracta,

bb = a section of normal flow beyond the obstruction.

There will be a loss of head due to the enlargement between the sections cc and bb equal to $\frac{(v_c - v)^2}{2g}$ (§ 4.12).

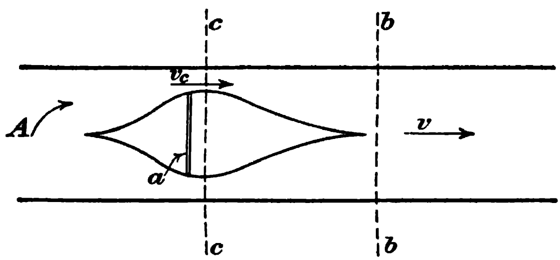


FIG. 59

Area of section of flow at $cc = C_c(A - a)$

where C_c = coefficient of contraction.

Also $$v_c C_c(A - a) = vA$$

Therefore $$v_c = \frac{Av}{C_c(A - a)}$$

Then, $$\text{loss of head} = \left[\frac{A}{C_c(A - a)} - 1\right]^2 \frac{v^2}{2g}$$

Assuming the coefficient of contraction = 0·66,

$$\text{loss of head due to obstruction} = \left[\frac{A}{0{\cdot}66(A - a)} - 1\right]^2 \frac{v^2}{2g}$$

This phenomenon is made use of in an instrument known as a pipe orifice which is used for measuring the flow of fluid (§ 4.16); the orifice forms an obstruction in the pipe and the loss of head is measured by means of pressure gauges.

EXAMPLE 9

The passage of water through a 6 in. pipe is restricted by a diaphragm with a 2 in. diameter hole in its centre. The loss of head at the diaphragm when the velocity in the pipe is 0·59 ft/sec equals 1·25 ft. Assuming the head lost $= k\frac{v^2}{2g}$ where v = the velocity of water in the pipe, find C_c the coefficient of contraction of the stream passing through the diaphragm. (*Lond. Univ.*)

In this case the area of flow at the obstruction $= (\pi/4)(2)^2$. Then,

$$\text{loss of head} = \left[\frac{A}{C_c(A-a)} - 1\right]^2 \frac{v^2}{2g}$$

$$1{\cdot}25 = \frac{\frac{\pi}{4}(6)^2}{C_c\frac{\pi}{4}(2)^2} - 1 \quad \frac{(0{\cdot}59)^2}{2g}$$

$$\frac{9}{C_c} = 15{\cdot}22 + 1$$

$$C_c = 0{\cdot}555$$

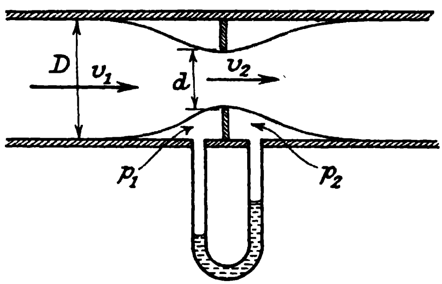


FIG. 60

4.16. Measurement of Flow by Pipe Orifice. If a diaphragm containing an orifice be inserted in a pipe through which a fluid is flowing (Fig. 60) there will be a loss of pressure in the fluid as it passes through the orifice. By measuring this loss of pressure it is possible to calculate the quantity of flow through the pipe. This principle is used in a type of flow meter, the quantity of flow being proportional to the measured pressure drop caused by passing through the orifice.

Let D = dia. of pipe bore,
d = dia. of orifice,
a_1 = cross-sectional area of pipe,
a_2 = area of orifice,
v_1 = velocity of fluid approaching orifice,
v_2 = velocity of fluid through orifice,
p_1 = pressure of fluid immediately before orifice,
p_2 = pressure of fluid immediately after orifice,
h = measured difference of pressure, feet of fluid.

Then, $$Q = a_1 v_1 = a_2 v_2 \quad . \quad . \quad . \quad . \quad (17)$$

Applying Bernoulli's equation to both sides of orifice,

$$\frac{p_1}{w} + \frac{v_1^2}{2g} = \frac{p_2}{w} + \frac{v_2^2}{2g}$$

or

$$\frac{p_1}{w} - \frac{p_2}{w} = h = \frac{(v_2^2 - v_1^2)}{2g}$$

Substituting for v_1 from eq. (17),

$$h = \left(v_2^2 - v_2^2 \frac{a_2^2}{a_1^2}\right) \frac{1}{2g}$$

$$= \frac{v_2^2}{2g}\left[1 - \left(\frac{a_2}{a_1}\right)^2\right]$$

$$= \frac{v_2^2}{2g}\left[1 - \left(\frac{d}{D}\right)^4\right]$$

from which

$$v_2 = \sqrt{\frac{2gh}{\left[1 - \left(\frac{d}{D}\right)^4\right]}}$$

Then

$$Q = a_2 v_2$$

$$= a_2 \sqrt{\frac{2gh}{\left[1 - \left(\frac{d}{D}\right)^4\right]}} \qquad . \quad . \quad . (18)$$

Owing to losses due to the passage of the fluid through the orifices, this equation must be multiplied by a coefficient k, the value of k being found from tests for the particular orifice under consideration. The true discharge is thus given by the equation

$$Q = ka_2 \sqrt{\frac{2gh}{\left[1 - \left(\frac{d}{D}\right)^4\right]}} \qquad . \quad . (19)$$

This solution can be applied to gases only if the pressure drop is small; that is, when the change in velocity is small, because eq. (17) assumes the density to be constant.

It is shown in § 11.7 that the coefficient k is a function of the Reynolds number (§ 3.17) and the ratio (d/D). In Fig. 61, curves showing the variation of k with the Reynolds number and with the ratio (d/D) are plotted; these curves were obtained from tests. It will be noticed that k tends to become constant at high values of R_e, when the flow is turbulent, and that its value varies slightly with the ratio (d/D).

The values of k for low values of R_e are probably affected by surface tension and by the fact that the flow is laminar.

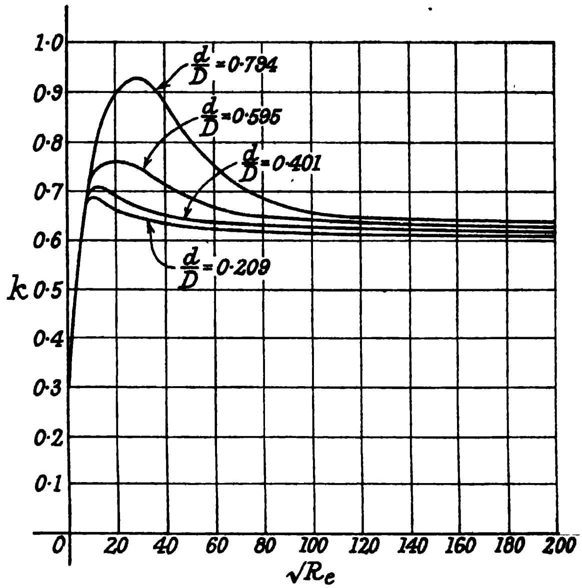


FIG. 61

EXAMPLE 10

Water having a viscosity of $\eta = 0{\cdot}015$ c.g.s. units flows through a pipe orifice. The diameter of the pipe is 2 in. and that of the orifice 1 in. If the pressure drop when passing through the orifice is found to be 0·85 ft of water, calculate the value of C_d from the curves of Fig. 61 and find the flow through the orifice.

As a first approximation a rough value of C_d is obtained from the curves of Fig. 61 for $d/D = 0{\cdot}5$. From these curves an approximate value of C_d is 0·63.

Using eq. (19),

$$Q = C_d a_2 \sqrt{\frac{2gh}{\left[1 - \left(\frac{d}{D}\right)^4\right]}}$$

Hence

$$v_2 = \frac{Q}{a_2} = C_d \sqrt{\frac{2gh}{\left[1 - \left(\frac{d}{D}\right)^4\right]}}$$

$$= 0{\cdot}63 \sqrt{\frac{64{\cdot}4 \times 0{\cdot}85}{[1 - (\frac{1}{2})^4]}}$$

$$= 0{\cdot}63 \times 7{\cdot}65 = 4{\cdot}82 \text{ ft/sec}$$

Using this value of v_2, the Reynolds number can now be calculated.
From § 3.16,

$$\eta = \frac{0{\cdot}015 \times 30{\cdot}5}{453{\cdot}6 \times 32{\cdot}2} \text{ slug-ft}^{-1}\text{-sec}^{-1}$$

$$= 3{\cdot}132 \times 10^{-5}$$

and

$$\rho = \frac{w}{g} - \frac{62{\cdot}4}{32{\cdot}2}$$

$$1{\cdot}935$$

Then

$$R_e = \frac{\rho v d}{\eta}$$

$$= \frac{1{\cdot}935 \times 4{\cdot}82 \times \frac{1}{12}}{3{\cdot}132 \times 10^{-5}}$$

$$= 24{,}810$$

Hence

$$\sqrt{R_e} = 157{\cdot}5$$

Using this value of $\sqrt{R_e}$, the actual value of C_d can now be obtained from the curves of Fig. 61. It will be found that $C_d = 0{\cdot}62$ when $\sqrt{R_e} = 157{\cdot}5$ and $d/D = 0{\cdot}5$.

Then

$$Q = C_d a_2 \sqrt{\frac{2gh}{\left[1 - \left(\frac{d}{D}\right)^4\right]}}$$

$$= 0{\cdot}62 \times \left(\frac{\pi}{4} \times \frac{1}{144}\right) \times 7{\cdot}65 \times 60$$

$$= 1{\cdot}55 \text{ ft}^3\text{/min}$$

4.17. External Mouthpiece. The discharge through an orifice may be increased by fitting a short length of pipe to the outside. Consider the vessel in Fig. 62 to be discharging liquid through a short length of pipe under a head H. The jet, on entering the pipe, will at first contract and then expand and fill the pipe. Let H_a be the atmospheric pressure in feet of liquid. The pressure at the outlet of the pipe will be at atmospheric; but, as the velocity of the vena contracta is larger than that at outlet, the pressure at the vena contracta will be less than atmospheric.

As the pipe is flowing full at outlet, the coefficient of contraction will be unity. The coefficient of velocity may be calculated by applying Bernoulli's equation to certain sections of the liquid.

Let a = area of pipe,

a_c = area of flow at vena contracta,

v = velocity at outlet of pipe,

v_c = velocity at vena contracta,

H_c = absolute pressure in feet of liquid at vena contracta.

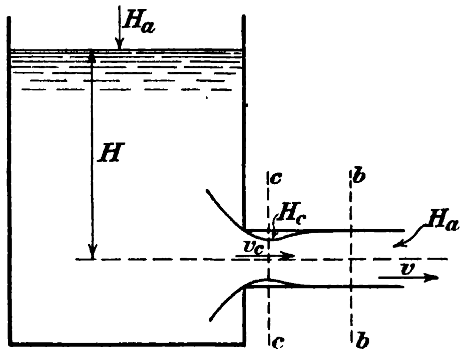


FIG. 62

Assuming coefficient of contraction at vena contracta to be 0·62,

$$a_c = C_c a$$
$$= 0{\cdot}62a$$

As quantity flowing at section *cc* equals quantity flowing at *bb*,

$$v_c a_c = va$$
$$v_c = v\frac{a}{a_c} \qquad . \quad (20)$$
$$\frac{v}{0{\cdot}62}$$

Owing to the enlarging of the section between *cc* and *bb*, there will be a loss of head of $\frac{(v_c - v)^2}{2g}$ (§ 4.12).

Substituting the value of v_c,

$$\text{loss of head} = \frac{\left(\frac{v}{0{\cdot}62} - v\right)^2}{2g}$$
$$= 0{\cdot}375 \times \frac{v^2}{2g}$$

Applying Bernoulli's equation to free liquid surface in tank and bb,

$$H_a + H = H_a + \frac{v^2}{2g} + \text{loss of head}$$

Therefore
$$H = \frac{v^2}{2g} + \left(0{\cdot}375 \times \frac{v^2}{2g}\right)$$
$$= 1{\cdot}375 \frac{v^2}{2g} \quad . \quad . \quad . \quad . \quad (21)$$

Therefore
$$C_v = \frac{1}{\sqrt{1{\cdot}375}} = 0{\cdot}855$$

Then
$$C_d = C_v \times C_c$$
$$= 0{\cdot}855$$

as $C_c = 1$.

The coefficient of discharge is thus considerably increased by fitting an external mouthpiece.

In order to find the pressure at the vena contracta, apply Bernoulli's equation to the liquid surface in the tank and to the section cc.

$$H_a + H = H_c + \frac{v_c^2}{2g}$$

But, from eq. (20),
$$v_c = \frac{v}{0{\cdot}62}$$

and from eq. (21),
$$H = 1{\cdot}375 \times \frac{v^2}{2g}$$

Then
$$H_a + 1{\cdot}375 \times \frac{v^2}{2g} = H_c + \left(2{\cdot}6 \times \frac{v^2}{2g}\right)$$

Therefore
$$H_c = H_a - 1{\cdot}225 \frac{v^2}{2g} \quad . \quad . \quad . \quad (22)$$
$$= H_a - 0{\cdot}89H$$

Or, the pressure at the vena contracta is $1{\cdot}225 \dfrac{v^2}{2g}$ or $0{\cdot}89H$ less than atmospheric.

The effect of the mouthpiece on the discharge is to decrease the pressure at the vena contracta and thus increase the effective head causing flow.

It is found by experiment that the frictional resistance at the entrance to the mouthpiece reduces the coefficient of discharge from 0·855 to 0·813. The effect of this frictional resistance on the pressure at the vena contracta is to reduce the vacuum pressure to about $0{\cdot}74H$.

It will be noticed that the pressure at the vena contracta will be zero when $0{\cdot}74H = 34$ ft of water. If this condition were reached, separation would take place and the flow of the liquid would no longer be steady. In practice this takes place before zero pressure is reached.

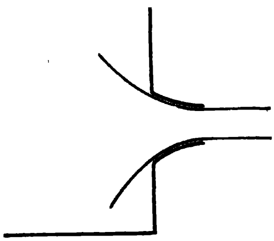


FIG. 63

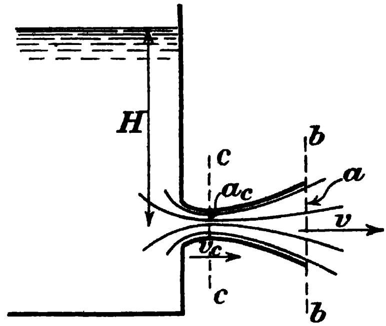


FIG. 64

In this type of mouthpiece the length of pipe must be at least three diameters in order for the pipe to run full.

By making the mouthpiece to the shape of the jet up to the vena contracts, as in Fig. 63, the loss due to the enlargement is eliminated. This will make the theoretical coefficient of discharge equal to unity. Such a mouthpiece is known as a convergent mouthpiece. Actually, owing to frictional loss, the coefficient of discharge for this mouthpiece is about 0·975.

By making the mouthpiece divergent, the loss due to the enlargement of the jet may be considerably reduced. In this type the mouthpiece is sometimes made convergent up to the vena contracta and then diverges as in Fig. 64. As the divergence increases, the velocity at *cc* increases; this will cause an increase in the vacuum pressure at the vena contracta; and, as this cannot be greater than 34 ft theoretically, or, if the liquid be water, 26 ft actually, there is a limit to the amount of divergence if a steady flow is to be maintained.

Applying Bernoulli's equation to liquid level and to sections *cc* and *bb* (Fig. 64),

$$H_a + H = H_c + \frac{v_c^2}{2g} = H_a + \frac{v^2}{2g}$$

from which
$$\frac{v^2}{2g} = H$$

and
$$\frac{v_c^2}{2g} = H + H_a - H_c$$

But
$$\frac{a}{a_c} = \frac{v_c}{v}$$
$$\frac{\sqrt{2g(H + H_a - H_c)}}{\sqrt{2gH}}$$
$$= \sqrt{1 + \frac{(H_a - H_c)}{H}}$$

Then, assuming the liquid to be water and the maximum vacuum pressure to be 26 ft,
$$\text{maximum ratio of } \frac{a}{a_c} = \sqrt{1 + \frac{26}{H}}$$

EXAMPLE 11

Water is discharged through an external cylindrical mouthpiece, of 4 in.^{2} area, under a head of 10 ft. Find the discharge and the pressure at the vena contracta. Coefficient of contraction = 0·64.

Applying Bernoulli's equation to water surface and outlet end of mouthpiece,
$$10 = \frac{v^2}{2g} + \frac{(v_c - v)^2}{2g}$$

But
$$v_c = \frac{v}{0{\cdot}64}$$

Then
$$10 = \frac{v^2}{2g} + \frac{\left(\frac{v}{0{\cdot}64} - v\right)^2}{2g}$$
$$= \frac{1{\cdot}316\, v^2}{2g}$$

Therefore
$$v = 22{\cdot}18 \text{ ft/sec}$$
$$\text{Discharge} = av = \frac{4}{144} \times 22{\cdot}18 = 0{\cdot}616 \text{ ft}^3\text{/sec}$$
$$v_c = \frac{v}{0{\cdot}64} = 34{\cdot}6 \text{ ft/sec}$$

Applying Bernoulli's equation to water surface and vena contracta,
$$34 + 10 = H_c + \frac{v_c^2}{2g}$$

Therefore $H_c = 44 - \dfrac{(34{\cdot}6)^2}{2g} = 25{\cdot}4$ ft of water absolute

4.18. Re-entrant or Borda's Mouthpiece. An internal mouthpiece, such as shown in Fig. 65, is known as a re-entrant or Borda mouthpiece. If the jet, after contraction, does not touch the sides of the mouthpiece, as in Fig. 65, it is said to be running free. If,

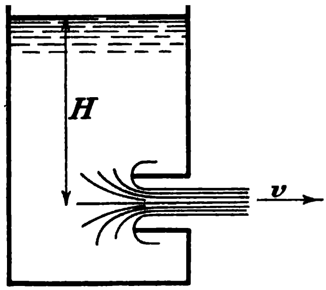


FIG. 65

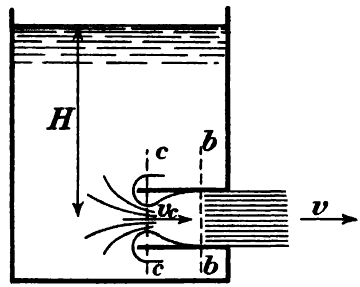


FIG. 66

after contraction, the jet expands and fills the mouthpiece, as in Fig. 66, it is said to be running full.

Consider the mouthpiece of Fig. 65. In this case the mouthpiece is running free.

Let H = height of liquid surface above centre of mouthpiece,
a = area of mouthpiece,
v = velocity of flow through mouthpiece,
a_c = contracted area of jet.

As force equals rate of change of momentum, total pressure at entrance = change of momentum per second. Or,

$$pa = \frac{(wa_cv)}{g}v$$

But
$$p = wH$$

Hence
$$waH = wa_c\frac{v^2}{g}$$

But $H = \dfrac{v^2}{2g}$.

Therefore
$$\frac{av^2}{2g} = a_c\frac{v^2}{g}$$

or
$$a_c = \frac{a}{2}$$

That is, the coefficient of contraction = 0·5.

This may be accounted for by the liquid surrounding the outside of the mouthpiece having to deviate through an angle of 180° in reaching the jet.

Next consider the mouthpiece running full, as in Fig. 66. This case is similar to an external mouthpiece. There will be a vacuum pressure at the vena contracta which will increase the velocity at that section. This will cause an increased discharge as the coefficient of contraction at the outlet is now unity.

Consider the sections *cc* and *bb*. There will be a loss of head due to the enlarging of the section.

Using the same notation as in § 4.17,

$$\text{loss of head due to enlargement} = \frac{(v_c - v)^2}{2g}$$

$$= \left(\frac{1}{C_c} - 1\right)^2 \frac{v^2}{2g}$$

$$= \frac{}{2g}$$

as C_c for the jet $= 0{\cdot}5$.

Applying Bernoulli's equation to the liquid surface and to the outlet end of the mouthpiece,

$$H_a + H = H_a + \frac{v^2}{2g} + \begin{pmatrix}\text{loss due to}\\ \text{enlargement}\end{pmatrix}$$

or
$$H = \frac{v^2}{2g} + \frac{v^2}{2g}$$

$$= \frac{v^2}{g}$$

Then
$$v = \sqrt{gH}$$

$$\text{Discharge, when running full} = av$$

$$= a\sqrt{gH}$$

$$\text{Discharge, when running free} = 0{\cdot}5a\sqrt{2gH}$$

Therefore, the discharge is increased by $\frac{1}{0{\cdot}5\sqrt{2}}$ when running full.

Coefficient of discharge when running full

$$= \frac{1}{\sqrt{2}} = 0{\cdot}707$$

In practice, the coefficient of discharge is found to be slightly greater than this amount.

The pressure at the vena contracta may be found by applying Bernoulli's equation to sections *cc* and *bb*.

$$H_c + \frac{v_c^2}{2g} = H_a + \frac{v^2}{2g} + \frac{v^2}{2g}$$

But
$$v_c = 2v$$

Therefore $$H_c + \frac{4v^2}{2g} = H_a + \frac{2v^2}{2g}$$

or $$H_c = H_a - H$$

as $H = v^2/g$.

Thus, the pressure at the vena contracta is less than atmospheric by an amount equal to the head of liquid in the vessel.

Assuming the liquid to be water and that separation takes place at a vacuum pressure of 26 ft of water, the maximum value of H for steady flow is when $H_c = H_a - 26$. Then,

$$H_a - 26 = H_a - H$$

or $$H = 26 \text{ ft of water}$$

EXAMPLE 12

Calculate the coefficient of discharge from a projecting cylindrical mouthpiece in the side of a water tank assuming that the only loss is that due to the sudden enlargement in the mouthpiece, taking a coefficient of contraction as 0·64. Compare the discharge through a Borda mouthpiece in the vertical side of a tank filled with water, and the jet running free, with that from a short cylindrical mouthpiece projecting from the vertical side of the tank if both are placed in similar positions, are 2 in. in diameter, and the constant head above the centre of each is 3 ft. Sketch the issuing jets in each case. (*Lond. Univ.*)

Applying Bernoulli's equation to water surface and outlet of mouthpiece,

$$H = \frac{v^2}{2g} + \frac{(v_c - v)^2}{2g}$$

But $$v_c = \frac{v}{0{\cdot}64}$$

Then $$H = \frac{v^2}{2g} + \frac{\left(\frac{v}{0{\cdot}64} - v\right)^2}{2g}$$

$$1{\cdot}316v^2$$

and $$v = \sqrt{\frac{2gH}{1{\cdot}316}}$$

$$\text{Discharge} = av = \frac{a}{\sqrt{1{\cdot}316}}\sqrt{2gH}$$

$$\text{Theoretical discharge} = a\sqrt{2gH}$$

$$\text{Coefficient of discharge} = \frac{\dfrac{a\sqrt{2gH}}{\sqrt{1\cdot316}}}{a\sqrt{2gH}} = \frac{1}{\sqrt{1\cdot316}}$$
$$= 0\cdot875$$

Coefficient of discharge for Borda mouthpiece running free
$$= 0\cdot5$$

Actual discharge for Borda mouthpiece
$$= 0\cdot5a\sqrt{2gH}$$
$$= 0\cdot5 \times \frac{\pi}{4} \times \frac{4}{144} \times \sqrt{64\cdot4 \times 3}$$
$$= 0\cdot1518 \text{ ft}^3/\text{sec}$$

Actual discharge for cylindrical mouthpiece
$$= C_d a\sqrt{2gH}$$
$$= 0\cdot875 \times \frac{\pi}{4} \times \frac{4}{144} \times \sqrt{64\cdot4 \times 3}$$
$$= 0\cdot268 \text{ ft}^3/\text{sec}$$

EXERCISES 4

1. The discharge through a sharp-edged circular orifice, 1 in. in diameter, under a constant head of 4 ft is 3·24 ft³/min. Find the coefficient of discharge. *Ans.* $C_d = 0\cdot615$.

2. If the jet in Question 1, when measured with a screw gauge, is found to have a diameter of 0·785 in., find the coefficient of velocity. *Ans.* $C_v = 0\cdot992$.

3. A jet of water issues from a sharp-edged vertical orifice under a constant head of 4 in. At a certain point of the issuing jet, the horizontal and vertical co-ordinates from the vena contracta are measured and found to be 16 in. and 16·8 in. respectively. Find the coefficient of velocity of the jet. *Ans.* $C_v = 0\cdot978$.

✓ 4. Find the discharge through a large rectangular vertical orifice, 6 ft wide and 4 ft deep, when the water level is 10 ft above the top edge of the orifice. $C_d = 0\cdot61$. *Ans.* 403 ft³/sec.

5. Water flows from a tank at the rate of 400 gal/min into a horizontal pipe of 6 in. diameter. The pipe suddenly changes to 8 in. diameter at a short distance from the tank and is then suddenly reduced back to 6 in. diameter. Find the loss of head at entrance to pipe, at enlargement, and at contraction. *Ans.* 0·231, 0·09, 0·231 ft of water.

✓ 6. A large tank has a circular sharp-edged orifice 1·44 in.² in area at a depth of 9 ft below constant water level. The jet issues horizontally, and in a horizontal distance of 7·8 ft it falls 1·8 ft. The measured discharge is 0·15 cusecs. Calculate the coefficients of velocity, contraction, and discharge. (*A.M.I.C.E.*) *Ans.* 0·97; 0·643; 0·624.

7. A reservoir is circular in plan, the diameter of the top water level is 300 ft, at a depth of 5 ft the diameter is 250 ft. The mouth of the outlet pipe, which is 24 in. in diameter, is 12 ft below top water level; how long will it take to lower the depth of the water in the reservoir 5 ft? (Take $Cd = 0{\cdot}8$.) (*Lond. Univ.*) *Ans.* 79·4 min.

8. A tank 20 ft long and 5 ft wide is divided into two parts, by a partition, so that one part is four times the other part. The water level in the large portion is 10 ft higher than that in the smaller. Find the time for the difference of water level in the two portions to reach 4 ft if the water flows through an orifice in the partition 3 in. square. $C_d = 0{\cdot}6$. *Ans.* 6·2 min.

9. A pipe 10 in. in diameter has a diaphragm fitted in it, in which there is a hole 4 in. in diameter concentric with the pipe. Investigate a formula for the loss of head at the diaphragm and show how the arrangement can be used to measure the flow along the pipe.

Show also how you would check experimentally the assumptions made. (*Lond. Univ.*)

10. Establish Bernoulli's equation for the streamline motion of a fluid. Show that when water is issuing steadily from a re-entrant orifice in the bottom of a tank, the area of the jet at the vena contracta is $\frac{1}{2}$ of the area of the orifice. (*Lond. Univ.*)

11. Water in a tank discharges through an external divergent mouthpiece. If the outlet area of the mouthpiece is four times the minimum area, find the maximum head in the tank at which steady flow through the mouthpiece can be obtained. Assume separation takes place at an absolute pressure of 8 ft of water. *Ans.* 1·735 ft of water.

12. Water under a constant head of 9 ft discharges through an external cylindrical mouthpiece of 2 in. diameter. $C_c = 0{\cdot}6$.

Find, (1) the discharge in cubic feet per second; (2) the coefficient of discharge; (3) the absolute pressure at the vena contracta in feet of water.

Ans. 0·437; 0·832; 25·7.

13. If the mouthpiece in Question (12) were a Borda mouthpiece running full, what would be the discharge? *Ans.* 0·372 ft³/sec.

14. Compensation water is to be discharged by two circular orifices under a constant head of 2 ft 6 in., measured to the centre of the orifices. What diameter will be required to give 3,000,000 gal a day? $C_c = 0{\cdot}62$; $C_v = 0{\cdot}97$. (*A.M.I.C.E.*) *Ans.* 8·18 in.

15. A pipe increases abruptly from diameter d to diameter D. Deduce an expression for the loss of head by shock when the discharge is Q. If $d = 12$ in., $D = 18$ in., and $Q = 5$ ft³/sec what is the loss of head? (*A.M.I.C.E.*)

Ans. 0·196 ft.

16. Deduce an expression for the loss of head at a sudden enlargement in a pipe line. Using your result, determine the loss of head when a 12 in. pipe line discharges directly through the side of a reservoir, the velocity of flow being 10 ft/sec. (*A.M.I.Mech.E.*) (Assume $C_c = 0{\cdot}6$.) *Ans.* 0·692 ft of water.

17. Two vertical-sided basins, each having a surface area of 2,000 ft², are connected by a sluice gate of area 2 ft². The initial difference of level in the basins is 9 ft. How long will it take to reduce this to 4 ft? The coefficient of discharge of the orifice is 0·8. (*A.M.I.C.E.*) *Ans.* 2 min 36 sec.

18. A sharp-edged orifice 1·9 in. in diameter is employed to measure the supply of air to an oil-engine. Prove that the volume in cubic feet per minute passing through the orifice is $13\sqrt{(h/\rho)}$, in which h is the drop of pressure between the two sides of the orifice measured in inches of water and ρ is the density of the air, assumed uniform, in pounds per cubic foot, and the coefficient of discharge for the orifice is 0·602.

Calculate the volume of air in cubic feet per minute at N.T.P. if $h = 0{\cdot}85$ in. of water, the pressure of the atmosphere 30·5 in. of mercury, and the temperature 15·8°C. For air $pV = 96T$. (*Lond. Univ.*) *Ans.* 43·0 ft³/min.

19. A tank with vertical sides and a horizontal cross-sectional area of 20 ft² is provided with a notch cut at the top of one of the sides. Water flowing into the tank at a constant rate was discharged over the notch, the head over the bottom of which was 6 in.

The supply of water was suddenly stopped and it was observed that the head over the notch started to fall at the rate of 0·14 in./sec. When the head had fallen to 3 in. it was found that it was falling at the rate of 0·05 in./sec. Estimate the rate of inflow to the tank when there is a steady head of 5 in. over the notch. (*Lond. Univ.*) *Ans.* 0·18 ft³/sec.

20. A cylindrical tank 5 ft in diameter and 20 ft high discharges through a circular orifice 2 in. in diameter in the base of the tank.

If there is inflow at a constant rate, and free discharge, find the rate of inflow if the head increases from 1 to 15 ft in 15 min. $C_d = 0{\cdot}62$.

Ans. 0·619 ft³/sec.

21. Air freely enters a heated drying shed through many openings at the base of the wall, and leaves through a circular sharp-edged orifice, area 4 ft², placed 12 ft higher up the wall. The temperature is 25°C inside, and 15°C outside. Estimate the rate of flow. $C_d = 0{\cdot}62$. (*I.Mech.E.*) *Ans.* 13·13 ft³/sec.

22. The flow through a 2 in. diameter pipe is measured by means of a pipe orifice of ½ in. diameter. The drop in pressure of the water in passing through the orifice was found to be 12·5 in. of water, and the coefficient for the orifice was 0·63. Calculate the flow in cubic feet per minute. *Ans.* 0·422 ft³/min.

23. Describe the orifice method for metering flow in a pipe. Describe how the coefficient varies with the Reynolds number, and explain why air and water may require different coefficients even at the same Reynolds number. (*I.Mech.E.*)

24. Water having a kinematic viscosity of $5{\cdot}05 \times 10^{-6}$ ft-sec units, flows through a pipe orifice of 2 in. diameter inserted in a 4 in. diameter pipe. The difference of pressure at the orifice was found to be 6·5 in. of water.

Using the curves of Fig. 61 obtain the value of C_d for the orifice under this condition of flow and, using this value of C_d, calculate the quantity of flow.

Ans. 0·63; 5·02 ft³/min.

25. A pipe orifice consists of a diaphragm, containing a 1·6 in. diameter orifice, fitted into a 2 in. pipe. From a calibration experiment on this meter the following results were obtained—

R_e	100	400	900	1,600	2,500	3,600	6,400	10,000	14,400	25,600
k	0·75	0·9	0·93	0·88	0·80	0·75	0·68	0·65	0·64	0·64

The values of the Reynolds number, R_e, apply to the conditions at the orifice and k is the ratio of the measured flow to the "theoretical" flow.

Plot the values of the coefficient k on a base representing $\sqrt{R_e}$.

When water, having a coefficient of viscosity of 0·000024 engineers' units, flows through the meter, the difference of pressure head between the two sides of the orifice measured 1·4 in. of Hg. Using the calibration curve, find by a method of successive approximations the flow through the pipe in cubic feet per minute. The results given by the second approximation are of sufficient accuracy. (*Lond. Univ.*) *Ans.* $Q = 10{\cdot}04$ ft^3/min.

CHAPTER 5

NOTCHES AND WEIRS

5.1. Notches and Weirs. A notch may be regarded as an orifice with the water surface below its upper edge. Notches are used for measuring the flow of water from a vessel or reservoir and are generally rectangular or triangular in shape.

A weir is the name given to a dam over which water is flowing. Theoretically, there is no difference between a simple rectangular weir and a rectangular notch, except the latter may have sharp edges.*

The sheet of water flowing through a notch or over a weir is known as the nappe or vein. The top of the weir over which the water flows is known as the sill or crest. Large weirs are sometimes divided into sections by vertical posts.†

Shallow rivers are often made navigable by building dams across the river at certain sections over which the water may flow. This has the effect of deepening the river on the upstream side of the dam by an amount equal to the height of the dam above the original water level. During a drought, little or no water will flow past the dam; but after heavy rains the water flows over the dam, thus converting it into a weir. It is necessary to make short canals, containing locks, around these dams in order that the shipping may pass.

5.2. Rectangular Notch. If water flows from a tank or reservoir over a notch there will be a contraction of the vein and a slight frictional resistance at the sides, as in the case of an orifice. This will cause the actual discharge to be less than the theoretical discharge; the ratio between them will be the coefficient of discharge for the notch. An average value of this coefficient is about 0·62.

Consider the rectangular notch in Fig. 67.

Let L = breadth of notch,
H = height of water surface above sill,
C_d = coefficient of discharge.

Consider a horizontal strip of water of thickness $\mathrm{d}h$ and of depth h. The theoretical velocity of the water flowing through strip will be $\sqrt{2gh}$.

$$\text{Discharge through strip} = L\,\mathrm{d}h\,\sqrt{2gh}$$

* The term "weir" is sometimes loosely applied to small notches.
† For non-dimensional factors for weirs, see § 11.5.

If the whole notch is considered to consist entirely of similar horizontal strips, the total discharge is given by the integration of this equation.

Then,
$$\text{total discharge} = L\sqrt{2g}C_d\int_0^H h^{1/2}\,dh$$
$$= \frac{2}{3}L\sqrt{2g}C_d\Big[h^{3/2}\Big]_0^H$$
$$= \frac{2}{3}C_d\sqrt{2g}LH^{3/2} \qquad (1)$$

This equation is not used for large weirs.

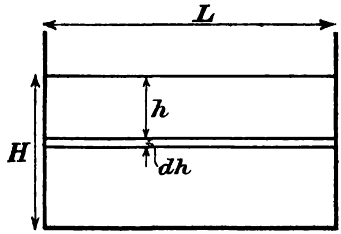


FIG. 67

If a notch of this type is used for measuring a quantity of water flowing, it must be calibrated experimentally. From eq. (1), the discharge for any given notch is equal to $kH^{3/2}$ where

$$k = \frac{2}{3}C_d\sqrt{2g}L$$

Then, by measuring the discharge per second for various heads, the value of k may be obtained by plotting the discharge and $H^{3/2}$. A perfect straight line will not be obtained, as C_d varies slightly with the head. This method of obtaining C_d is demonstrated in Example 2.

An alternative method is to assume $Q = kH^n$; then taking logs of both sides of this equation,

$$\log Q = \log k + n\log H \qquad . \quad . \quad . \quad (2)$$

which is a straight-line law.

By plotting from experimental results $\log H$ as base and $\log Q$ as vertical ordinate, a straight line is obtained from which k and n can be found. For, when $H = 1$, $\log H = 0$; then

$$\log k = \log Q$$

and hence k can be found.

Also, by choosing any convenient point on the graph,

$$n = \frac{\log Q - \log k}{\log H} \qquad \text{[from eq. (2)]}$$

This method is demonstrated in Example 1.

Example 1

The following observations were made during measurements on a weir whose crest b is 3 ft long.

Head H (ft)	0·2	0·4	0·6	0·8	1·0	1·2	1·5
Q (ft^3/sec) .	0·846	2·34	4·24	6·48	9·00	11·78	16·35

If the discharge is given by $Q = KbH^n$, determine K and n. (*A.M.I.Mech.E.*)

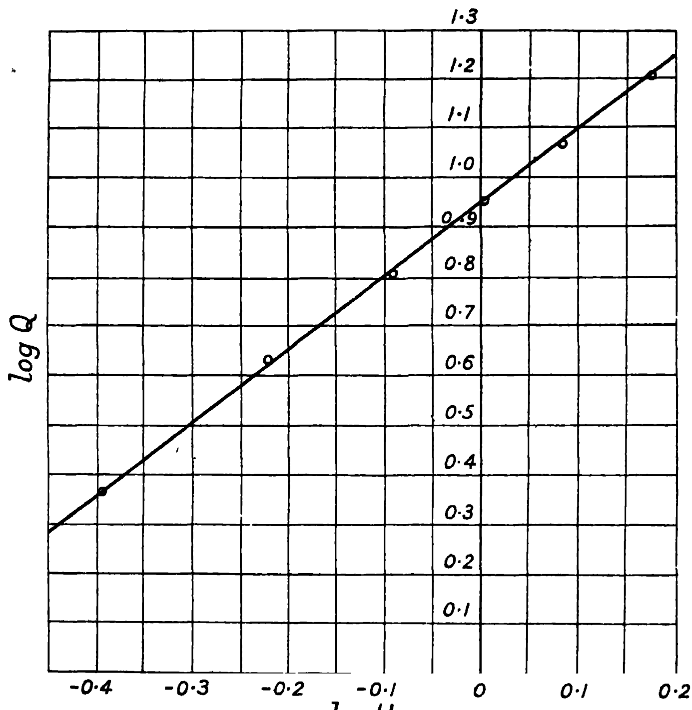


Fig. 68

First plot $\log H$ and $\log Q$ and draw a straight line a mean through the points, as shown in Fig. 68.

Let $k = Kb$.

When $H = 1$, $\log H = 0$; then

$$\log k = \log Q$$
$$= 0{\cdot}955$$

from which $k = 9$

Then $$K = k/b$$
$$= 9/3 = 3$$

Also $$n = \frac{\log Q - \log k}{\log H}$$

Using the values for the point at which $\log H = 0{\cdot}1$,

$$n = \frac{1{\cdot}105 - 0{\cdot}955}{0{\cdot}1}$$
$$= 1{\cdot}5$$

Hence, the equation is

$$Q = 3bH^{3/2}$$

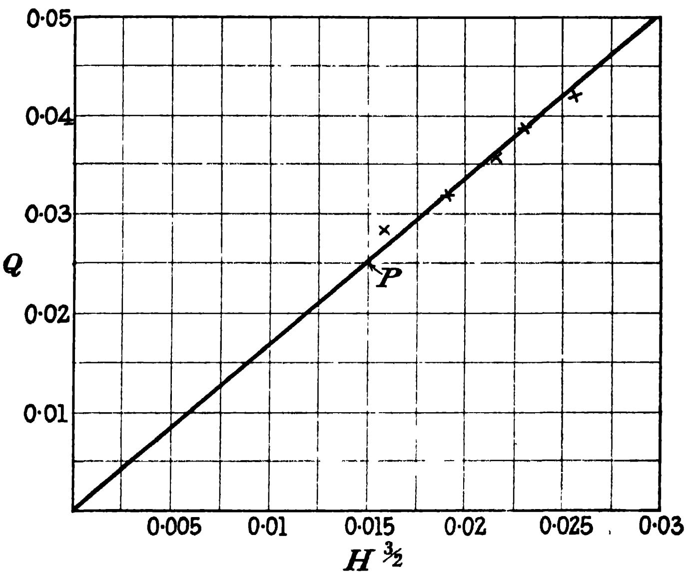


FIG. 69

EXAMPLE 2

In order to find the coefficient of discharge for a small rectangular notch, the discharge was measured experimentally for different heads for a rectangular notch 6 in. wide. The following results were obtained—

Head (ft) .	0·0651	0·0716	0·0775	0·0827	0·0870
Discharge (ft^3/sec)	0·02813	0·03180	0·03535	0·03872	0·0419

Find the average value of C_d for the notch.

Applying eq. (1),

$$\text{Discharge} = Q = \frac{2}{3} C_d \sqrt{2g} L H^{3/2}$$
$$= kH^{3/2}$$

The value of the constant k may be found by plotting Q and $H^{3/2}$, thus obtaining a straight line. This is done in Fig. 69; a straight line is drawn a mean through the points and passing through the origin.

Taking the values of Q and $H^{3/2}$ from a point P on the straight line,

$$k = \frac{Q}{H^{3/2}} = \frac{0\cdot025}{0\cdot015} = 1\cdot667$$

As $$k = \frac{2}{3} C_d \sqrt{2g} L$$

$$\frac{2}{3} C_d \sqrt{2g} = \frac{k}{L} = \frac{1\cdot667}{0\cdot5} = 3\cdot333$$

and $$C_d = \frac{3\cdot333}{\frac{2}{3}\sqrt{2g}} = 0\cdot624$$

5.3. Triangular or V-Notch. In the case of a rectangular notch, it will be noticed that the total wetted edge of the notch does not vary directly with the head, as the length of the base is the same for all heads. Therefore, the coefficient of contraction, which depends on the length of wetted edge, is not a constant for all heads. But in the case of a triangular notch, there is no base to cause contraction, which will be due to the sides only. The coefficient of contraction will, therefore, be a constant for all heads. For this reason, the triangular notch is the most satisfactory type for measuring the quantity of water flowing.

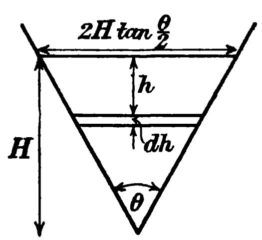


Fig. 70

Consider the triangular notch in Fig. 70.

Let H = height of water surface
and θ = angle of notch.

Then, width of notch at water surface $= 2H \tan \frac{\theta}{2}$

Consider a horizontal strip of the notch of thickness dh and of depth h.

$$\text{Width of strip} = 2(H - h) \tan \frac{\theta}{2}$$

Theoretical velocity of flow through strip

$$= \sqrt{2gh}$$

Discharge through strip = area of strip × velocity

$$= 2(H - h) \tan \frac{\theta}{2}\, dh\, \sqrt{2gh}\, C_d$$

Total discharge through notch

$$= 2C_d\sqrt{2g} \tan \frac{\theta}{2} \int_0^H (H - h)h^{1/2}\, dh$$

$$= 2C_d\sqrt{2g} \tan \frac{\theta}{2} \left[\frac{2}{3} Hh^{3/2} - \frac{2}{5} h^{5/2}\right]_0^H$$

$$= \frac{8}{15} C_d\sqrt{2g} \tan \frac{\theta}{2} H^{5/2}$$

Assuming $C_d = 0{\cdot}6$,

$$\text{discharge} = 2{\cdot}56 \tan \frac{\theta}{2} H^{5/2} \qquad (3)$$

For a 90° notch, $\tan \frac{\theta}{2} = 1$. Then,

$$\text{discharge} = 2{\cdot}56H^{5/2} \qquad (4)$$

EXAMPLE 3

In order to find the constant for a 90° triangular notch, the discharge through the notch was measured for different heads. The following readings were obtained—

Head (ft)	0·0407	0·0491	0·0550	0·0692	0·0798	0·0919	0·101
Discharge (ft³/sec)	0·00095	0·00156	0·00207	0·00361	0·00490	0·00702	0·00867

Find the constant for the notch and the value of C_d.

As discharge for a 90° notch $= \frac{8}{15} C_d\sqrt{2g}H^{5/2}$

$$Q = kH^{5/2}$$

where $k = \frac{8}{15} C_d\sqrt{2g}$.

A straight line should, therefore, be obtained by plotting Q and $H^{5/2}$. This has been done in Fig. 71; it will be noticed that the points lie approximately on a straight line passing through the origin.

From point P on this straight line,

$$k = \frac{Q}{H^{5/2}} = \frac{0{\cdot}0055}{0{\cdot}002} = 2{\cdot}75$$

Then $$Q = 2{\cdot}75H^{5/2}$$

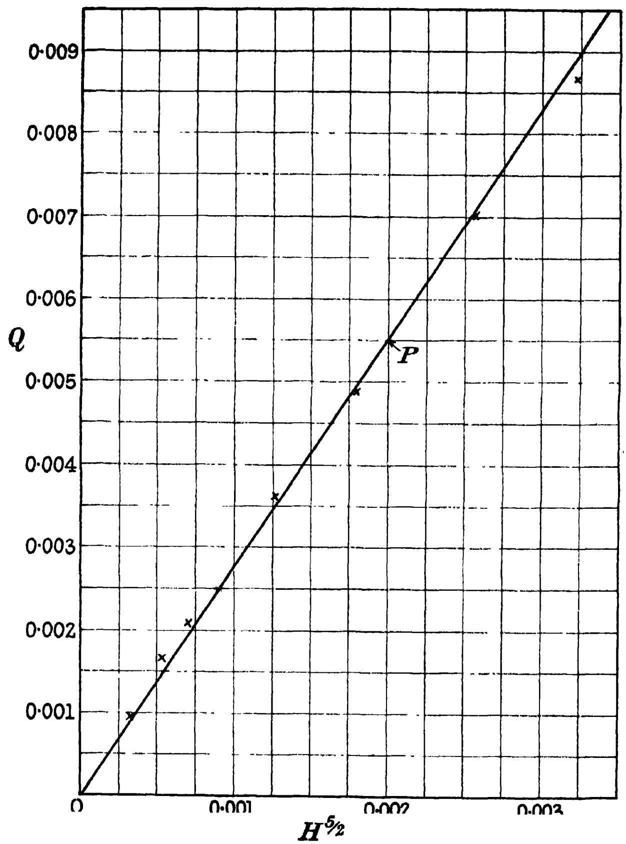


FIG. 71

As $$k = \frac{8}{15} C_d\sqrt{2g}$$

$$C_d = \frac{2{\cdot}75 \times 15}{\sqrt{2g} \times 8} = 0{\cdot}642$$

EXAMPLE 4

A trapezoidal notch has a base L and a head H, and the sides make an angle of θ to the vertical. Deduce an expression for the discharge through the notch.

A notch of this type may be divided into a rectangular notch of breadth L, and a triangular notch subtending an angle of 2θ. Then, the total discharge may be found by adding together the discharges from these two.

Discharge through rectangular notch

$$= \frac{2}{3} C_d \sqrt{2g} L H^{3/2}$$

Discharge through triangular notch

$$= \frac{8}{15} C_d \sqrt{2g} \tan\theta \,.\, H^{5/2}$$

$$\text{Total discharge} \quad = \frac{2}{3} C_d \sqrt{2g} L H^{3/2} + \frac{8}{15} C_d \sqrt{2g} \tan\theta \,.\, H^{5/2}$$

$$= C_d \sqrt{2g} H^{3/2} \left(\frac{2}{3} L + \frac{8}{15} \tan\theta \,.\, H\right)$$

This may also be obtained from first principles by producing the sloping sides to their point of intersection, and integrating between the limits of the head H.

5.4. Thomson's Principle of Geometric Similarity. Geometrically similar weirs or notches may be defined as notches which may be represented by drawings of the same notch but to a different scale. The discharge through similar notches will depend on their linear dimensions raised to power of 5/2.

Consider two similar triangular notches.

$$\text{Discharge} \propto \text{area} \times \text{velocity}$$

But $\qquad \text{area} \propto H^2$

and $\qquad \text{velocity} \propto \sqrt{H}$

Then, $\qquad \text{discharge} \propto H^2 \times \sqrt{H} = kH^{5/2}$

The constant k should be the same for all similar notches.

In the case of similar rectangular notches, let the breadth of the notches be L and nL, and let the corresponding heads be H and nH.

Then,

$$\frac{\text{discharge of one weir}}{\text{discharge of other}} = \frac{(\text{area} \times \text{velocity}) \text{ of one}}{(\text{area} \times \text{velocity}) \text{ of other}}$$

$$= \frac{nL \times nH \times \sqrt{nH}}{L \times H \times \sqrt{H}}$$

$$= n^{5/2}$$

5.5. Francis' Formula for Rectangular Weirs. An empirical formula for the discharge of a rectangular weir is given by Francis as—

$$Q = 3{\cdot}33(L - 0{\cdot}1nH)H^{3/2} \text{ ft}^3/\text{sec}$$

where L = total breadth of weir in feet,

H = head in feet,

and n = number of end contractions.

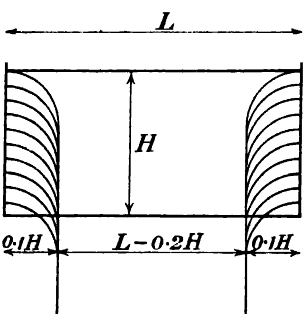


FIG. 72

For a simple rectangular weir, $n = 2$. For a large weir which is split up into bays by vertical posts, n will depend on the number of bays into which the weir is divided.

Although Francis deduced this formula experimentally, it can be proved to be quite rational. As the water flows over the weir, the vein is contracted at the sides (Fig. 72) by an amount which is found, experimentally, to average $0{\cdot}1H$ for each side. If there are two contractions, as in the case of Fig. 72, the effective breadth of the weir is $(L - 0{\cdot}2H)$. If there were n contractions the effective length would be $(L - 0{\cdot}1nH)$.

Substituting the effective length in eq. (1),

$$Q = \frac{2}{3} C_d\sqrt{2g}(L - 0{\cdot}1nH)H^{3/2}$$

Assuming $C_d = 0{\cdot}623$,

$$Q = 3{\cdot}33(L - 0{\cdot}1nH)H^{3/2}$$

If the end contractions are suppressed, as in the case of a weir having the same width as the channel by which the water approaches, n will be zero. Then,

$$Q = 3{\cdot}33LH^{3/2}$$

EXAMPLE 5

Show that a rational formula for the flow Q over a rectangular weir of width B can be expressed as

$$Q = A(B - CH)H^{3/2}$$

where H is the depth of water at a point near the weir which is not affected by the curvature of the surface.

In a rectangular weir, 5 ft in breadth, the discharge is $5{\cdot}91$ ft^3/sec when the head of water is $0{\cdot}51$ ft, and is $14{\cdot}99$ ft^3/sec when the head is $0{\cdot}96$ ft. Find the values of the constants in the above expression, and estimate the discharge when the head is $0{\cdot}75$ ft. (*Lond. Univ.*)

This is Francis' formula. Substituting the values of B and H in the given two cases,

$$5{\cdot}91 = A(5 - C \times 0{\cdot}51)0{\cdot}51^{3/2} \quad . \quad . \quad (5)$$

also
$$14{\cdot}99 = A(5 - C \times 0{\cdot}96)0{\cdot}96^{3/2} \quad . \quad . \quad (6)$$

From (5)
$$16{\cdot}2 = 5A - 0{\cdot}51AC \quad . \quad . \quad . \quad . \quad (7)$$

From (6)
$$16{\cdot}0 = 5A - 0{\cdot}96AC$$

Subtracting
$$0{\cdot}2 = 0{\cdot}45AC$$

Then
$$C = \frac{0{\cdot}2}{0{\cdot}45A}$$

Substituting in (7),

$$16{\cdot}2 = 5A - 0{\cdot}51A \times \frac{0{\cdot}2}{0{\cdot}45A}$$
$$= 5A - 0{\cdot}2265$$

from which,
$$A = 3{\cdot}29$$

and
$$C = \frac{0{\cdot}2}{0{\cdot}45 \times 3{\cdot}29} = 0{\cdot}135$$

Then
$$Q = 3{\cdot}29(5 - 0{\cdot}135H)H^{3/2}$$

When $H = 0{\cdot}75$ ft,

$$Q = 3{\cdot}29(5 - 0{\cdot}101)0{\cdot}75^{3/2}$$
$$= 10{\cdot}46 \text{ ft}^3/\text{sec}$$

5.6. Bazin's Formula for Rectangular Weirs. Another type of equation used for obtaining the discharge over a rectangular weir without end contractions is known as Bazin's formula. Using eq. (1),

$$Q = \frac{2}{3} C_d \sqrt{2g} L H^{3/2}$$
$$= m\sqrt{2g} L H^{3/2}$$

where
$$m = \frac{2}{3} C_d$$

The coefficient m was found by Bazin to vary with the head, its value being obtained from the following equation—

$$m = 0{\cdot}405 + \frac{0{\cdot}00984}{H}$$

EXAMPLE 6

Find the discharge, using Bazin's formula, for a rectangular weir with end contractions suppressed, when the head is 6 in. and the length 4 ft.

As $H = 0{\cdot}5$ ft,

$$m = 0{\cdot}405 + \frac{0{\cdot}00984}{0{\cdot}5}$$
$$= 0{\cdot}42468$$

$$\text{Discharge} = m\sqrt{2g}LH^{3/2}$$
$$= 0{\cdot}42468 \times \sqrt{2g} \times 4 \times (0{\cdot}5)^{3/2}$$
$$= 4{\cdot}82 \text{ ft}^3/\text{sec}$$

5.7. Velocity of Approach. If the area of the channel through which the water approaches the weir is larger than the weir itself, the water will have a velocity on reaching the weir known as the velocity of approach. This velocity may be assumed to be uniform over the whole weir.

Let A = cross-sectional area of channel behind weir,
v_1 = velocity of approach
Q = discharge over weir in cubic feet per second

Then, as quantity of water passing over weir per second equals quantity flowing along channel per second,

$$v_1 = \frac{Q}{A}$$

The quantity Q is determined, as a first approximation, from the ordinary weir equation, ignoring the velocity of approach.

Additional head due to velocity of approach $= v_1^2/2g$ and acts over whole of weir.

Consider the horizontal strip of the weir in Fig. 67.

$$\text{Discharge through strip} = C_d\sqrt{2gh} \times L\,dh$$

$$\text{Total discharge} = C_d\sqrt{2g}L\int_{\frac{v_1^2}{2g}}^{H+\frac{v_1^2}{2g}} h^{1/2}\,dh$$

$$= \frac{2}{3}C_d\sqrt{2g}L\left[h^{3/2}\right]_{\frac{v_1^2}{2g}}^{H+\frac{v_1^2}{2g}}$$

$$= \frac{2}{3}C_d\sqrt{2g}L\left[\left(H+\frac{v_1^2}{2g}\right)^{3/2} - \left(\frac{v_1^2}{2g}\right)^{3/2}\right] \quad . \quad . \quad . \quad . \quad (8)$$

As the value of v_1 was obtained only approximately in the first case, it should be corrected to suit the new discharge obtained from eq. (8). Then, by substituting this new value of v_1 in eq. (8), a more accurate value of the actual discharge may be obtained. If the value of v_1 is small, this correction of the first approximation will make very little difference to the discharge obtained from eq. (8).

Francis' formula for velocity of approach becomes—

$$Q = 3\cdot33(L - 0\cdot1nH_1)\left[H_1^{3/2} - \left(\frac{v_1^2}{2g}\right)^{3/2}\right]$$

where $H_1 = v_1^2/2g + H$, and is known as the still water head.

From the results of experiments, Bazin found that the discharge could be obtained by increasing the actual measured head, H, by the amount $a(v_1^2/2g)$, where a is a constant having a mean value of 1·6.

Then, equivalent static head, or still water head

$$= H + \frac{av_1^2}{2g}$$
$$= H_1$$

Bazin's formula then becomes—

$$Q = m\sqrt{2g}L\left(H + \frac{av_1^2}{2g}\right)^{3/2}$$

where $m = 0\cdot405 + 0\cdot00984/H_1$.

EXAMPLE 7

Find the discharge over a weir 10 ft long under a measured head of 2 ft, if the channel approaching the weir is 20 ft wide and 3 ft deep.

First find the discharge, ignoring velocity of approach. Using Francis' formula,

$$Q = 3\cdot33(L - 0\cdot2H)H^{3/2}$$
$$= 3\cdot33(10 - 0\cdot4)2^{3/2}$$
$$= 90\cdot4 \text{ ft}^3/\text{sec}$$
$$v_1 = \frac{Q}{A} = \frac{90\cdot4}{20 \times 3} = 1\cdot51 \text{ ft/sec}$$

$$\text{Still water head} = H_1 = H + \frac{v_1^2}{2g}$$
$$= 2 + \frac{1\cdot51^2}{2g}$$
$$= 2\cdot035 \text{ ft}$$

Substituting in Francis' formula for velocity of approach,

$$Q = 3\cdot33(L - 0\cdot1nH_1)\left[H_1^{3/2} - \left(\frac{v_1^2}{2g}\right)^{3/2}\right]$$
$$= 3\cdot33(10 - 0\cdot407)\left[2\cdot035^{3/2} - \left(\frac{1\cdot51^2}{2g}\right)^{3/2}\right]$$
$$93\cdot0 \text{ ft}^3/\text{sec}$$

In this example the value of v_1 is so small that no adjustment is necessary.

5.8. Time of Emptying Reservoir with Rectangular Weir. Consider a reservoir of area A ft^2 in plan from which water is flowing over a rectangular weir of breadth L. It is required to find the time taken for the water level in the reservoir to fall from a height H_1 to a height H_2 above the level of the sill.

Suppose at any instant the height of water level above the sill is h. Then let a small quantity dq flow over the weir in a time dt, and let this cause the water level in reservoir to fall by amount dh. As dh is measured downwards, it is negative.

$$\text{Discharge through weir} = \mathrm{d}q = \frac{2}{3} C_d\sqrt{2g}Lh^{3/2}\,\mathrm{d}t$$

$$\text{Discharge from reservoir} = \mathrm{d}q = -A\,\mathrm{d}h$$

$$\text{Then} \qquad \frac{2}{3} C_d\sqrt{2g}Lh^{3/2}\,\mathrm{d}t = -A\,\mathrm{d}h$$

$$\text{and} \qquad \mathrm{d}t = -\frac{A\,\mathrm{d}h\,h^{-3/2}}{\frac{2}{3}C_d\sqrt{2g}L}$$

$$\text{Total time} = T = \int_0^T \mathrm{d}t = -\frac{A}{\frac{2}{3}C_d\sqrt{2g}L}\int_{H_1}^{H_2} h^{-3/2}\,\mathrm{d}h$$

$$= \frac{2A}{\frac{2}{3}C_d\sqrt{2g}L}\Big[h^{-1/2}\Big]_{H_1}^{H_2}$$

$$= \frac{2A}{\frac{2}{3}C_d\sqrt{2g}L}\left(\frac{1}{H_2^{1/2}} - \frac{1}{H_1^{1/2}}\right)$$

If Bazin's coefficient is used, this equation becomes

$$T = \frac{2A}{m\sqrt{2g}L}\left(\frac{1}{H_2^{1/2}} - \frac{1}{H_1^{1/2}}\right)$$

Using Francis' formula the equation becomes

$$T = \frac{2A}{3{\cdot}33(L - 0{\cdot}1nH)}\left(\frac{1}{H_2^{1/2}} - \frac{1}{H_1^{1/2}}\right) \tag{9}$$

the value of H being taken as a mean of H_1 and H_2.

It will be noticed that if it is required to empty the reservoir by means of a weir, $H_2 = 0$, then the time given by eq. (9) becomes infinity. Actually, the equation would not hold for extremely small values of H_2 because a layer of water would adhere to the sill owing to surface tension.

EXAMPLE 8

Show that the discharge over a sharp-edged V notch is theoretically proportional to the head raised to an index power of 2·5.

A sharp-edged V notch inserted in the side of a rectangular tank, 12 ft long and 4 ft broad, gives a calibration $Q = 2{\cdot}64H^{2{\cdot}5}$ where Q is measured in cubic

feet per second and H is measured in feet. Find how long it will take to reduce the head in the tank from 12 in. to 1 in. if the water discharges freely over the notch and there is no inflow into the tank. (*Lond. Univ.*)

Consider the water level at any instant to be h ft above bottom of notch. Let a small quantity dq flow through in time dt, thereby reducing the water level by dh.

Then $$\mathrm{d}q = 2{\cdot}64h^{2\cdot5}\,\mathrm{d}t$$

Also $$\mathrm{d}q = -A\,\mathrm{d}h \text{ (as d}h\text{ is negative)}$$

Therefore $$2{\cdot}64h^{2\cdot5}\,\mathrm{d}t = -A\,\mathrm{d}h$$

or $$\mathrm{d}t = -\frac{A\,\mathrm{d}h}{2{\cdot}64h^{2\cdot5}}$$

$$\text{Total time} = T = \int_0^T \mathrm{d}t = -\frac{A}{2{\cdot}64}\int_1^{1/12} h^{-2\cdot5}\,\mathrm{d}h$$

$$= \frac{2}{3} \times \frac{A}{2{\cdot}64}\left[h^{-1\cdot5}\right]_1^{1/12}$$

$$= \frac{2}{3} \times \frac{A}{2{\cdot}64}\left(\frac{1}{(\frac{1}{12})^{3/2}} - \frac{1}{(1)^{3/2}}\right)$$

$$= \frac{2}{3} \times \frac{4 \times 12}{2{\cdot}64}\left(\frac{1}{0{\cdot}0241} - 1\right) \text{sec}$$

$$= 8{\cdot}2 \text{ min}$$

5.9. The Syphon Spillway. It is necessary with reservoirs to have an automatic device for keeping the water level in them at a constant height. The simplest way is to fit a weir at the side, with the sill at the same height as the required water level. An increase in the water level will then cause the excess water to flow over the sill into the overflow channel below. It will thus flow away to waste. As the discharge by this method is due to the head over the sill only, the method is not sensitive unless the weir is very long, which is not a practical proposition.

A more sensitive method is to fit syphon spillways to the reservoir; these employ the whole head between the reservoir level and the water level in the overflow channel, thus creating a high velocity and a consequently large discharge.

A cross-sectional view of an automatic syphon spillway is shown in Fig. 73. It consists of an ordinary weir sill surrounded by an air-tight cover, as shown, thus converting the discharge face of the weir into a large rectangular-sectioned pipe.

As soon as the water level in the reservoir rises above the sill A by a measurable amount, the water flows over the sill and strikes the inside of the cover, thus completely filling the cross-section of

the pipe. The air is now trapped in the upper portion of the cover B, and is immediately sucked away by the stream of flowing water. This causes a negative pressure at B which sucks up the water from the reservoir and completely fills the pipe; the syphon action is thus started. The water will now rush down the pipe to waste, with

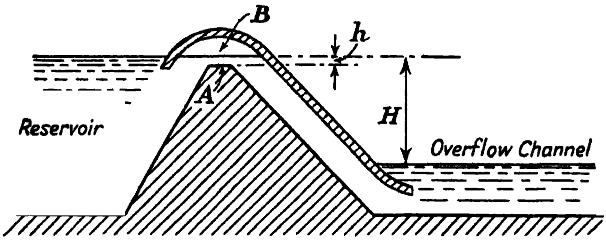


FIG. 73. AUTOMATIC SYPHON SPILLWAY

a velocity caused by the total head H, thus causing a large discharge. Had this operation been performed by an ordinary open weir, the velocity of discharge would have been due to the small head h only, and would have been consequently very small.

It will be seen from this that the syphon spillway has a much greater discharge for a given length than an ordinary open weir, as it utilizes the whole difference of head between the reservoir and the overflow channel.

Sometimes several syphon spillways are fitted side by side with their sills at different levels. Then, for a small rise in water level, the lowest spillway only is in action. If this discharge is not sufficient to maintain the correct level, the water in the reservoir will rise higher and thus bring the next spillway into action.

Let A_1 = area of cross-section of spillway pipe in square feet

and H = difference in water level of reservoir and overflow channel, in feet.

Then
$$Q = C_d A_1 \sqrt{2gH}$$

where C_d is the coefficient of discharge, to be determined by test.

5.10. Broad-crested Weir. A weir having a broad sill is known as a broad-crested weir. The discharge of a weir of this type depends on the head H, the breadth b, and the length l of the sill; it will also depend on the roughness of the sill's surface, on the viscosity, and on the temperature.

As the water flows over the sill there is a loss of head due to the frictional resistance. If the sill is very long, this resistance will be similar to that of the bed of an open channel.

Let Fig. 74 represent the water flowing over a broad-crested weir and let h be the head of water at the downstream edge of the sill. Assume the sill is of sufficient length to allow the velocity of the water to be uniform throughout its depth at the downstream edge. Let this uniform velocity be v ft/sec.

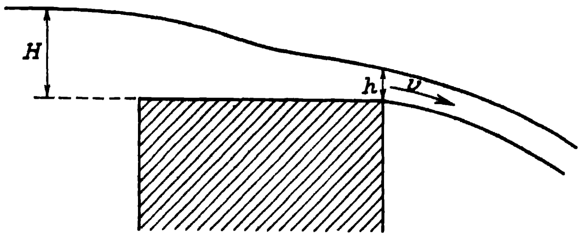


FIG. 74

Then, neglecting losses, $H = h + \dfrac{v^2}{2g}$

from which $v = \sqrt{2g(H - h)}$

Then discharge per second $= Q = C_d bhv$

$$= C_d bh\sqrt{2g(H - h)}$$

$$= C_d\sqrt{2g}\,b(Hh^2 - h^3)^{1/2} \qquad . \text{ (10)}$$

where C_d is the experimental coefficient of discharge for the weir. C_d will be a function of the head, the length of sill, the roughness of surface, the coefficient of viscosity and of the temperature. Its value can be obtained only from test.

From eq. (10) it will be noticed that the discharge is a maximum when $(Hh^2 - h^3)$ is a maximum. Hence, differentiating,

$$\frac{dQ}{dh} = 2Hh - 3h^2 = 0$$

or $2H - 3h = 0$

Hence $h = \frac{2}{3}H$

Substituting this value in eq. (10),

$$\text{Maximum } Q = C_d\sqrt{2g}\,b\tfrac{2}{3}H\sqrt{\tfrac{1}{3}H}$$

$$= 3{\cdot}09C_d bH^{3/2} \qquad . \text{ (11)}$$

For a large range of head H the weir seems to set itself automatically to this condition of maximum discharge; it thus appears that the stable condition of equilibrium is when the discharge is a maximum.

When $h = 2H/3$,

$$v = \sqrt{2g\frac{H}{3}}$$

and
$$d = h = \frac{2H}{3}$$

Hence
$$\text{Froude number} = \frac{v}{\sqrt{gd}} = \frac{\sqrt{2g\frac{H}{3}}}{\sqrt{g\frac{2H}{3}}} = 1$$

Hence, the condition of maximum discharge is when the Froude number is unity. Thus, the flow over the crest cannot exceed a Froude number of 1. This is analogous to the flow of gases through a pipe (§ 18.4) where the Mach number cannot exceed unity.

If the sill is long, it is possible for a hydraulic jump to occur. The reason for the occurrence of the jump follows the same argument as in § 10.11 and § 10.12.

5.11. Flow over a Submerged Weir. The discharge over a submerged weir can be obtained by dividing it into two horizontal sections, as shown in Fig. 75. The portion between the upstream and downstream water surfaces may be treated as a free weir; the portion between the downstream water surface and the top of the sill may be treated as a drowned orifice.

Let Q_1 = discharge per second through the free portion of the weir. Consider a horizontal strip at h ft below the upper surface and of thickness dh. Then,

discharge through strip = dq = $C_d B\sqrt{2gh}\,\mathrm{d}h$

Integrating between the limits of the free weir, $H_1 - H_2$ and 0,

$$\begin{aligned} Q_1 &= C_d\sqrt{2g}B\int_0^{(H_1-H_2)} h^{1/2}\,\mathrm{d}h \\ &= \tfrac{2}{3}C_d\sqrt{2g}B\Big[h^{3/2}\Big]_0^{(H_1-H_2)} \\ &= \tfrac{2}{3}C_d\sqrt{2g}B(H_1 - H_2)^{3/2} \end{aligned}$$

Next consider the portion of weir below the downstream water surface.

Let Q_2 = discharge per second through this drowned portion or weir.

Treating this portion as a drowned orifice (§ 4.6),

$$\begin{aligned} Q_2 &= C_d \times \text{area} \times \text{velocity} \\ &= C_d B H_2\sqrt{2g(H_1 - H_2)} \end{aligned}$$

Then total discharge $= Q_1 + Q_2$

The above simple solution can be applied if the values of C_d for the two sections of the weir are known. The problem may be complicated by the formation of a standing wave (§ 10.11) on the lower water surface.

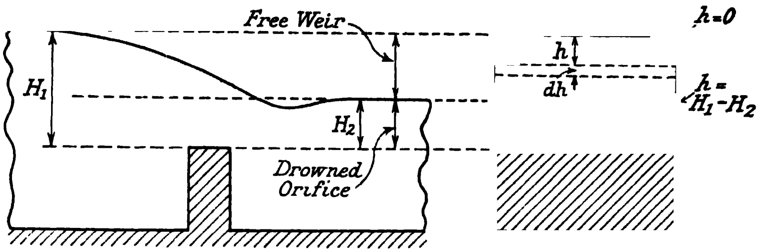


FIG. 75

EXAMPLE 9

A submerged weir spans the entire width of a rectangular channel 20 ft wide, the sharp edge of the weir being 3 ft above the bottom of the channel. Estimate the mean velocity of flow in the channel when the depth of water is 5 ft on the upstream side, and 3·25 ft on the downstream side of the weir. Allow for the velocity of approach, and take $C_d = 0{\cdot}62$ for the weir. (*Lond. Univ.*)

First the discharge over the weir is calculated, neglecting the velocity of approach, as explained in § 5.7.

$$Q_1 = \tfrac{2}{3}C_d\sqrt{2g}B(H_1 - H_2)^{3/2}$$
$$= \tfrac{2}{3} \times 0{\cdot}62\sqrt{64{\cdot}4} \times 20\,(2 - 0{\cdot}25)^{3/2}$$
$$= 153 \text{ ft}^3/\text{sec}$$
$$Q_2 = C_d B H_2\sqrt{2g(H_1 - H_2)}$$
$$= 0{\cdot}62 \times 20 \times 0{\cdot}25\sqrt{64{\cdot}4(2 - 0{\cdot}25)}$$
$$= 33 \text{ ft}^3/\text{sec}$$
$$\text{Total quantity} = Q = Q_1 + Q_2$$
$$= 153 + 33$$
$$= 186 \text{ ft}^3/\text{sec}$$

Let v_1 = velocity of approach

$$= \frac{Q}{\text{area of channel}}$$
$$\frac{186}{5 \times 20} = 1{\cdot}86 \text{ ft/sec}$$

Now calculate the discharge over the weir using this approximate value of the velocity of approach. Then, as explained in § 5.7,

$$Q_1 = \frac{2}{3} C_d \sqrt{2g} B \left[\left\{ (H_1 - H_2) + \frac{v_1^2}{2g} \right\}^{3/2} - \left(\frac{v_1^2}{2g} \right)^{3/2} \right]$$

$$= \frac{2}{3} \times 0{\cdot}62 \sqrt{64{\cdot}4} \times 20 \left[\left\{ (2 - 0{\cdot}25) + \frac{1{\cdot}86^2}{2g} \right\}^{3/2} - \left(\frac{1{\cdot}86^2}{2g} \right)^{3/2} \right]$$

$$= 159{\cdot}9 \text{ ft}^3/\text{sec}$$

$$Q_2 = C_d B H_2 [\sqrt{2g(H_1 - H_2)} + v_1]$$

$$= 0{\cdot}62 \times 20 \times 0{\cdot}25[\sqrt{64{\cdot}4(2 - 0{\cdot}25)} + 1{\cdot}86]$$

$$= 38{\cdot}3 \text{ ft}^3/\text{sec}$$

Then, total quantity $= Q = Q_1 + Q_2$

$$= 159{\cdot}9 + 38{\cdot}3$$

$$= 198{\cdot}2 \text{ ft}^3/\text{sec}$$

Mean velocity of flow in channel

$$= \frac{Q}{\text{area of channel}}$$

$$= \frac{198{\cdot}2}{20 \times 5}$$

$$= 1{\cdot}982 \text{ ft/sec}$$

If a more accurate result is required, the calculation should be repeated using $v_1 = 1{\cdot}982$ ft/sec.

EXERCISES 5

1. Find the discharge through a rectangular weir, 8 ft wide, under a head of 8 in., when the side contractions are suppressed (1) by Bazin's formula and (2) by Francis' formula. *Ans.* (1) 14·7 ft³/sec. (2) 14·5 ft³/sec.

2. A rectangular weir is 6 ft broad and has a head of 2 ft 3 in. Find the discharge taking into account the two end contractions. *Ans.* 62·2 ft³/sec.

3. A rectangular weir, 20 ft long, is divided into three bays by two vertical posts, each 1 ft wide. Find the discharge when the head is 1 ft 6 in. *Ans.* 104·7 ft³/sec.

4. Find the discharge through a triangular notch under a constant head of 10 in. if the angle of the notch is 120°. $C_d = 0{\cdot}62$. *Ans.* 2·94 ft³.

5. A stream approaching a waterfall having a fall of 60 ft, is gauged by a weir. The measured head over the weir is 11 in. and the length of the weir is 10 ft. The velocity of approach v_1 is 4 ft/sec, and, due to this, the head may be supposed to be increased by $1{\cdot}5v_1^2/2g$. Determine the power available from the fall, assuming that 50 per cent of the energy can be used. (*Lond. Univ.*) *Ans.* 165·2 h.p.

6. Obtain a formula for the discharge over a rectangular weir, taking into account the effect of lateral contractions.

Determine the discharge over a sharp-crested weir, 15 ft long, with no lateral contractions, the measured head over the crest being 17·9 in. The width of the channel of approach is 25 ft, and its depth below the crest of the weir is 3 ft. (*Lond. Univ.*) *Ans.* 93 ft³/sec.

7. During a test in a laboratory, the water which has passed through a Venturi meter flows over a right-angled V notch, the head at the notch being registered. The larger diameter of the Venturi meter is 10 in., and the diameter of the throat is 4 in. When a steady head over the V notch of 0·604 ft is maintained, the difference of pressure head at the Venturi meter is found to be 1·075 ft of water. Determine the coefficient of this Venturi meter on the assumption that the V notch results are correct, the coefficient being 0·60. (*Lond. Univ.*) *Ans.* $k = 0{\cdot}99$.

8. A reservoir has an area of 100,000 yd² and is provided with a weir 15 ft long; find how long it will take for the level at the sill to fall from 2 ft to 1 ft.

Deduce the formula you use and note any assumptions made. (*Lond. Univ.*) *Ans.* 179·5 min.

9. State the principle of similarity and show how it can be used to prove that the discharge from a triangular notch is

$$Q = CH^{5/2}$$

The compensation water from a waterworks of 12,000,000 gal per day is discharged over a rectangular weir. Find the length of the weir if the head is not to be more than 15 in. (*Lond. Univ.*) *Ans.* 5·05 ft.

10. Explain why a sharp-edged V notch gives a coefficient of discharge which is practically independent of the head. (*Lond. Univ.*)

11. Deduce an expression for the discharge over a triangular notch. What does this become if the angle of the notch is 90°? (*A.M.I. Mech. E.*)

12. Find the depth and top width of a triangular notch capable of discharging a maximum quantity of 25 ft³/sec and such that the head shall be 3 in. when the discharge is 0·2 ft³/sec. For a right-angled notch, $C = 2{\cdot}54$. (*A.M.I.C.E.*) *Ans.* 1·725 ft. 8·7 ft.

13. Find the discharge over 100 ft of broad-crested weir when the head is 3 ft. Prove any formula used. Assume $C_d = 0{\cdot}61$. (*I. Mech. E.*) *Ans.* 980 ft³/sec.

14. Deduce an expression for the discharge over a right-angled triangular notch. If the coefficient of discharge is 0·61, what will be the discharge under a head of 12 in.? (*A.M.I. Mech. E.*) *Ans.* 2·6 ft³/sec.

15. Deduce an expression for the discharge over a right-angled triangular notch. If the coefficient of discharge is 0·61, what will be the discharge if the head is 18 in.? (*A.M.I.C.E.*) *Ans.* 7·16 ft³/sec.

16. Calculate the maximum discharge over a broad-crested weir of length 12 ft having no end contractions. The head above the sill of the weir is 1·6 ft and C_d is 0·97. *Ans.* 72·8 ft³/sec.

17. Find the discharge over a flat-topped broad-crested weir 100 ft long with rounded entrance, when the upstream level is 2·25 ft above the crest. Deduce any formula used. Point out the limitations of the treatment. [Assume $C_d = 0{\cdot}66$.] (*Lond. Univ.*) *Ans.* 687·0 ft³/sec.

18. Explain why the depth of the stream flowing over the top of a broad-crested weir is theoretically two-thirds of the height of the level of the water in the reservoir above the sill.

If the ordinary formula for a rectangular notch, without end contractions, is used to give the flow across a broad-crested weir, what should be the value of the coefficient of discharge in this formula? (*Lond. Univ.*) *Ans.* $\sqrt{(1/3)}$.

19. Show that the flow over a submerged weir is given approximately by the expressions,

$$Q = (1/3)\, C_d b \sqrt{2g(H_1 - H_2 + h)}\,.\,[2(H_1 + h) + H_2]$$

in which b is the width, H_1 and H_2 are the heights of the free surfaces above the sill, and h is a supplementary head which takes into account the velocity of approach, C_d is a coefficient of discharge. State the assumptions made in obtaining the formula. (*Lond. Univ.*)

20. Obtain an equation for the discharge over a rectangular notch and show how the Francis' formula for large weirs can be derived from it.

A channel, 8 ft wide, containing flowing water 3 ft deep, is gauged by a rectangular weir 4 ft long. The water surface in the channel is 2 ft above the sill of the weir. Using Francis' formula, calculate the discharge in cubic feet per second over the weir, allowing for the velocity of approach in the channel. (*Lond. Univ.*) *Ans.* $Q = 34{\cdot}7$ ft³/sec.

CHAPTER 6

IMPACT OF JETS

6.1. Pressure on Stationary Flat Plate. When a jet of fluid impinges normally on a flat plate (Fig. 76) the force on the plate is equal to the rate of change of the momentum of the jet, or to the change of momentum per second.

Let a = cross-sectional area of jet in square feet,
V = velocity of jet in feet per second,
and W = weight of fluid striking the plate per second.

Then
$$W = waV$$

The jet strikes the plate and leaves it tangentially, so that all its momentum in a direction normal to the plate is destroyed.

$$\begin{aligned}\text{Force on plate} &= \text{change of momentum per second}\\ &= \text{mass of fluid striking plate per second}\\ &\qquad \times \text{ change of velocity normal to plate}\\ &= \frac{W}{g} \times V\\ &\ \ \frac{waV^2}{g}\text{ Lb}\end{aligned}$$

If the plate is inclined to the jet, as in Fig. 77, the force of the jet may be resolved into a normal component.

Let θ = angle of inclination of plate to jet.

$$\begin{aligned}\text{Normal force on plate} &= (\text{change of momentum/sec}) \sin\theta\\ &= \frac{W}{g} \times V \times \sin\theta\\ &= \frac{waV^2}{g} \sin\theta \text{ Lb}\end{aligned}$$

EXAMPLE 1

A jet of water 2 in. in diameter impinges on a fixed plate and has a velocity of 100 ft/sec. Find the normal force on the plate (1) when the jet is normal to the plate; (2) when the jet is inclined at 60° to the plate.

Weight of water per second striking plate

$$= waV$$

$$= 62{\cdot}4 \times \frac{\pi}{4}\left(\frac{2}{12}\right)^2 \times 100$$

$$= 136 \text{ Lb}$$

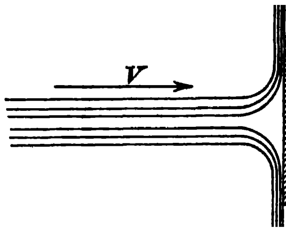


FIG. 76

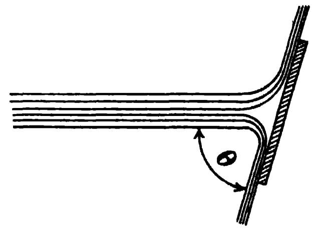


FIG. 77

(1) $$\text{Force} = \frac{WV}{g} = \frac{136 \times 100}{32{\cdot}2}$$

$$= 422{\cdot}0 \text{ Lb}$$

(2) $$\text{Normal force} = \frac{WV}{g} \sin \theta$$

$$= 422 \times 0{\cdot}866 = 366 \text{ Lb}$$

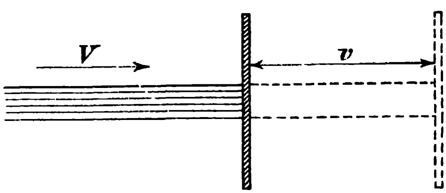


FIG. 78

6.2. Pressure on Moving Flat Plate. If a jet of fluid impinges on a plate which is moving in the same direction as the jet, the velocity with which the jet strikes the plate will be the relative velocity between the jet and the plate.

Referring to Fig. 78, let v = velocity of plate.

Weight of fluid striking plate per second

$$= W = wa(V - v)$$

$$\text{Force on plate} = \frac{W}{g}(V - v)$$

$$= \frac{wa(V - v)^2}{g}$$

This case would not be possible in practice as there would be a continually lengthening jet, the distance between the plate and nozzle increasing by v ft every second.

If, instead of a single plate, there is a continuous series of plates at a fixed distance apart and all moving in the same direction as

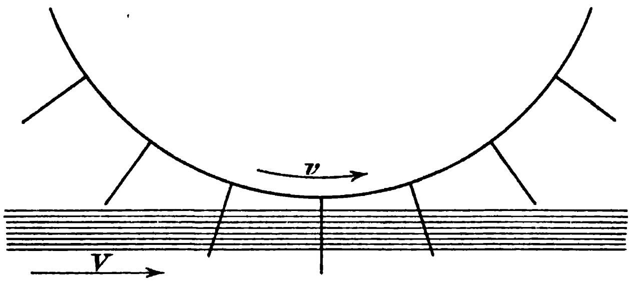


FIG. 79

the jet with a velocity v, the weight of fluid striking the plates is now equal to waV. This condition would be obtained if the plates are all fixed radially around the circumference of a large wheel on which the jet impinged tangentially (Fig. 79).

$$\text{Then} \quad \text{force on plates} = \frac{W}{g}(V - v)$$

$$= \frac{waV}{g}(V - v)$$

Work done per second on plates

$$= \frac{waV}{g}(V - v)v$$

$$\frac{(V - v)}{g}\, v \text{ per Lb of fluid}$$

$$\text{Energy supplied by jet} = \text{kinetic energy of jet per second}$$

$$= \frac{WV^2}{2g}$$

$$= \frac{V^2}{2g} \text{ per Lb of fluid}$$

$$\text{Efficiency of plates} = e = \frac{\text{work done per Lb of fluid}}{\text{kinetic energy of jet pér Lb}}$$

$$\frac{\dfrac{(V - v)v}{g}}{\dfrac{V^2}{2g}} \qquad \frac{2(V - v)v}{V^2}$$

Differentiating and equating to zero for maximum efficiency,

$$\frac{de}{dv} = V - 2v = 0$$

from which $$v = \frac{V}{2}$$

Then maximum efficiency $$= \frac{2\left(V - \frac{V}{2}\right)\frac{V}{2}}{V^2} = \frac{1}{2}$$

Flat plates used in this manner are called vanes, and a wheel of the type shown in Fig. 79 is known as an undershot water wheel.

EXAMPLE 2

A jet of water 3 in. in diameter and moving with a velocity of 40 ft/sec strikes a series of flat plates normally. If the plates are moving in the same direction as the jet with a velocity of 30 ft/sec, find the pressure on the plates, the work done per second, and the efficiency.

$$\text{Pressure on plates} = \frac{waV}{g}(V - v)$$

$$\frac{62{\cdot}4}{32{\cdot}2} \times \frac{\pi}{4} \times \left(\frac{1}{4}\right)^2 \times 40(40 - 30)$$

$$= 38 \text{ Lb}$$

$$\text{Work done per second} = 38 \times 30$$

$$= 1{,}140 \text{ ft-Lb}$$

$$\text{Efficiency} = \frac{2(V - v)v}{V^2}$$

$$= \frac{2(40 - 30)30}{1{,}600}$$

$$= 37{\cdot}5 \text{ per cent}$$

6.3. Pressure on a Fixed Curved Vane. Consider the curved fixed vane of Fig. 80, and let ab be the normal at the centre of the vane. The jet strikes the vane at an angle of α to ab and leaves at an angle of β, the vane deflecting the jet through an angle of $180 - (\alpha + \beta)$. The velocity of the jet is not changed in magnitude as it flows over the vane; it is the direction only which is changed. The velocity of the entering jet in the direction ab is $V \cos \alpha$, and it leaves the vane with a velocity component of $- V \cos \beta$ in the direction ab.
Force on vane in direction ab

$$= \text{change of momentum per second}$$

$$= \frac{W}{g} (\text{change of velocity in direction } ab)$$

$$= \frac{W}{g}[V \cos \alpha - (-V \cos \beta)]$$

$$= \frac{W}{g}(V \cos \alpha + V \cos \beta)$$

where $W = waV$.

If the vane is semicircular, the angles α and β are each equal to 0. Then,

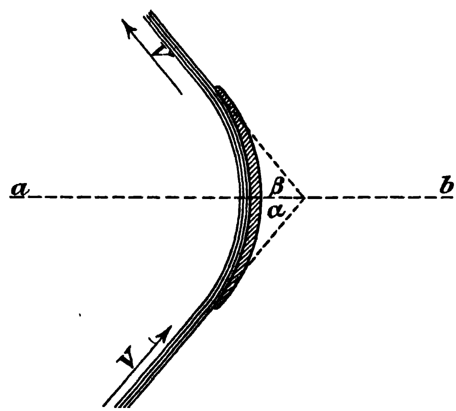


FIG. 80

force on vane in direction *ab*

$$= \frac{2W}{g} V$$

The force of a jet on a semicircular vane is thus twice as great as that on a flat plate. This is due to the fact that, with a semicircular vane, use is made of the reaction of the leaving fluid which exerts the same force on the vane in leaving as in entering. This principle is made use of in the Pelton wheel.

There will be a tangential force on the vane at right angles to *ab*. This will be equal to mass of fluid per second × change of velocity in a direction at right angles to *ab*. Or,

$$\text{tangential force} = \frac{W}{g}(V \sin \alpha - V \sin \beta)$$

6.4. Pressure on a Moving Curved Vane. Suppose the curved vane of Fig. 80 is moving in the direction *ab* with a velocity v, and let the jet impinge on the vane with a velocity V as before. The velocity of the fluid over the vane will be equal to the relative

velocity of the jet to the vane, and may be found by subtracting the vectors of V and v.

Let V_r = relative velocity between jet and vane at entrance.

Referring to Fig. 81, draw *ab* to represent the velocity of the jet at entrance in magnitude and direction. Next draw *ac* to represent the velocity of the vane in magnitude and direction. Then *cb* represents the relative velocity between the jet and the vane. If the fluid is to enter without shock, the vane at entrance must be parallel to *cb*.

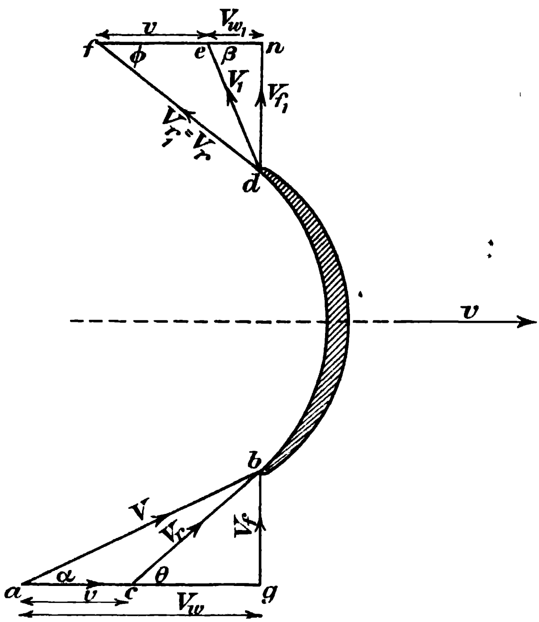


FIG. 81

The fluid will pass over the vane and leave with the velocity V_{r_1}. The absolute velocity of the leaving fluid may be found by drawing the triangle of velocities at exit.

Let V_1 = absolute velocity with which fluid leaves vane. Draw *df* to represent the relative velocity V_{r_1}; if the fluid leaves the vane without shock, V_{r_1} will be parallel to the vane at exit. Draw *fe* to represent v in magnitude and direction. Then *de* gives the absolute velocity of the leaving fluid.

The velocity of the entering fluid may be resolved into two components, one parallel to the direction of motion of the vane and known as the velocity of whirl, the other perpendicular to the direction of motion of the vane and known as the velocity of flow. The same terms are also applied to the components of the velocity of the leaving fluid.

Let V_w = velocity of whirl at entrance,
V_{w_1} = velocity of whirl at exit,
V_f = velocity of flow at entrance,
V_{f_1} = velocity of flow at exit.

These are represented in Fig. 81 by *ag*, *he*, *gb*, and *dh* respectively.

Let θ = angle between relative velocity and direction of motion at inlet,

and ϕ = angle between relative velocity and direction of motion at outlet.

Then, if the fluid is to enter and leave the vane without shock, the angles of the blade at inlet and outlet must be made equal to θ and ϕ respectively.

The force on the vane in the direction of motion is equal to the change of momentum per second of the fluid in this direction. Or,

$$\text{force on vane} = \frac{W}{g}(V_w + V_{w_1}) \quad . \quad . \quad . \quad (1)$$

where W = weight of fluid flowing over vane per second.

If the friction between the fluid and vane be neglected, the relative velocity at exit equals the relative velocity at entrance. Or,

$$V_{r_1} = V_r$$

From eq. (1),

$$\text{work done on vane per second} = \frac{W}{g}(V_w + V_{w_1})v \quad . \quad . \quad . \quad (2)$$

If V_{w_1} is in the same direction as the velocity of the vane, the equation then becomes—

$$\text{work done per second} = \frac{W}{g}(V_w - V_{w_1})v$$

The work done is also equal to the change of kinetic energy of the jet per second. Or,

$$\text{work done per second} = \frac{WV^2}{2g} - \frac{WV_1^2}{2g}$$

$$= \frac{W}{2g}(V^2 - V_1^2)$$

Then

$$\text{efficiency} = \frac{\frac{W}{2g}(V^2 - V_1^2)}{\frac{W}{2g}V^2} = \frac{(V^2 - V_1^2)}{V^2}$$

$$= 1 - \left(\frac{V_1}{V}\right)^2 \quad . \quad . \quad . \quad (3)$$

It follows from this equation that, for a given angle α, the efficiency is a maximum when V_1 is a minimum. This occurs when the angle ϕ is zero, in which case

$$V_1 = V_{w_1} = V_r - v$$

If α also equals zero,

$$V_r = V - v$$

Then $$V_1 = V - 2v$$

Therefore $$V_1 = 0 \text{ when } v = V/2$$

in which case the efficiency is unity; also the vane is semicircular.

EXAMPLE 3

A vane has a velocity of 40 ft/sec. Water impinges on the vane at an angle of 30° and leaves at an angle of 160° to the direction of motion. If the entering water has an absolute velocity of 80 ft/sec, find (1) the angles of the blade tips at inlet and outlet; (2) the work done on the vane per pound of water; and (3) the efficiency.

(1) Referring to Fig. 81, $V = 80$ ft/sec, $v = 40$ ft/sec, $\alpha = 30°$ and $\beta = 20°$.

From diagram of velocities at inlet,

$$V_w = 80 \cos 30° = 69{\cdot}3 \text{ ft/sec}$$
$$V_f = 80 \sin 30° = 40 \text{ ft/sec}$$
$$\tan\theta = \frac{V_f}{V_w - v} = \frac{40}{69{\cdot}3 - 40} = 1{\cdot}36$$

and $$\theta = 53{\cdot}7°$$

$$V_r = \frac{V_f}{\sin\theta} = \frac{40}{\sin 53{\cdot}7°} = 49{\cdot}6 \text{ ft/sec}$$

From diagram of velocities at outlet,

$$V_{r_1} = V_r = 49{\cdot}6 \text{ ft/sec}$$
$$\tan\beta = \frac{V_{r_1}\sin\phi}{V_{r_1}\cos\phi - v}$$

or $$\tan 20° = \frac{49{\cdot}6 \sin\phi}{49{\cdot}6\cos\phi - 40}$$

from which $$\tan\phi = 0{\cdot}364 - \frac{0{\cdot}294}{\cos\phi}$$

Therefore $$\phi = 4°$$

Also $$V_1 = \frac{V_r \sin 4°}{\sin 20°}$$
$$= \frac{49{\cdot}6 \times 0{\cdot}0698}{0{\cdot}342} = 10{\cdot}12 \text{ ft/sec}$$

These results might also have been obtained by drawing the velocity diagrams to scale.

(2) Work done per pound of water

$$= \frac{1}{g}(V_w + V_{w_1})v$$

$$= \frac{1}{32\cdot2}(69\cdot3 + 10\cdot12 \cos 20°)40$$

$$= 97\cdot9 \text{ ft-Lb/sec}$$

(3) $$\text{Efficiency} = \frac{\text{work done per second}}{\text{kinetic energy supplied per second}}$$

$$= \frac{97\cdot9}{\frac{V^2}{2g}} = \frac{97\cdot9 \times 64\cdot4}{(80)^2}$$

$$= 98\cdot5 \text{ per cent}$$

The work done might also have been found from the change of kinetic energy. Or,

work done per pound of water

$$= \frac{V^2}{2g} - \frac{V_1^2}{2g}$$

$$\text{Efficiency} \quad \frac{V^2 - V_1^2}{V^2}$$

6.5. Flow over a Radial Vane. Suppose the blade of Fig. 82 to be one of a series of blades fixed radially to the rim of a rotating wheel.

Let r = radius of wheel at entrance.
r_1 = radius of wheel at exit,
ω = angular velocity of wheel,
v = tangential velocity of blade tip at entrance,
v_1 = tangential velocity of blade tip at exit.

Consider 1 Lb of fluid and treat all velocities in direction of motion of wheel as positive.

Tangential momentum of fluid striking blade at entrance

$$= \frac{V_w}{g} \text{ per Lb of fluid per sec}$$

Moment of momentum at entrance

$$= \frac{V_w}{g} r \text{ per Lb of fluid per sec}$$

Tangential momentum of fluid leaving blade

$$= \frac{V_{w_1}}{g} \text{ per Lb of fluid per sec}$$

Moment of momentum at exit

$$= \frac{V_{w_1}}{g} r_1 \text{ per Lb of fluid per sec}$$

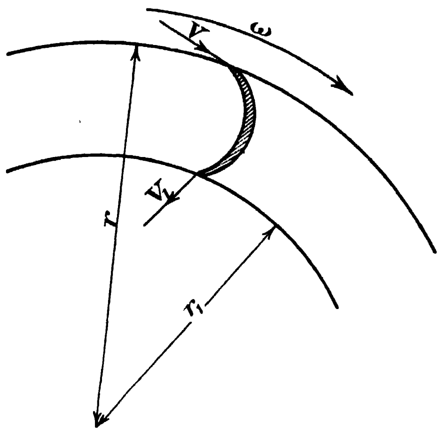


FIG. 82

Torque on wheel = change of moment of momentum per Lb per sec

$$= \frac{V_w r}{g} - \frac{V_{w_1} r_1}{g}$$

Work done by torque per Lb of fluid

$$= \left(\frac{V_w r}{g} - \frac{V_{w_1} r_1}{g}\right)\omega$$

But $\quad v = \omega r$

and $\quad v_1 = \omega r_1$

Then

work done on wheel per Lb of fluid

$$= \frac{V_w v}{g} - \frac{V_{w_1} v_1}{g} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If the fluid leaves against the direction of motion of the wheel, V_{w_1} will be negative, and eq. (4) becomes

$$\frac{V_w v}{g} + \frac{V_{w_1} v_1}{g}$$

This equation is called the momentum equation and is very important in problems dealing with the blading design of water, gas and steam turbines, centrifugal pumps and rotary air compressors

All blading designed with the use of this equation is known as momentum blading, in order to distinguish it from aerofoil blading (§ 15.5 to § 15.11). The momentum equation holds when the space between the blades is relatively narrow; then the fluid can be regarded as a curved jet flowing parallel to the face of the vane. The aerofoil conception holds when the space is relatively large; that is, when the blades are spaced a relatively large distance apart (§ 15.8)

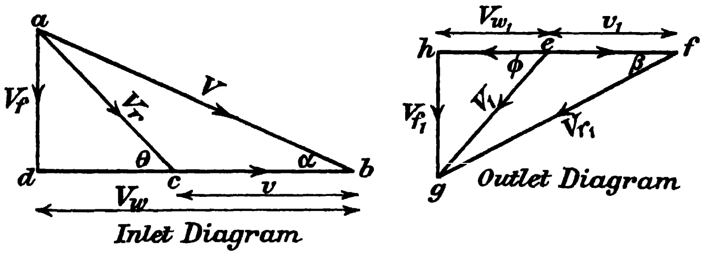


FIG. 83

EXAMPLE 4

A wheel having radial blades is 2 ft radius at the outer tip of the blades and 1 ft at the inner. Water enters the blades at the outer tip with a velocity of 100 ft/sec at an angle of 30° to the tangent, and leaves the blade with a velocity of flow of 14 ft/sec. The blade has an angle of 40° at entrance and 35° at exit. Find the work done per pound of water entering the wheel, the speed of the wheel, and the efficiency.

The diagrams of velocities are shown in Fig. 83.

Consider the diagram at inlet.

$$V = 100 \text{ ft/sec}$$
$$V_f = 100 \sin 30° = 50 \text{ ft/sec}$$
$$V_w = 100 \cos 30° = 86{\cdot}6 \text{ ft/sec}$$
$$dc = V_w - v = \frac{50}{\tan 40°} = 59{\cdot}6$$

Then $$v = 86{\cdot}6 - 59{\cdot}6 = 27 \text{ ft/sec}$$

Also $$\frac{v}{v_1} = \frac{r}{r_1} = 2$$

Therefore $$v_1 = \frac{27}{2} = 13{\cdot}5 \text{ ft/sec}$$

Consider the diagram at outlet.

$$hf = \frac{14}{\tan 35°} = 20$$
$$= v_1 + V_{w_1}$$

and $$V_{w_1} = 20 - 13{\cdot}5 = 6{\cdot}5 \text{ ft/sec}$$

and is negative, as it is against the direction of motion of the wheel.

From eq. (4),
work done per pound of water

$$= \frac{V_w v}{g} - \frac{V_{w_1} v_1}{g}$$

$$= \frac{(86{\cdot}6 \times 27) - (-\,6{\cdot}5 \times 13{\cdot}5)}{32{\cdot}2}$$

$$= 74{\cdot}8 \text{ ft-Lb}$$

$$\omega = \frac{v}{r} = \frac{27}{2} = 13{\cdot}5 \text{ rad/sec}$$

$$\text{Speed} = \frac{13{\cdot}5 \times 60}{2\pi}$$

$$= 129 \text{ r.p.m.}$$

$$\text{Efficiency} = \frac{\text{work done}}{\text{kinetic energy supplied}}$$

$$= \frac{74{\cdot}8}{\frac{V^2}{2g}} = \frac{74{\cdot}8 \times 64{\cdot}4}{100^2}$$

$$= 48{\cdot}2 \text{ per cent}$$

6.6. Jet Propulsion. This term is applied to the propulsion of bodies by the discharge of a jet of fluid astern; the method is not new and is used in nature by several types of sea organisms. It is used in surface ships and aeroplanes, the kinetic energy of the propulsion jet being produced by some type of heat engine. Its efficiency in ships is less than that obtained from the screw propeller, and for this reason it is very rarely used. With aeroplanes at low altitudes its propulsion efficiency is between 50 to 60 per cent compared with an efficiency of 88 per cent of an ordinary propeller. But, at very high altitudes, the propeller efficiency is considerably reduced and may be as low as 30 per cent, whilst that of the jet remains almost unaltered.

With surface ships the entrance orifices through which the water is drawn are usually situated amidships, so that any entering velocity of the water is lost. This reduces the theoretical propulsion efficiency. With aeroplanes the entrance orifices are in the nose to face the direction of the plane's motion. The air thus enters with a relative velocity equal to the speed of the plane; this increases the theoretical propulsion efficiency. These two methods will be dealt with separately.

Let v = velocity of craft in feet per second,

V = absolute velocity of issuing jet in feet per second,

V_r = relative velocity between issuing jet and craft $= v + V$,

w = density of fluid forming jet (Lb/ft^3),

a = area of jet in square feet,

e = propulsion efficiency of complete jet plant,

P = propulsive force of jet in pounds,

W = weight of fluid discharged per second $= waV_r$.

1. When Inlet Orifices Face Direction of Motion

Propulsive force = momentum of jet

Hence
$$P = \frac{WV}{g}$$
$$= \frac{W(V_r - v)}{g} \quad . \quad . \quad . \quad (5)$$

$$\text{Work done per second by jet} = \frac{W(V_r - v)v}{g} \quad . \quad . \quad . \quad (6)$$

$$\text{Propulsion h.p. of jet} = \frac{W(V_r - v)v}{550g} \quad . \quad . \quad . \quad (7)$$

In this case the fluid leaves the engine plant with a kinetic energy of $V_r^2/2g$ per pound, relative to the craft. But the fluid enters the engine plant with a kinetic energy of $v^2/2g$ per pound relative to the craft, due to the craft's motion and the fact that the inlet orifices face ahead. Hence,

energy supplied per pound of fluid by engines
$$= \frac{V_r^2}{2g} - \frac{v^2}{2g} \quad . \quad . \quad . \quad . \quad (8)$$

$$\text{Propulsion efficiency} = \frac{\text{work done by jet}}{\text{energy supplied by engines}}$$
$$= \frac{\dfrac{(V_r - v)v}{g}}{\dfrac{V_r^2 - v^2}{2g}}$$
$$= \frac{2v}{V_r + v} \quad . \quad . \quad . \quad . \quad (9)$$

It will be noticed from eq. (9) that the propulsion efficiency is unity when $V_r = v$, but the propulsive force is reduced to zero when this condition is reached, as $V = 0$ when $V_r = v$ (eq. (5)). Hence,

in order to obtain a reasonable propulsive horse-power, the propulsion efficiency cannot be high. This fact is the chief disadvantage of jet propulsion, because the higher the propulsion efficiency the smaller is the thrust per pound of fluid.

2. WHEN INLET ORIFICES ARE AMIDSHIPS. In this case the orifices are at right-angles to the direction of motion. The work done by jet and the propulsion horse-power are the same as in Case (1), and eqs. (6) and (7) can be used.

No gain is obtained from the motion of the craft in this case, because the orifices are at right-angles to the direction of its motion; hence, the engine plant must supply the whole of the jet's kinetic energy. Then,

energy supplied per pound of fluid by engines

$$= \frac{V_r^2}{2g}$$

$$\text{Propulsion efficiency} = \frac{\dfrac{(V_r - v)v}{g}}{\dfrac{V_r^2}{2g}}$$

Then

$$e = \frac{2(V_r - v)v}{V_r^2} \quad . \quad . \quad . \quad (10)$$

Differentiating and equating to zero for a maximum,

$$\frac{de}{dv} = V_r - 2v = 0$$

from which

$$v = \frac{V_r}{2}$$

Substituting this value in eq. (10) for maximum efficiency,

$$\text{maximum propulsive efficiency} = \frac{2(2v - v)v}{(2v)^2}$$

$$= 50 \text{ per cent}$$

The main advantage of jet propulsion with ships is that it overcomes the difficulty of the racing of the propeller in rough seas. In the late nineteenth century several naval ships were fitted with jet propulsion as an experiment. It was found that, although the actual jet was more efficient than the screw propeller, the mechanical efficiency of the pumps reduced the overall efficiency of the whole plant to a much lower value than that of the screw propeller plant. Jet propulsion is sometimes used in lifeboats.

It should be noticed that, by fitting the entrance of the suction pipe to face the direction of the craft's motion, a vacuum pressure is produced in front of the pipe entrance which causes an increased

resistance to the craft. If the suction pipe entrance is placed at a suitable position at the side of the craft, the effect of the suction may act on the boundary layer and prevent break-away (§ 16.9); this should reduce the craft's resistance.

EXAMPLE 5

A jet-propelled boat has a velocity of 12 m.p.h. when the jet has a velocity of 35 ft/sec relative to the boat. If the area of the jet is 25 in.^{2}, find the brake horse-power required to work the pumps. Assume that full advantage of the boat's motion is obtained when scooping in the water.

Weight of water discharged per second

$$= waV_r$$

$$= 62{\cdot}4 \times \frac{25}{144} \times 35$$

$$= 379 \text{ Lb}$$

$$v = \frac{12 \times 88}{60} = 17{\cdot}6 \text{ ft/sec}$$

$$\text{Work done by pumps per second} = \frac{W(V_r^2 - v^2)}{2g}$$

$$= \frac{379}{64{\cdot}4}(35^2 - 17{\cdot}6^2)$$

$$= 5{,}380 \text{ ft-Lb}$$

$$\text{Horse-power} = \frac{5{,}380}{550}$$

$$= 9{\cdot}8$$

EXERCISES 6

1. A jet of water 2 in. in diameter, having a velocity of 60 ft/sec, impinges normally on a flat plate. Find the pressure on the plate (1) when the plate is at rest; (2) when the plate is moving in the same direction as the jet with a velocity of 20 ft/sec. Find, also, the work done per second in the second case. *Ans.* (1) 152 Lb. (2) 67·5 Lb; 1,350 ft-Lb.

2. A 3-in. diameter jet, having a velocity of 80 ft/sec, strikes a flat plate, the normal of which is inclined at 30° to the jet. Find the normal pressure on the plate (1) when the plate is stationary; (2) when the plate has a velocity of 40 ft/sec away from jet. *Ans.* (1) 526 Lb. (2) 127 Lb.

3. A jet of water impinges on a series of hemispherical cups and is deflected through 180°. If the velocity of the jet is 100 ft/sec, and that of the cups 40 ft/sec, find the work done per pound of water striking the cups. *Ans.* 149 ft-Lb.

4. A jet of water having a velocity of 100 ft/sec impinges on a series of vanes moving with a velocity of 50 ft/sec. The jet makes an angle of 30° to the direction of motion of the vanes when entering, and leaves at an angle of 120°.

Draw the triangle of velocities for inlet and outlet and find (1) the angles of the vane tips so that the water enters and leaves without shock; (2) the work done per pound of water entering the vanes; and (3) the efficiency.

Ans. (1) 53°, $15\frac{1}{2}$°. (2) 149 ft-Lb. (3) 96 per cent.

5. Water flows inwards over a series of curved vanes which are fixed to the rim of a revolving wheel. The outer diameter of the vanes is 4 ft and the inner diameter 2 ft. The angle between the jet and the wheel tangent at inlet is 30°, and the water leaves the wheel with a velocity of 10 ft/sec at an angle of 120° to wheel tangent. Draw the velocity triangles at inlet and outlet, and find the best angles of the blades and the work done per pound of water if the jet has a velocity of 120 ft/sec and the wheel makes 300 r.p.m.

Ans. 55°; 14°; 208 ft-Lb.

6. A vessel provided with a jet propeller is driven at a speed of v ft/sec. The water is discharged astern with a relative exit velocity of V ft/sec and the total jet area is a ft^2. Find in terms of these quantities (a) the propelling force on the vessel; (b) the energy expended by the jet in propulsion; (c) the efficiency of the jet. State what conclusion can be drawn from these results, and why a vessel can be more efficiently driven by means of a screw propeller. In a jet-propelled vessel the water is discharged through two 9-in. orifices. The jet efficiency is 73 per cent, and the combined efficiency of the engine and pumps 45 per cent. Find the indicated horse-power required to drive the vessel at 13 knots. [Weight of sea water, 64 Lb/ft^3; 1 knot = 1·69 ft/sec.] (*Lond. Univ.*) *Ans.* 133·2 h.p.

7. A 42-in. pipe is deflected through 90°, the ends being anchored by tie rods at right angles to the pipe at the ends of the bend. If the pipe is delivering 63 ft^3/sec, find the tension in each tie rod. (*Lond. Univ.*) *Ans.* 796 Lb.

8. A jet of water having a velocity of 50 ft/sec, and making an angle of 45° with the horizontal, impinges on a vane moving horizontally with a velocity of 25 ft/sec. Find the shape of vane to give the best results and the angles at the entering and leaving tips.

Find the horizontal pressure on the vane per pound of water striking per second. (*Lond. Univ.*) *Ans.* 73·6°; 0; 1·47 Lb.

9. A square plate weighing 28 Lb, and of uniform thickness and 12 in. edge, is hung so that it can swing freely about the upper horizontal edge. A horizontal jet $\frac{3}{4}$ in. in diameter and having a velocity of 50 ft/sec impinges on the plate. The centre-line of the jet is 6 in. below the upper edge of the plate, and when the plate is vertical the jet strikes the plate normally and at its centre. Find what force must be applied at the lower edge of the plate in order to keep the plate vertical.

If the plate is allowed to swing freely, find the inclination to the vertical which the plate will assume under the action of the jet. (*Lond. Univ.*)

Ans. 7·425 Lb; 32°.

10. A motor-boat with jet propulsion draws 10 ft^3/sec through orifices amidships and discharges it astern through orifices having an effective area of 0·5 ft.^{2} If the boat travels at 10 m.p.h., find the propelling force. (*A.M.I. Mech. E.*) *Ans.* 103 Lb.

11. A circular jet of water delivers 2 ft^3/sec with a velocity of 80 ft/sec, and impinges tangentially on a vane moving in the direction of the jet with a velocity of 40 ft/sec. The vane is so shaped that, if stationary, it would deflect the jet through an angle of 45°. Though what angle will it deflect the jet? What driving force will be exerted on the vane in its direction of motion? (*A.M.I Mech. E.*) *Ans.* 22·5°; 45·5 Lb.

12. A tank, from which water is discharging under a constant head H, is mounted on frictionless wheels, so that the direction of motion is opposite to that of the jet, which issues from an orifice a ft^2 in area in one side. What force in pounds applied horizontally would just prevent movement of the tank? If the tank moved with velocity v, and the jet issued with velocity V, what would then be the force causing motion, the work done per second, and the efficiency? For maximum efficiency, what will be the ratio of V to v? (*A.M.I.C.E.*)

13. A free jet, whose sectional area is 3 in.^{2}, and whose velocity is 80 ft/sec, impinges tangentially on a smooth vane which diverts its direction through 120°. What is the magnitude and direction of the resultant force on the vane? (*A.M.I. Mech. E.*) *Ans.* 448 Lb at 30°.

14. A vessel is propelled by the reaction of jets discharged astern, the water being drawn in initially at the side. Establish expressions for the theoretical efficiency and for the input of power to the pumps in terms of the speed of the ship, v; the velocity through the jets, V; the weight of water pumped per second, W, and the combined efficiency of the pump and pipe system, e.

In an actual case a small ship is fitted with jets of total outlet area 7 ft.^{2} The velocity through the jets is 30 ft/sec and the ship speed is 10 knots. The engine efficiency is 85 per cent, the pump efficiency is 65 per cent, and the pipe losses may be taken as equivalent to 10 per cent of the kinetic energy at the jets. Determine the propelling force and the overall efficiency. (1 knot = 6,080 ft/hr; sea water, 64 Lb/ft.^{3} (*Lond. Univ.*)

Ans. $\dfrac{2(V-v)v}{V^2}$; $\dfrac{WV^2}{2g \times 550e}$; 5,480 Lb; 24·43 per cent.

15. A lifeboat is propelled by jet propulsion at a speed of 15 m.p.h. It has two jets, each of 1 ft^2 cross-sectional area; the intake by the pumps is 80 ft^3/sec and the inlet orifices are situated amidships. If the efficiency of the pumps is 75 per cent, find: (1) the propelling force of the jets, (2) the required b.h.p. of the engine driving the pumps, and (3) the propulsion efficiency of the jets. $W = 64$ Lb/ft^3 for sea water. Assume influx velocity of water is lost by shock and friction.

What is the propulsion efficiency if all the influx velocity is recovered?

Ans. (1) 2,860 Lb; (2) 308 b.h.p.; (3) 49·5 per cent; 71 per cent.

16. A jet-propelled aircraft, having a speed of 500 m.p.h., has a propulsion jet area of 1·46 ft.^{2} The weight of air discharged is 43 Lb/sec and the velocity of the jet relative to the aircraft is 1,400 ft/sec. Calculate, under these conditions, the propulsion force, the propulsion efficiency of the jet and the thrust horse-power developed by the jet, if the intake of the air is at the same speed as the aircraft.

Ans. Propulsion force = 890 Lb. Eff. = 68·7 per cent; thrust h.p. = 1,185.

17. A jet-propelled aeroplane has a speed of 600 m.p.h. and a propulsion efficiency of 60 per cent. If the total resistance of the plane at this speed is 2,500 Lb, calculate (i) the weight of gases discharged per second; (ii) the necessary diameter of the jet orifice at outlet in feet, if the pressure and temperature at discharge are 10 Lb/in.^{2} and 600°F respectively; the pressure of the atmosphere at the altitude of the flight is 10 Lb/in.^{2} $pV = 53T$. (*Lond. Univ.*) *Ans.* (i) 68·6 Lb/sec; (ii) 1·29 ft.

CHAPTER 7

FRICTION AND FLOW THROUGH PIPES

7.1. Fluid Friction. When a fluid flows past a solid surface, or when a solid body moves through a fluid, the frictional resistance is due to a viscous drag between the streambands of the fluid. The streamband of fluid adjacent to the solid surface is always at rest relative to the wetted surface. The viscous drag is due to the molecular attraction between the molecules of the fluid.

For laminar flow (§ 3.16), when the Reynolds number is small (§ 3.17), the frictional coefficient is proportional to $R_e^{1/2}$. At a certain velocity the flow will develop transverse eddies and thus be converted into turbulent flow; this velocity is known as the *critical* velocity and occurs at a certain value of the Reynolds number, the value depending on the shape of the solid surface and on its roughness.

When turbulent flow has developed, it is found that the coefficient of friction is still a function of the Reynolds number, but the value of this function now varies between $R_e^{1/5}$ and $R_e^{1/7}$. The former value is termed the 1/5 power law and holds for low velocities of turbulent flow. The latter value is the 1/7 power law and applies to high velocities of turbulent flow. For intermediate velocities the index of the Reynolds number function lies between 1/5 and 1/7 (§ 9.1).

It is found that the total frictional resistance to fluid flow depends on the following—

1. The area of the wetted surface.
2. The density of the fluid.
3. The surface roughness.
4. It is independent of the fluid pressure.
5. It increases with the square of the velocity.
6. It depends on a friction coefficient which is a function of R_e.

The critical velocity occurs when the Reynolds number reaches a certain value, the value depending on the type of flow problem. The value is independent of the type of fluid and holds for all gases, vapours and liquids.

7.2. Froude's Experiments. The frictional resistances of surfaces moving in water were investigated by Froude.* An experimental tank, about 300 ft long, containing water was used. Thin wooden boards were towed endwise in this tank by connecting them to a

* *British Association Reports,* 1872–1874.

carriage running on rails at the side. The carriage was hauled along at various speeds by means of a wire rope passing around a drum, the force required to tow the boards being measured. Boards of lengths varying from 2 ft to 50 ft were used, their surfaces being covered with varnish, tinfoil, calico and sand, in turn.

From the results of these experiments Froude concluded—

1. The frictional resistance varies approximately with the square of the velocity.
2. The frictional resistance varies with the nature of the surface.
3. The frictional resistance per square foot of surface decreases as the length of the board increases, but is constant for long lengths.

It will be noticed that Froude's conclusions are in agreement with the more recent knowledge given in § 7.1.

Let f' = frictional resistance per square foot of a given surface at unit velocity,

$A \doteq$ area of wetted surface in square feet,

V = velocity of surface in feet per second.

Then, total frictional resistance $= f'AV^n$

Assuming the index $n = 2$,

$$\text{total frictional resistance} = f'AV^2 \quad . \quad . \quad . \quad (1)$$

It will be proved later that Froude's frictional coefficient f' is a function of the Reynolds number (§ 9.1).

7.3. Resistance of Ships. The resistance of a ship to motion is due to the frictional resistance of its wetted surface and to head resistance. The latter will depend on the shape of ship and can be reduced by making the immersed portion of the ship a streamline form. The energy utilized in overcoming the head resistance is wasted in the formation of surface waves, known as the wash. The best form of ship can be determined only by experiment, and it is usual, before building a large ship, to make a small model of the same proportions and to measure its resistance in an experimental tank. By so doing, the head or wave resistance of the proposed ship may be calculated from that of the model.*

Total resistance of ship = frictional resistance + wave resistance

Let R_f = frictional resistance of ship,

R_w = wave resistance of ship,

and R = total resistance of ship.

Then $$R = R_f + R_w$$

* For a fuller account of the resistance of ships see Sir William White's *Naval Architecture.*

Let f_s = frictional resistance per square foot of wetted surface area of ship at unit velocity,

A_s = area of wetted surface of ship,

and V_s = speed of ship.

From eq. (1),

$$R_f = f_s A_s V_s^2$$

Then

$$R_w = R - f_s A_s V_s^2$$

It is known from experiments with ships that the surface wave resistance of a ship is in proportion to the square of the speed and to the wetted surface area, and it may be calculated from the measured wave resistance of the model.*

Let f_m = frictional resistance per square foot of wetted surface area of model at unit velocity,

A_m = area of wetted surface of model,

V_m = speed of model,

r = total resistance of model,

r_f = frictional resistance of model,

r_w = wave resistance of model.

Then

$$r_w = r - r_f$$
$$= r - f_m A_m V_m^2$$

The model is made the same form as the ship and, therefore, represents the ship to a given scale. Let n be the ratio between the linear dimensions of the ship and model.

Then

$$A_s = n^2 A_m$$

The speed of the model should be $\frac{1}{\sqrt{n}} V_s$, in order to compare with that of the ship. This is known as the corresponding speed.† Or,

$$V_s = \sqrt{n}\, V_m$$

As the wave resistance is in proportion to the area and to the velocity squared,

$$\frac{R - R_f}{r - r_f} = \left(\frac{V_s}{V_m}\right)^2 \frac{A_s}{A_m} = \frac{n V_m^2 n^2 A_m}{V_m^2 A_m}$$
$$= n^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But

$$\frac{R_f}{r_f} = \frac{f_s A_s V_s^2}{f_m A_m V_m^2} = \frac{f_s n^2 A_m n V_m^2}{f_m A_m V_m^2}$$
$$= \frac{f_s}{f_m} n^3$$

* See also § 12.1 to § 12.3.

† For explanation of this see § 12.1.

Substituting in eq. (2),

$$R - r_f \frac{f_s}{f_m} n^3 = n^3(r - r_f)$$

Then

$$R = n^3 \left[r + r_f \left(\frac{f_s}{f_m} - 1 \right) \right] \quad . \quad . \quad (3)$$

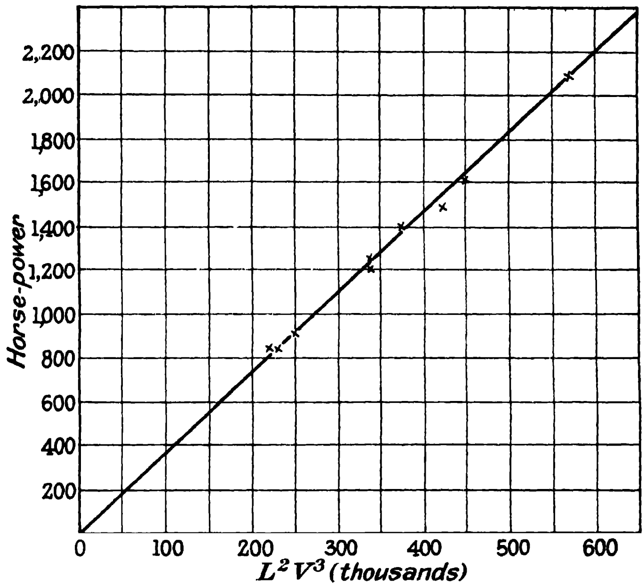


FIG. 84

If the surface of the model is the same as that of the ship, $f_s = f_m$. Eq. (3) then becomes

$$R = n^3 r$$

With ships of similar form and of the same surface, and using the notation of § 7.2,

$$\text{total resistance} = \text{frictional resistance} + \text{wave resistance}$$
$$= f'L^2V^2 + cL^2V^2$$

where c is a coefficient depending on the form and L is the linear dimension.

Then $\quad$ total resistance $= (f' + c)L^2V^2$

$$\text{Horse-power} = \frac{\text{resistance} \times V}{550}$$

where resistance is in pounds and velocity in feet per second.

Then $$\text{horse-power} = \frac{(f' + c)}{550} L^2 V^3$$
$$= kL^2V^3$$

where k is a constant for the type of ship considered.

The same laws will hold for any other fluid; this will be seen from the curve in Fig. 84. This curve was plotted from the maximum speed and maximum horse-power of nine rigid airships, covered with the same material and of almost similar form. There was a considerable variation in their sizes. As the horse-power equals kL^2V^3, a straight line passing through the origin should be obtained if the horse-power is plotted against L^2V^3. The slope of this line gives the constant k. This has been done in Fig. 84; it will be noticed that the points lie approximately on a straight line, thus proving the above laws to hold.

EXAMPLE I

Show how the total resistance and the power required for propulsion of a ship can be deduced from experiments on a scale model. Determine the indicated horse-power to drive a ship 300 ft long, having 13,500 ft² wetted surface, at 20 knots, if the resistance of the model, one-sixteenth the size of the ship, is 20 Lb at the corresponding speed. Take f for the ship as 0·0091, and for the model 0·0094, and assume that 60 per cent of the i.h.p. is available for propulsion. [1 knot = 1·69 ft/sec.] (*Lond. Univ.*)

$$\text{Area of wetted surface of model} = A_m = \frac{A_s}{n^2} = \frac{13{,}500}{(16)^2}$$

$$\text{Corresponding speed of model} = \frac{V_s}{\sqrt{n}} = \frac{20}{4}\ \text{knots}$$

Assuming the units of f' are for velocities in knots,

$$\text{frictional resistance of model} = r_f = f_m A_m V_m^2$$
$$= 0{\cdot}0094 \times \frac{13{,}500}{256} \times \left(\frac{20}{4}\right)^2$$
$$= 12{\cdot}4\ \text{Lb}$$

Then $$r_w = 20 - 12{\cdot}4 = 7{\cdot}6\ \text{Lb}$$

Using eq. (3),

$$\text{total resistance of ship} = n^3\left[r + r_f\left(\frac{f_s}{f_m} - 1\right)\right]$$
$$= 16^3\left[20 + 12{\cdot}4\left(\frac{0{\cdot}0091}{1{\cdot}0094} - 1\right)\right]$$
$$= 80{,}100\ \text{Lb}$$

$$\text{Horse-power} = \frac{RV}{550} \times \frac{100}{60}$$
$$= \frac{80{,}100 \times 20 \times 1{\cdot}69 \times 100}{550 \times 60}$$
$$= 8{,}200$$

7.4. Friction of Revolving Disc. Froude's experiments gave the true coefficient of friction only when very long boards were used. Professor Unwin overcame this difficulty by revolving a disc at a known speed in the liquid and obtained the coefficient of friction of the disc's surface by measuring the work done.

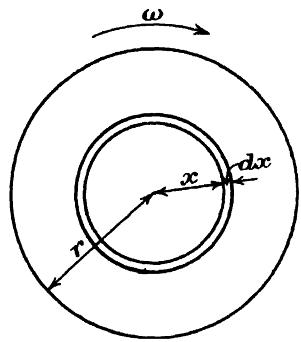


FIG. 85

Consider the disc in Fig. 85.

Let ω = angular velocity of disc in radians per second,

r = radius of disc,

and μ = coefficient of friction at unit velocity.*

Then

frictional force $= \mu \times \text{area} \times (\text{velocity})^2$

Consider a thin ring of the disc of a radius x and let thickness of ring be dx.

Area of ring (both sides) $= 4\pi x \, dx$

Tangential velocity of ring $= \omega x$

Frictional resistance of ring $= \mu \times 4\pi x \, dx \times \omega^2 x^2$

Moment of resistance about centre

$$= 4\pi\mu\omega^2 x^3 \, dx \times x$$

$$\text{Total moment of disc} = 4\pi\mu\omega^2 \int_0^r x^4 \, dx$$

$$= \frac{4}{5} \pi\mu\omega^2 r^5$$

$$\text{Work done per second} = \text{moment} \times \text{angle turned through}$$

$$= \frac{4}{5} \pi\mu\omega^3 r^5$$

If the frictional resistance is assumed to vary with $(\text{velocity})^n$, this expression becomes

$$\text{work done per second} = \frac{4\pi\mu\omega^{n+1} r^{n+3}}{n+3}$$

7.5. Friction in Pipes—Hydraulic Gradient. Fluids flowing through pipes are subjected to a frictional resistance depending on the velocity, the area of the wetted surface, and the nature of the surface. In long pipes the frictional resistance is so large that all other resistances are rendered insignificant in comparison, and the

* Actually μ will vary with the temperature and velocity; see § 3.16 on Viscous Flow.

total energy of the fluid is absorbed in overcoming it. The energy lost in overcoming the frictional resistance is expressed in feet of fluid and is known as the head lost in friction.

Consider water to flow through a long pipe from a reservoir A into a reservoir B (Fig. 86). Take the line xx through the water level in B as the datum line. Let H be the height of water in A above datum, and let l be the length of the pipe. If p is the intensity

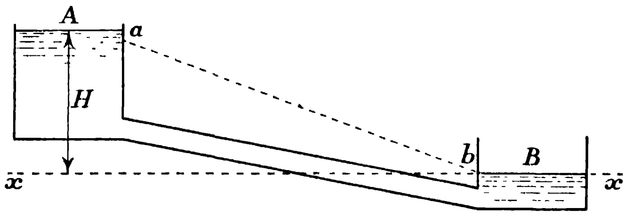


FIG. 86

of pressure of the water at any section of the pipe, the pressure head at that section will be p/w. Supposing the pressure heads of the water at all sections of the pipe are plotted as vertical ordinates, using the centre-line of the pipe as a base-line, a straight sloping line ab will be obtained. This line falls off uniformly from A, as there is a uniform loss of head due to friction as the water flows along the pipe. This line is called the hydraulic gradient, and its slope is equal to the total loss of head divided by length of pipe.

In practice the slope is small, so that either the sine or the tangent of the angle may be used as the slope of the hydraulic gradient.

Let i = slope of pressure head line ab,

and h = total head lost.

Then, slope of hydraulic gradient $= i = h/l$

It will be noticed that, if the pipe is uniform and if the whole of the available head is lost in friction, the slope of the hydraulic gradient will be the difference of level of water surfaces divided by length of pipe.

Next consider the pipe line shown in Fig. 87. A and B are two reservoirs separated by a hill; a uniform pipe is laid over the hill so that water from A may flow into B. Fig. 87 is drawn to greatly enlarged vertical scale; actually, the length of the pipe may be taken as the length of its horizontal projection. As the pipe is long, the loss of head due to friction will be very large and all other losses may be neglected; hence, taking the water level at B as datum, the water will lose head at a uniform rate from the water level in A to the water level in B. From this it follows that the hydraulic gradient will be a straight line joining the water surface in A and B.

The pressure head at any section of the pipe is represented by the vertical distance between the hydraulic gradient and the pipe centre-line at that section. If the hydraulic gradient is above the centre-line of pipe the pressure is above atmospheric; if below the centre-line of pipe the pressure is below atmospheric.

It will be seen from Fig. 87 that at C and D the water pressure is atmospheric, whilst between C and D it is less than atmospheric.

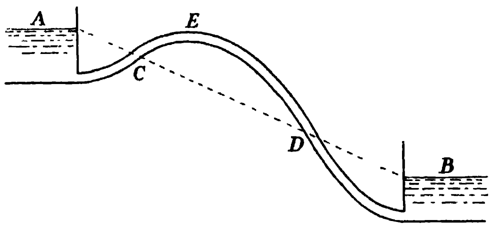


FIG. 87

The highest point of the pipe above the hydraulic gradient is E; at this point the water pressure is least. If the absolute pressure at E is less than 8 ft of water, or 26 ft vacuum, separation will occur, for at this pressure the water commences to vaporize and the dissolved gases are also released; large cavities of gas will occur causing the flow to cease. It follows from this that engineers must lay their pipe lines so that no section of the pipe will be more than 26 ft above the hydraulic gradient at that section.

A pipe which rises above its hydraulic gradient is known as a syphon. It will be noticed from Fig. 87 that a pipe may be above the hydraulic gradient and yet be below the water surface at A; such a pipe would still be a syphon.

Consider next the short pipe line shown in Fig. 88. Let the water flow from A to B along a pipe of varying section *abcd*. At any section of the pipe the total head of the water will be the datum head + the velocity head + the pressure head. Choose any horizontal line as the datum line, and, starting from the water level in A, mark off the losses of head in the pipe from all sources, to the same vertical scale as the figure. The line thus obtained is the total-energy line, and is shown dotted. The height of this line above the datum line, at any section, will give the total energy of the water at that section.

Let v_1 = velocity of flow in *ab*,

v_2 = velocity of flow in *bc*,

v_3 = velocity of flow in *cd*.

The following are the losses to be taken into account—

At a, a loss due to entrance to pipe $= 0{\cdot}5\,\dfrac{v_1^2}{2g}$

Between a and b a uniform loss due to friction

$$= \frac{4flv_1^2}{2gd} \qquad (\S\ 7.6)$$

At b, a loss due to sudden contraction $= 0{\cdot}5\ \dfrac{v_2^2}{2g}$

Between b and c, a uniform loss due to friction.

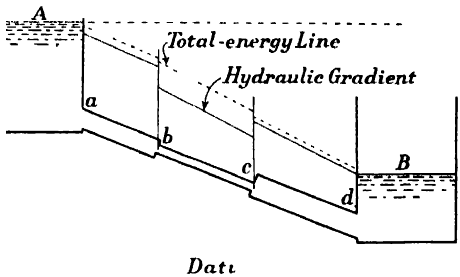


FIG. 88

At c, a loss due to sudden enlargement $= \dfrac{(v_2 - v_3)^2}{2g}$

Between c and d, a uniform loss due to friction.

At d, a loss due to velocity head being destroyed

$$= \frac{v_3^2}{2g}$$

The sum of all these losses will equal the difference of level between the water surfaces in A and B.

As the height of the hydraulic gradient above the centre-line of the pipe represents the pressure head of the water, it follows that if the velocity head is deducted from the total-energy line the hydraulic gradient will be obtained. For,

pressure head above centre-line of pipe =

total energy above datum − velocity head

$$\text{Velocity head between } a \text{ and } b = \frac{v_1^2}{2g}$$

$$\text{Velocity head between } b \text{ and } c = \frac{v_2^2}{2g}$$

$$\text{Velocity head between } c \text{ and } d = \frac{v_3^2}{2g}$$

These amounts have been subtracted from the total-energy line of Fig. 88, and the full line representing the hydraulic gradient is obtained.

It will be noticed that, if the diameter of the pipe bc is much smaller than that of cd, the velocity v_2 will be large compared with v_3; hence the hydraulic gradient of the pipe cd may be higher than that of bc.

7.6. Loss of Head due to Friction in Pipes. A rational formula for the loss of head in a pipe due to friction may be obtained by assuming Froude's experimental results of fluid friction to hold.

Consider liquid flowing along a uniform horizontal pipe, of cross-sectional area A, with a velocity v. Let l be the length of the pipe and let the intensity of pressure be reduced by the frictional resistance from p_1 to p_2 over the length l.

Let f' = frictional resistance per unit area at unit velocity,
P = wetted perimeter of pipe.

Resolving horizontally,

$$p_1 A = p_2 A + \text{frictional resistance}$$

But $$\text{frictional resistance} = f' \times \text{area} \times v^n$$
$$= f' P l v^n$$

Therefore $$(p_1 - p_2)A = f' P l v^n$$

Dividing through by the density of the liquid w,

$$\frac{p_1}{w} - \frac{p_2}{w} = f' \frac{Plv^n}{Aw}$$

Let h_f = head lost due to friction

$$= \frac{p_1}{w} - \frac{p_2}{w}$$

Then $$h_f = f' \frac{Plv^n}{Aw}$$

The ratio A/P is called the hydraulic mean depth and is represented by m.

Then $$h_f = \frac{f' l v^n}{mw} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Assuming $n = 2$, and as $i = h_f/l$,

$$i = \frac{f' v^2}{mw}$$

Or $$v = C\sqrt{mi} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $C = \sqrt{(w/f')}$.

This form is known as the Chezy formula, the constant C being found experimentally. It will be shown later that C is a function of R_e.

A more convenient form of this formula is obtained by expressing the head lost in terms of the velocity head.

For a pipe flowing full,

$$m = \frac{A}{P} = \frac{\frac{\pi}{4}d^2}{\pi d} = \frac{d}{4}$$

Substituting this in eq. (4), and assuming $n = 2$,

$$h_f = \frac{4f'}{wd}lv^2$$

Putting $$f' = \frac{fw}{2g}$$

where f is a constant found experimentally

$$h_f = \frac{4flv^2}{d2g} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This form of the Chezy formula is due to Darcy and is known as the Darcy formula; the coefficient f is known as Darcy's frictional coefficient.

This modified form of the Chezy formula is usually used for pipes running full, as it is convenient to have the frictional head lost in terms of the velocity head.

Darcy found that the coefficient f of eq. (6) varied with the surface of the pipe and with the diameter, and gives the following formula for f—

For new pipes,

$$f = 0{\cdot}005\left(1 + \frac{1}{12d}\right)$$

For old pipes,

$$f = 0{\cdot}01\left(1 + \frac{1}{12d}\right)$$

where d is the diameter of the pipe.

These equations give approximate values only which hold over a very limited range of turbulent flow. It will be shown in § 8.2 that f is a function of R_e; hence it varies with the temperature, velocity, the surface roughness, and diameter of the pipe.

In using eq. (6), care should be taken that all the dimensions are in feet and seconds units.

Although eq. (6) is used by engineers for calculations on pipe flow, the results obtained can be only very approximate. As f

varies greatly with the temperature it follows that there will be a large variation in the flow during the year, a greater flow being obtained in summer.* In this country the summer flow may increase by 18 per cent of the winter flow.

EXAMPLE 2

Water flows through a pipe, 8 in. in diameter, 150 ft long, with a velocity of 8 ft/sec. Find the head lost in friction—(*a*) using the formula

$$h_f = \frac{4flv^2}{d \times 2g}$$

assuming f to be 0·0056; (*b*) using the formula $v = C\sqrt{mi}$, assuming $C = 106$.

(*a*)
$$h_f = \frac{4 \times 0{\cdot}0056 \times 150 \times 8^2}{\frac{8}{12} \times 64{\cdot}4}$$

$$= 5{\cdot}0 \text{ ft of water}$$

(*b*)
$$m = \frac{d}{4} \text{ for a circular pipe running full}$$

$$i = \frac{h_f}{l}$$

Then
$$v = 106\sqrt{\frac{d}{4} \times \frac{h_f}{150}}$$

Squaring both sides,

$$8^2 = 11{,}200 \times \frac{8}{12 \times 4} \times \frac{h_f}{150}$$

Therefore
$$h_f = \frac{12 \times 4 \times 64 \times 150}{8 \times 11{,}200}$$

$$= 5{\cdot}15 \text{ ft of water.}$$

7.7. Reynolds' Experiments on Flow through Pipes. Reynolds† obtained the loss of head in a pipe by measuring the fall of pressure over a known length of the pipe; from this i, the slope of the hydraulic gradient, was obtained. For,

$$i = \frac{h_f}{l}$$

Reynolds' apparatus is shown in Fig. 89.

* For results of experiments on pipe flow made by Darcy and others, see Barnes' *Hydraulic Flow Reviewed*.

† For complete account of Reynolds' experiments, see *Phil. Trans.* (1883).

The velocity of water in the pipe was obtained by measuring the discharge over a known time; then

$$v = \frac{\text{discharge per second}}{\text{area of cross-section of pipe}}$$

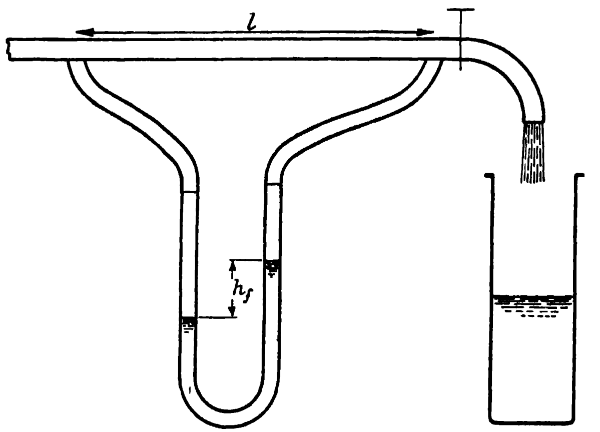


Fig. 89

This was repeated for several velocities, commencing with extremely small values, and the results were then plotted as shown in Fig. 90, the base of the graph representing v and the ordinate representing i. The graph obtained was found to be a straight line up to a certain velocity; beyond this velocity the graph was curved.

The graph is evidently following a law of the type

$$i = kv^n$$

where k and n are constants. For the straight-line portion of the graph, n equals unity. The value of n for the curved portion of the graph can be found by plotting $\log i$ and $\log v$.

For $\qquad i = kv^n$

Then $\qquad \log i = \log k + n \log v \qquad (7)$

When $v = 1$, $\log v = 0$; then $\log i = \log k$, from which the value of k can be found.

Also, from eq. (7),

$$n = \frac{\log i - \log k}{\log v}$$

These logs are shown plotted in Fig. 91, and show that two distinct types of flow occur. For the first portion of the graph the straight line AB was obtained; the remaining portion of the graph gave the straight line CD. The line BC, which joins the other two lines, follows no defined law and is due to the changing from one type of flow to the other.

It follows from this graph that the flow of the water consists of two types—

1. A steady or viscous flow up to the point B, known as laminar flow. Between A and B it is found that $n = 1$.

2. An unsteady or eddy flow for the higher velocities beyond B. This is known as a turbulent flow.

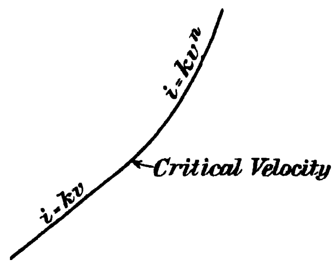


FIG. 90

The point B, at which point the change from laminar to turbulent flow takes place, is known as the critical velocity.

Thus, the line AB represents laminar flow and line CD turbulent flow. The line BC is a transition curve during which the flow changes from laminar to turbulent; the position of this line varies with the conditions at the entrance to the pipe and may move to the left or right if this condition is altered, but the slope of BC does not appreciably change. Reynolds concluded that the path EBC was due to the inertia of the water in changing from steady flow to turbulent flow and that the point E is the true critical velocity. The point E is known as the lower critical velocity, and is assumed to be the true critical velocity.

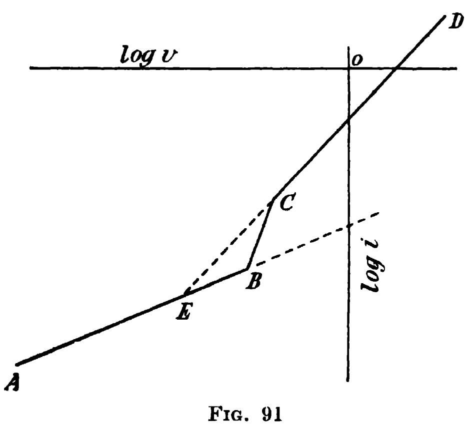


FIG. 91

Reynolds repeated these experiments with pipes of different diameters and with water at different temperatures, and then repeated the experiments with other fluids, including gases. From these results he found that the value of the critical velocity varies inversely with the diameter of the pipe and inversely with the temperature of the water.

These results hold for all other fluids; the value of the critical velocity of any fluid will also depend on the density of the fluid and on its viscosity.

Reynolds found from his experimental results that, for the lower critical velocity, represented by point E of Fig. 91,

$$\frac{\rho v d}{\eta} = 2{,}000$$

For the higher critical velocity represented by point C,

$$\frac{\rho v d}{\eta} = 2{,}500$$

Between these two values is the unstable transition state between laminar and turbulent flow.*

The values of R_e of 2,000 to 2,500 hold for all problems on fluid flow through round pipes and apply to all fluids. These values do not apply to any other sectioned pipe or to other types of surface. The value of R_e at the critical velocity for fluid flow past flat plates is 300,000 for the lower critical velocity and 500,000 for the higher (§ 9.5).

In all civil engineering problems on flow of water it is found that the velocities used are all above the critical velocity; and in all large pipes, such as used in practice, the suffix n approximates to 2 which agrees with the practical friction formula given in § 7.6.

EXAMPLE 3

An experiment was carried out on an 8 in. diameter wrought iron pipe over a length of 8 ft. The velocity of flow through the pipe was varied and the loss of head for each velocity was measured. The following values of i were obtained—

v (ft/sec) . .	4·7	6·5	8·72	10 6	12·8	14·6
i . . .	0·0134	0·0250	0·0425	0·0629	0·0975	0·1171

Find the values of k and n in the formula $i = kv^n$.

First plot $\log i$ and $\log v$.

$\log v$ . . .	0·672	0·813	0·941	1·025	1·107	1·164
$\log i$ . . .	− 1·873	− 1·602	− 1·371	− 1·201	− 1·013	− 0·931

These are shown plotted in Fig. 92.

When $\log v = 0$, $\log k = \log i$

$$= -3{\cdot}19$$

$$= \bar{4}{\cdot}81$$

* The ratio $\rho v d/\eta$ was first derived analytically by Poiseuille in 1840 (§ 8.2). Reynolds, in the publication of his own results in 1883, does not state how he derived this ratio.

Therefore $k = 0{\cdot}000645$

From eq. (7), $$n = \frac{\log i - \log k}{\log v}$$

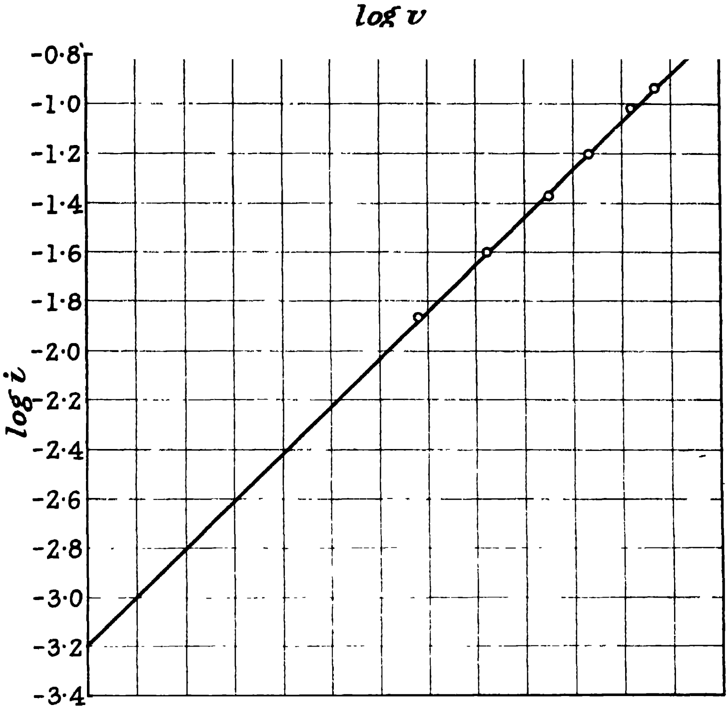


FIG. 92

$$= \frac{-2{\cdot}6 + 3{\cdot}19}{0{\cdot}302}$$

$$= 1{\cdot}955$$

Then $i = 0{\cdot}000645v^{1{\cdot}955}$

7.8. Determination of Critical Velocity. Besides the method given in § 7.7 there are two other methods of obtaining the critical velocity of water.

1. COLOUR BANDS (REYNOLDS' METHOD). The critical velocity may be determined by allowing water to flow through a glass tube and injecting a thin stream of coloured liquid into the centre of the stream (Fig. 93). As long as the velocity in the glass tube is below the critical velocity, the colour band will remain a thin straight

line flowing along the centre of the stream. But for velocities above the critical velocity, the coloured band is broken up by eddies and mixes with the water, as in Fig. 94.

2. CHANGE OF TEMPERATURE. Barnes and Coker* determined the critical velocity by measuring the temperature of the stream for

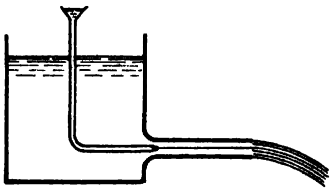


FIG. 93

FIG. 94

various velocities. As the frictional resistance below the critical velocity is proportional to v and, above the critical velocity, to v^n, it follows that more heat will be generated above the critical velocity. If the temperature of the water is plotted on a base representing the velocity, the curve will become much steeper beyond the critical

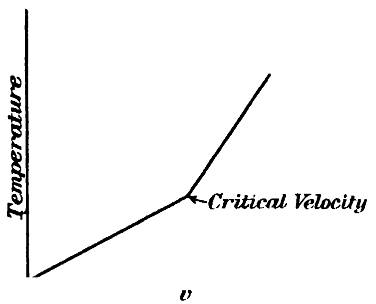


FIG. 95

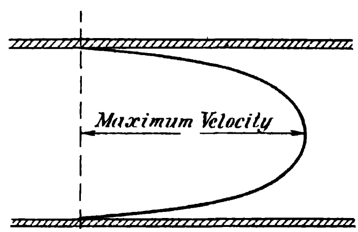


FIG. 96

velocity, as shown in Fig. 95. The critical velocity will be represented by the kink in the curve.

7.9. Distribution of Velocity in a Pipe. The velocity of a fluid flowing along a pipe having turbulent flow will vary at different points of the cross-section, its magnitude depending on the radius. The velocity of flow at any radius may be measured with a Pitot tube. It is found that the velocity is a maximum at the centre and a minimum at the circumference. The variation is shown in the curve of Fig. 96, the velocity being plotted horizontally on the diameter of the pipe as a base.

It is found that the maximum velocity is about 1·2 times the mean velocity.

* *Proc. Roy. Soc.*, **74.**

7.10. Flow through Pipe Lines. The velocity of a liquid flowing through a pipe may be found by applying Bernoulli's equation to the two ends of the pipe and allowing for any loss of head in the pipe. In all such problems the most convenient formula for the frictional head lost is the Darcy equation

$$h_f = \frac{4flv^2}{d2g}$$

as it is necessary to express all unknown terms as a function of the velocity head.

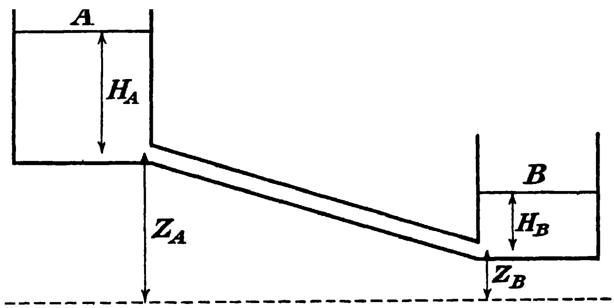


FIG. 97

Suppose a liquid flows from a reservoir A (Fig. 97) under a constant head H_A into a reservoir B in which there is a constant head of H_B. Let the height of centre of pipe at A be Z_A, and at B be Z_B. Let v be the velocity of flow through the pipe, l be the length, and d the diameter.

Then $$h_f = \frac{4flv^2}{d2g}$$

and head lost at entrance of pipe

$$= \frac{0{\cdot}5v^2}{2g}$$

Applying Bernoulli's equation to points just beyond each end of the pipe,

$$H_A + Z_A = H_B + Z_B + \frac{0{\cdot}5v^2}{2g} + \frac{4fl}{d}\frac{v^2}{2g} + \frac{v^2}{2g}$$

The term $v^2/2g$ will be lost on entering B.

It will be noticed that $(H_A + Z_A) - (H_B + Z_B)$ is the difference in level of the liquid surfaces in A and B; hence,

difference in level of liquid surfaces

$$= \frac{v^2}{2g}\left(1{\cdot}5 + \frac{4fl}{d}\right) \tag{8}$$

From this equation the unknown velocity may be obtained. If the length of the pipe is short, the second term of eq. (8) is small, and the full equation should be used; but if the pipe is long, the head lost in friction will be very large compared with the head lost at the two ends of the pipe, in which case the term 1·5 may be neglected as small.

Then $$h_f = \frac{4flv^2}{2gd}$$

$$= \text{difference of liquid level}$$

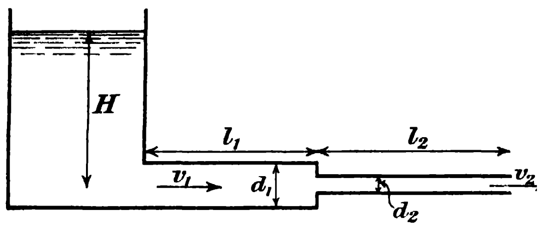


FIG. 98

If the pipe in Fig. 97, instead of discharging into the reservoir B, discharged into the atmosphere, the equation would then be

$$H_A + Z_A = Z_B + \frac{0{\cdot}5v^2}{2g} + \frac{4fl}{d}\frac{v^2}{2g} + \frac{v^2}{2g}$$

the last term being the velocity head of the discharging liquid.

This may be written

$$H = \frac{v^2}{2g}\left(1{\cdot}5 + \frac{4fl}{d}\right)$$

where H is the height of liquid level in A above outlet of pipe.

Suppose a liquid flows from a tank through a pipe of which the diameter is varied as in Fig. 98.

As quantity of liquid flowing per second is constant,

$$v_1 \frac{\pi}{4} d_1^2 = v_2 \frac{\pi}{4} d_2^2$$

Then $$v_1 \quad v_2 \left(\frac{d_2}{d_1}\right)^2$$

Head lost in friction in large pipe

$$= \frac{4fl_1v_1^2}{d_1 2g}$$

$$\frac{4fl_1v_2^2\left(\frac{d_2}{d_1}\right)^4}{d_1 2g}$$

Head lost in friction in small pipe

$$= \frac{4fl_2v_2^2}{d_2 2g}$$

$$\text{Total head lost in friction} = 4f\left[\frac{l_1\left(\dfrac{d_2}{d_1}\right)^4}{d_1} + \frac{l_2}{d_2}\right]\frac{v_2^2}{2g}$$

Applying Bernoulli's equation to points just outside each end of pipe,

$$H = \frac{0 \cdot 5v_1^2}{2g} + \frac{v_2^2}{2g} + \text{head lost in friction} + \text{head lost at contraction}$$

$$= \frac{0 \cdot 5\left(\dfrac{d_2}{d_1}\right)^4}{2g}v_2^2 + \frac{v_2^2}{2g} + 4f\left[\frac{l_1\left(\dfrac{d_2}{d_1}\right)^4}{d_1} + \frac{l_2}{d_2}\right]\frac{v_2^2}{2g} + \left(\frac{1}{C_c} - 1\right)^2\frac{v_2^2}{2g}$$

From this equation, the velocity v_2 may be found. If the pipe is long, the velocity head, the head lost at entrance, and the head lost at the sudden contraction may be neglected as small.

EXAMPLE 4

A cast-iron pipe, 6 in. in diameter and 1,500 ft long, connects two reservoirs. If the difference of water level in the two reservoirs is 96 ft, find the discharge through the pipe; $f = 0 \cdot 01$. Ignore loss at entrance to pipe.

$$\text{Total head} = \text{velocity head} + \text{head lost in friction}$$

$$96 = \frac{v^2}{2g} + \frac{4flv^2}{d2g}$$

$$96 = \frac{v^2}{2g}\left(1 + \frac{4fl}{d}\right)$$

$$= \frac{v^2}{64 \cdot 4}\left(1 + \frac{4 \times 0 \cdot 01 \times 1{,}500}{0 \cdot 5}\right)$$

$$= \frac{v^2}{64 \cdot 4}(1 + 120)$$

Then $$v^2 = \frac{64 \cdot 4 \times 96}{121} = 51 \cdot 1$$

$$v = 7 \cdot 15 \text{ ft/sec}$$

$$\text{Discharge} = \frac{\pi}{4}(0 \cdot 5)^2 \times 7 \cdot 15 = 1 \cdot 4 \text{ ft}^3\text{/sec}$$

EXAMPLE 5

Two reservoirs are connected by a straight pipe 1 mile long. For the first half of its length the pipe is 6 in. in diameter; its diameter is then suddenly reduced to 3 in. The surface of the water in the upper reservoir is 100 ft above

that in the lower. Tabulate the losses of head which occur, including that at the sharp-edged entry, and determine the flow in gallons per minute. Assume $f = 0{\cdot}01$. (*Lond. Univ.*)

Let v_1 = velocity in 6 in. pipe
and v_2 = velocity in 3 in. pipe.

As quantity of flow is the same in both pipes

$$\frac{\pi}{4}(0{\cdot}5)^2 v_1 = \frac{\pi}{4}(0{\cdot}25)^2 v_2$$

Then
$$v_1 = \frac{v_2}{4}$$

Head lost at entrance

$$= \frac{0{\cdot}5 v_1^2}{2g} = 0{\cdot}03125\,\frac{v_2^2}{2g}$$

Head lost in 6 in. pipe due to friction

$$= \frac{4flv_1^2}{d_1 2g}$$

$$= \frac{4 \times 0{\cdot}01 \times 2{,}640 \times v_2^2}{0{\cdot}5 \times 16 \times 2g}$$

$$= 13{\cdot}2\,\frac{v_2^2}{2g}$$

Head lost at sudden contraction

$$= \frac{0{\cdot}5 v_2^2}{2g} \text{ (as the value of } C_c \text{ is not given)}$$

Head lost in 3 in. pipe due to friction

$$= \frac{4flv_2^2}{d_2 2g}$$

$$= \frac{4 \times 0{\cdot}01 \times 2{,}640 v_2^2}{0{\cdot}25 \times 2g}$$

$$= 422\,\frac{v_2^2}{2g}$$

$$\text{Head lost at exit} = \frac{v_2^2}{2g}$$

$$\text{Total head lost} = \frac{v_2^2}{2g}(0{\cdot}03125 + 13{\cdot}2 + 0{\cdot}5 + 422 + 1)$$

$$= 436{\cdot}73125\,\frac{v_2^2}{2g}$$

$$= 100 \text{ ft}$$

Then
$$v_2^2 = \frac{64{\cdot}4 \times 100}{436{\cdot}73125} = 14{\cdot}73$$
$$v_2 = 3{\cdot}835 \text{ ft/sec}$$
$$\text{Discharge} = \frac{\pi}{4}(0{\cdot}25)^2 \times 3{\cdot}835 \times 60 \times 6{\cdot}24$$
$$= 70{\cdot}5 \text{ gal/min}$$

EXAMPLE 6

The difference of surface level in two reservoirs connected by a syphon is 25 ft. The length of the syphon is 2,000 ft; its diameter is 12 in.; and $f = 0{\cdot}01$. If the barometric height is 34 ft and if air is liberated from solution when the absolute pressure is less than 4 ft of water, what will be the maximum length of inlet leg of the syphon to run full, if the vertex is 18 ft above the surface level in the upper reservoir? What will then be the discharge? (*Lond. Univ.*)

The problem is represented by Fig. 87, E being 18 ft above the water level in A.

Let l = length of pipe between A and E.

First find the velocity of water in the pipe by applying Bernoulli's equation to points A and B, taking the water level in B as datum. Then,
$$25 = \frac{v^2}{2g} + \frac{4f \times 2{,}000v^2}{2gd}$$
$$= \frac{v^2}{2g}\left(1 + \frac{4 \times 0{\cdot}01 \times 2{,}000}{1}\right)$$
from which
$$v = 4{\cdot}45 \text{ ft/sec}$$

Next apply Bernoulli's equation to points A and E, taking the water level in A as datum. The limiting condition for the pipe to run full is when the absolute pressure at E is 4 ft of water.

Take into account the atmospheric pressure at A.

Total energy at A = total energy at E (above absolute zero pressure)

Hence
$$34 = 18 + \frac{v^2}{2g} + \frac{4flv^2}{2gd} + 4$$
or
$$12 = \frac{v^2}{2g}\left(1 + \frac{4fl}{d}\right)$$
$$= \frac{(4{\cdot}45)^2}{2g}\left(1 + \frac{4 \times 0{\cdot}01l}{1}\right)$$
from which
$$l = 947 \text{ ft}$$
$$\text{Discharge} = \text{area of pipe} \times \text{velocity}$$
$$= \frac{\pi}{4} \times 1 \times 4{\cdot}45$$
$$= 3{\cdot}49 \text{ ft}^3/\text{sec}$$

7.11. Parallel Flow through Pipes, or Forked Pipes. Suppose a liquid to be flowing along a pipe which, at a certain point, divides into two branches as in Fig. 99. Then, a particular particle of the liquid will flow along either the route ABD or the route ABC.

Let H_A, H_D, and H_C be the total head at A, D, and C respectively.

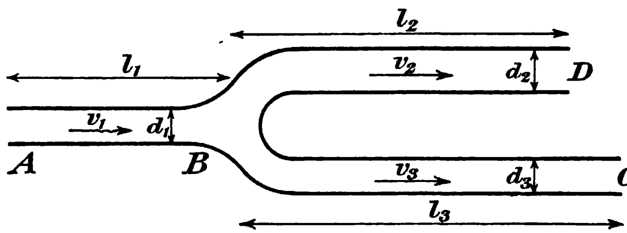


FIG. 99

Then, head causing flow along route $ABD = H_A - H_D$ — head lost in friction.

And, head causing flow along route $ABC = H_A - H_C$ — head lost in friction.

Let l_1, d_1 and v_1 refer to pipe AB,
l_2, d_2 and v_2 refer to pipe BD,
and l_3, d_3 and v_3 refer to pipe BC.

Then, for route ABD,

$$H_A - H_D - \frac{4fl_1v_1^2}{d_1 2g} - \frac{4fl_2v_2^2}{d_2 2g} = \frac{v_2^2}{2g} \quad . \quad . \quad (9)$$

And, for route ABC,

$$H_A - H_C - \frac{4fl_1v_1^2}{d_1 2g} - \frac{4fl_3v_3^2}{d_3 2g} = \frac{v_3^2}{2g} \quad . \quad . \quad (10)$$

If the pipes are long, the velocity heads given on the right of these equations are relatively small and may be written as zero.

Also, quantity flowing per second through AB equals sum of quantities through BD and BC. Then,

$$v_1d_1^2 = v_2d_2^2 + v_3d_3^2 \quad . \quad . \quad . \quad (11)$$

From eqs. (9), (10), and (11) the three unknowns v_1, v_2, and v_3 may be obtained.

EXAMPLE 7

Two pipes A and B, each 6 in. in diameter, branch from a point C to a point D, which is 20 ft below C. Pipe A is 300 yd long and pipe B is 500 yd long. Water is supplied at C under a head of 100 ft. A short pipe 3 in. in diameter is fitted at D. Find the delivery when this pipe is fully open to the atmosphere. Take $v = 80\sqrt{mi}$ for pipes A and B. (*Lond. Univ.*)

Let v_A, v_B, and v be velocities in pipes A, B, and 3 in. pipe respectively.

$$\text{Total head} = 100 + 20 = 120 \text{ ft}$$

Consider pipe A.

$$m = \frac{d}{4} = \frac{0{\cdot}5}{4} = \frac{1}{8}$$

$$v_A = 80\sqrt{\frac{1}{8} \times \frac{h_f}{900}}$$

Then $$h_f = 1{\cdot}125 v_A^2$$

Consider pipe B.

$$v_B = 80\sqrt{\frac{1}{8} \times \frac{h_f}{1{,}500}}$$

Then $$h_f = 1{\cdot}875 v_B^2$$

As pressure at C and D is the same in both pipes,

$$\frac{v_A^2}{2g} + 1{\cdot}125 v_A^2 = \frac{v_B^2}{2g} + 1{\cdot}875 v_B^2$$

from which $$v_A = 1{\cdot}285 v_B \quad . \quad . \quad . \quad . \quad (12)$$

Consider route B.

$$\text{Total head} = \frac{v^2}{2g} + \text{frictional head lost in } B$$

or $$120 = \frac{v^2}{2g} + 1{\cdot}875 v_B^2 \quad . \quad . \quad . \quad (13)$$

Also, quantity flowing through 3 in. pipe equals sum of quantities through A and B. That is,

$$\frac{\pi}{4}(0{\cdot}25)^2 v = \frac{\pi}{4}(0{\cdot}5)^2 (v_A + v_B)$$

Substituting from eq. (12),

$$v = 4(1{\cdot}285 v_B + v_B) = 9{\cdot}14 v_B$$

Substituting in eq. (13),

$$120 = \frac{v^2}{64{\cdot}4} + 1{\cdot}875\left(\frac{v}{9{\cdot}14}\right)^2,$$

from which $$v = 56{\cdot}2 \text{ ft/sec}$$

$$\text{Discharge} = \frac{\pi}{4}(0{\cdot}25)^2 \times 56{\cdot}2 = 2{\cdot}76 \text{ ft}^3\text{/sec}$$

7.12. Time of Emptying Tank through Pipe. Let a reservoir or tank be emptied by means of a long pipe of length l and diameter d. Let the area of liquid surface in the reservoir be A, and the height of the liquid level above the outlet of pipe be H_1 ft. Let v be the velocity of flow in the pipe. It is required to find the time taken to lower the liquid level in the reservoir from H_1 ft to H_2 ft above the outlet of pipe. Ignore all losses but friction.

$$\text{Head lost in friction in pipe} = \frac{4flv^2}{d2g}$$

Consider the instant when the liquid level is h ft above the outlet of pipe, and let the liquid level fall by a small amount dh in the time dt, the term dh being negative as the liquid level is falling.

Then, quantity per second flowing from reservoir equals quantity per second passing along pipe. Or,

$$-A\,\mathrm{d}h = \frac{\pi}{4}d^2v\,\mathrm{d}t \quad . \quad . \quad . \quad . \quad . \quad (14)$$

But
$$h = \frac{v^2}{2g} + \frac{4flv^2}{d2g}$$
$$= \frac{v^2}{2g}\left(1 + \frac{4fl}{d}\right)$$

from which
$$v = \sqrt{\frac{2gh}{\left(1 + \frac{4fl}{d}\right)}}$$

Substituting this value of v in eq. (14),

$$-A\,\mathrm{d}h = \frac{\pi}{4}d^2 \cdot \frac{\sqrt{2gh}}{\sqrt{1 + \frac{4fl}{d}}}\,\mathrm{d}t$$

Therefore
$$\mathrm{d}t = -\frac{4A\sqrt{1 + \frac{4fl}{d}}\,h^{-1/2}\,\mathrm{d}h}{\pi d^2\sqrt{2g}}$$

$$\text{Total time} = T = \int_0^T \mathrm{d}t = -\frac{4A\sqrt{1 + \frac{4fl}{d}}}{\pi d^2\sqrt{2g}}\int_{H_1}^{H_2} h^{-1/2}\,\mathrm{d}h$$

$$= \frac{8A\sqrt{1 + \frac{4fl}{d}}}{\pi d^2\sqrt{2g}}\left(H_1^{1/2} - H_2^{1/2}\right) \quad . \quad (15)$$

EXAMPLE 8

Two tanks, the bottom of which are on the same level, are connected with one another by a horizontal pipe 3 in. in diameter, 1,000 ft long, and bell-mouthed at each end. One tank is of size 20 by 20 ft and contains water to a depth of 20 ft; the other tank is of size 15 by 15 ft and holds water to a depth of 10 ft.

If the tanks are put in communication with one another by means of the pipe (which is full of water), how long will it be before the water level in the larger tank falls from a height of 19 ft to 17 ft? Assume $f = 0{\cdot}01$. (*Lond. Univ.*)

This question is shown diagrammatically in Fig. 100.

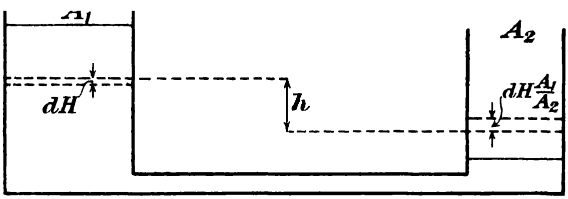


FIG. 100

Let A_1 and A_2 be the areas of the large and small tanks respectively.

Let H_1 and H_2 be the initial and final difference of water level in the two tanks.

Let h = difference of water level in tanks at any instant.

Let water level in A_1 fall by amount dH in time dt.

Then, level in small tank rises by $\mathrm{d}H(A_1/A_2)$. Let dh be difference in head causing flow due to this change. Then,

$$\mathrm{d}h = \mathrm{d}H + \mathrm{d}H\,\frac{A_1}{A_2}$$

$$= \mathrm{d}H\left(1 + \frac{A_1}{A_2}\right) \quad . \qquad . \text{ (16)}$$

Let a, v, d, and l be the area, velocity, diameter, and length of pipe respectively.

As quantity flowing from large tank equals quantity flowing along pipe, and as dH is negative,

$$-A_1\,\mathrm{d}H = av\,\mathrm{d}t \quad . \quad . \quad . \quad . \quad . \text{ (17)}$$

But
$$h = \frac{v^2}{2g} + \frac{4flv^2}{d2g}$$

$$= \frac{v^2}{2g}\left(1 + \frac{4fl}{d}\right) \quad . \text{ (18)}$$

Substituting eqs. (16) and (18) in eq. (17),

$$-A_1 \frac{dh}{\left(1+\frac{A_1}{A_2}\right)} = a\sqrt{\frac{2gh}{\left(1+\frac{4fl}{d}\right)}}\,dt$$

Therefore
$$dt = -\frac{A_1\sqrt{1+\frac{4fl}{d}}\,h^{-1/2}\,dh}{\left(1+\frac{A_1}{A_2}\right)a\sqrt{2g}}$$

Integrating between the limits of H_2 and H_1,

$$T = \int_0^T dt = \frac{2A_1\sqrt{1+\frac{4fl}{d}}\,(H_1^{1/2} - H_2^{1/2})}{\left(1+\frac{A_1}{A_2}\right)a\sqrt{2g}} \quad . \quad (19)$$

In this question,
$$A_1 = 20 \times 20 = 400 \text{ ft}^2$$
$$A_2 = 15 \times 15 = 225 \text{ ft}^2$$
$$a = \frac{\pi}{4}\left(\frac{1}{4}\right)^2 = 0{\cdot}0491 \text{ ft}^2$$
$$H_1 = 19 - \left(10 + \frac{400}{225}\right) = 7{\cdot}22 \text{ ft}$$
$$H_2 = 17 - \left(10 + \frac{3 \times 400}{225}\right) = 1{\cdot}66 \text{ ft}$$

Substituting these values in eq. (19)

$$T = \frac{2 \times 400\sqrt{1+\frac{4 \times 0{\cdot}01 \times 1{,}000}{0{\cdot}25}}\,(7{\cdot}22^{1/2} - 1{\cdot}66^{1/2})}{\left(1+\frac{400}{225}\right) \times 0{\cdot}0491\sqrt{64{\cdot}4}}$$
$$= 12{,}960 \text{ sec}$$
$$= 216 \text{ min}$$

7.13. Flow of Gases through Pipes. The frictional resistance of gas flowing along a pipe may be found from the same frictional formula as for liquids, providing the pressure drop is small so that the density can be assumed to remain constant. The head causing flow must be in feet of gas, and, if the pipe has a large slope, the slight difference of atmospheric pressure due to change of altitude must be taken into account.

Let gas be flowing up a sloping uniform pipe of length l and diameter d. Then,

$$\text{head lost in friction} = \frac{4flv^2}{d2g} \text{ ft of gas}$$

Let w = weight of 1 ft³ of water,
w_1 = weight of 1 ft³ of gas,
and w_2 = weight of 1 ft³ of air.

Let h_1 be height of upper end of pipe above lower end.

Then, atmospheric pressure at lower end is h_1 ft of air greater than at upper end. But pressure of gas at lower end is greater than pressure at higher end by h_1 ft of gas.

Then, head causing flow due to change of altitude

$$= h_1 \text{ ft of air} - h_1 \text{ ft of gas}$$

$$\left(h_1 \frac{w_2}{w_1} - h_1\right) \text{ ft of gas}$$

This term can be neglected if the change of altitude is small, such as within the height of a room.

Suppose the pressure of gas above atmosphere be measured with a U-tube containing water.

Let y_1 = pressure of gas at lower end in feet of water,
and y_2 = pressure of gas at higher end in feet of water.

Then, head causing flow due to difference of pressure

$$= (y_1 - y_2) \text{ ft of water}$$

$$= (y_1 - y_2) \frac{w}{w_1} \text{ ft of gas}$$

$$\text{Total head causing flow} = h_1 \frac{w_2}{w_1} - h_1 + (y_1 - y_2) \frac{w}{w_1} \text{ ft of gas}$$

Equating this amount to the frictional head plus velocity head, then

$$h_1 \frac{w_2}{w_1} - h_1 + (y_1 - y_2) \frac{w}{w_1} = \frac{4flv^2}{d2g} + \frac{v^2}{2g} \qquad . \quad . \quad . \ (20)$$

If the gas is flowing down the pipe h_1 will be negative.

If the pipe is horizontal $h_1 = 0$, and eq. (20) then becomes

$$(y_1 - y_2) \frac{w}{w_1} = \frac{4flv^2}{d2g} + \frac{v^2}{2g} \qquad . \quad . \quad . \quad . \ (21)$$

Unwin found Darcy's coefficient f to be equal to $0{\cdot}0044 \left(1 + \frac{1}{7d}\right)$ for coal gas.

Example 9

Gas is supplied from a holder at a gauge pressure of 4 in. of water to a pipe 6 in. in diameter, 300 ft long, which rises to and discharges at a height of 50 ft above the level of the outlet of the holder. The pressure at the pipe outlet must be not less than 1 in. of water by gauge. Find the delivery in cubic feet per hour. Take the weights of the gas and air as 0·045 and 0·08 Lb/ft³ respectively and the coefficient of friction as 0·008. (*Lond. Univ.*)

Applying eq. (20),

$$\left(50 \times \frac{0{\cdot}08}{0{\cdot}045}\right) - 50 + (0{\cdot}333 - 0{\cdot}0834)\,\frac{62{\cdot}4}{0{\cdot}045}$$

$$= \frac{v^2}{2g}\left(1 + \frac{4 \times 0{\cdot}008 \times 300}{0{\cdot}5}\right)$$

or $$88{\cdot}9 - 50 + 346{\cdot}5 = \frac{v^2}{2g}(1 + 19{\cdot}2)$$

from which $$v = 35 \text{ ft/sec}$$

$$\text{Delivery} = \frac{\pi}{4}(0{\cdot}5)^2 \times 35 \times 3{,}600 = 24{,}750 \text{ ft}^3/\text{hr}$$

7.14. Transmission of Power through Pipes. If power is transmitted through a considerable distance by means of water under pressure, the power supplied will be in proportion to the quantity of water per second passing through the pipe, and to the total head of the water. As the water flows along the pipe it will be subjected to a loss of head due to friction. It can be shown that the maximum power is transmitted by a pipe when the frictional loss of head is one-third of the total head supplied.

Let H = total head supplied at entrance to pipe,
h_f = head lost due to friction,

and let v, d, and l be the velocity of flow through pipe, the diameter of pipe, and length of pipe respectively.

Then $$h_f = \frac{4flv^2}{d2g}$$

Total head available at outlet of pipe $= H - h_f$

$$= H - \frac{4flv^2}{d2g}$$

$$\text{Available horse-power} = \frac{w\pi d^2 v}{4 \times 550}\left(H - \frac{4flv^2}{d2g}\right)$$

as $w(\pi/4)d^2v$ = weight of water flowing per second.

Therefore $$\text{h.p.} = \frac{w\pi d^2}{4 \times 550}\left(Hv - \frac{4flv^3}{d2g}\right)$$

This will be a maximum when the amount inside the bracket is a maximum. Differentiating this with respect to v and equating to zero for a maximum,

$$\frac{\mathrm{d(h.p.)}}{\mathrm{d}v} = H - 3\left(\frac{4flv^2}{d2g}\right) = 0$$

or $$H - 3h_f = 0$$

Therefore $$H = 3h_f$$

That is, the horse-power transmitted is a maximum when the head lost in friction is one-third of total head supplied.

For any pipe line transmitting power,

$$\text{efficiency of transmission} = \frac{H - h_f}{H}$$

Power is transmitted through water pipes for working hydraulic machines. The supply of water under pressure for power purposes was being developed in large cities during the latter half of the nineteenth century; but the commercializing of electricity has mainly displaced this method of power transmission.

EXAMPLE 10

A hydraulic machine is supplied with water through a horizontal pipe 3,000 ft long. The brake horse-power of the hydraulic machine is 50, and its mechanical efficiency is 80 per cent. Gauges fitted to the supply pipe show that the pressure at the power station end is 750 Lb/in.^{2}; and at the machine 680 Lb/in.^{2} If the coefficient of resistance, f, for the pipe is 0·008, determine (1) the diameter of the supply pipe, (2) the velocity of flow. (*Lond. Univ.*)

Let a, d, and v be the area, diameter, and velocity respectively of the pipe.

Horse-power supplied by machine

$$= 50 \times \frac{100}{80} = 62{\cdot}5$$

$$= \frac{WH}{550}$$

$$= \frac{62{\cdot}4av}{550} \times \frac{680 \times 144}{62{\cdot}4}$$

from which $$av = 0{\cdot}351 = \frac{\pi}{4}\,d^2v$$

Then $$d^2v = 0{\cdot}447 \qquad . \quad . \quad . \quad . \quad (22)$$

Head lost in friction in pipe $= (750 - 680)\,\dfrac{144}{62{\cdot}4}$

$$= 161{\cdot}8 \text{ ft of water}$$

$$= \frac{4flv^2}{d2g}$$

Therefore
$$161{\cdot}8 = \frac{4 \times 0{\cdot}008 \times 3{,}000v^2}{d \times 64{\cdot}4}$$

from which
$$\frac{v^2}{d} = 108{\cdot}6 \qquad . \quad . \quad . \quad . \quad (23)$$

Substituting for d from eq. (22),

$$v^5 = 5{,}280$$

Then
$$v = 5{\cdot}55 \text{ ft/sec}$$

Substituting this value of v in eq. (23),

$$d = \frac{5{\cdot}55^2}{108{\cdot}6} = 0{\cdot}284 \text{ ft}$$

7.15. Flow of Liquids through Nozzles. A nozzle is a tapering mouthpiece which is fitted to the outlet end of a pipe for the purpose of converting the total head of the liquid into velocity head. They are used on the end of hose pipes and in some forms of turbines. As the pressure of the jet issuing from the nozzle is atmospheric, the whole of the energy will be kinetic. The loss of energy in the nozzle itself will be small compared with the frictional loss in the pipe to which the nozzle is fixed, and may, therefore, be neglected.

In § 7.14 it was proved that for the maximum power to be transmitted along the pipe, the loss of head in the pipe due to friction must be one-third of the total head supplied. In which case, the loss of head due to friction will be one-half of the total head in the nozzle. By making use of this fact it is possible to obtain the ratio of the area of nozzle to area of supply pipe for maximum transmission of power.

Consider the pipe and nozzle of Fig. 101.

Let l = length of supply pipe,
D = diameter of supply pipe,
A = cross-sectional area of pipe,
V = velocity in supply pipe,
f = Darcy's coefficient of frictional resistance of supply pipe,
a = area of outlet end of nozzle,
v = velocity of jet issuing from nozzle.

For maximum transmission of power,

head lost in friction in supply pipe = $\frac{1}{3}$ total head supplied
= $\frac{1}{2}$ velocity head at nozzle

or
$$\frac{4flV^2}{D2g} = \frac{1}{2} \times \frac{v^2}{2g} \qquad . \quad . \quad . \quad (24)$$

But, as quantity flowing through pipe equals quantity passing through nozzle,

$$VA = va$$

Therefore $$\frac{v}{V} = \frac{A}{a} \quad . \quad . \quad . \quad . \quad . \quad (25)$$

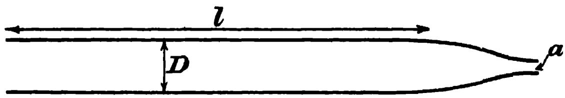


FIG. 101

From eq. (24),

$$\frac{8fl}{D} = \frac{v^2}{V^2}$$

Substituting from eq. (25),

$$\frac{8fl}{D} = \left(\frac{A}{a}\right)^2$$

Therefore $$\frac{A}{a} = \sqrt{\frac{8fl}{D}} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

This gives the ratio between the areas of nozzle and supply pipe for maximum transmission of power.

$$\text{Horse-power of jet} = \frac{\text{kinetic energy per second}}{550}$$

$$= \frac{wav \times v^2}{2g \times 550} = \frac{wav^3}{2g \times 550}$$

If the jet were projected vertically upwards the height the liquid would reach $= v^2/2g$.

If the area of the supply pipe and jet do not conform with eq. (26), the pipe will not be transmitting the maximum horse-power possible. If such be the case,

$$\text{head transmitted by pipe} = H - \frac{4flV^2}{2gD}$$

where H is the head supplied at the source.

Substituting from eq. (25) for V,

$$\text{head transmitted by pipe} = H - \frac{4fl}{2gD}\left(\frac{va}{A}\right)^2$$

But $$\text{head transmitted} = \text{kinetic energy at nozzle}$$

$$= \frac{v^2}{2g} \text{ per Lb of fluid}$$

Hence $$H - \frac{4fl}{2gD}\frac{a^2}{A^2}v^2 = \frac{v^2}{2g}$$

from which $$v = \sqrt{\frac{2gH}{1 + \dfrac{4fla^2}{DA^2}}}$$

$$\text{Horse-power transmitted} = \text{horse-power of jet}$$
$$= \frac{(wav)v^2}{550 \times 2g}$$

$$\text{Efficiency of transmission} = \frac{\text{head transmitted}}{\text{head supplied}}$$
$$= \frac{v^2}{2gH}$$

EXAMPLE 11

The head of water at one end of a pipe 200 yd long, 3 in. in diameter, is 100 ft, and f for the main is 0·01. What diameter of nozzle fitted to the end will give the maximum power, and what will the power then be? If a formula is used it must be proved. (*Lond. Univ.*)

Using eq. (26) let d be the diameter of nozzle.

$$\frac{A}{a} = \sqrt{\frac{8fl}{D}}$$

Then $$\frac{D^2}{d^2} = \sqrt{\frac{8 \times 0{\cdot}01 \times 600}{0{\cdot}25}} = 13{\cdot}85$$

and $$d = \sqrt{\frac{(0{\cdot}25)^2}{13{\cdot}85}} = 0{\cdot}067 \text{ ft} = 0{\cdot}804 \text{ in.}$$

$$h_f = \frac{H}{3} = \frac{4flV^2}{2gD}$$

That is $$\frac{100}{3} = \frac{4 \times 0{\cdot}01 \times 600V^2}{64{\cdot}4 \times \frac{1}{4}}$$

from which $$V = 4{\cdot}72 \text{ ft/sec}$$

Then $$v = 4{\cdot}72 \times 13{\cdot}85 = 65{\cdot}4 \text{ ft/sec}$$

and $$\text{h.p.} = \frac{wav^3}{550 \times 2g} = \frac{62{\cdot}4 \times \left(\frac{\pi}{4} \times 0{\cdot}067^2\right) \times 65{\cdot}4^3}{550 \times 64{\cdot}4} = 1{\cdot}738$$

EXAMPLE 12

A horizontal pipe, 6 in. internal diameter and 540 ft long, conducts water from a reservoir. When the water level in the reservoir is 4 ft above the axis of the pipe the discharge through the pipe is 29·7 ft³/min. If a nozzle tapering from 6 to $1\frac{1}{2}$ in. internal diameter were fitted to the free end of the pipe, what would be the horse-power of the jet if the level of water in the reservoir were increased to 40 ft above the axis of the pipe? (*Lond. Univ.*)

In the first case, $$V = \frac{29{\cdot}7}{60 \times \frac{\pi}{4} \times \frac{1}{4}} = 2{\cdot}52 \text{ ft/sec}$$

Also $$H = \frac{V^2}{2g}\left(1 + \frac{4fl}{D}\right)$$

That is $$4 = \frac{2{\cdot}52^2}{64{\cdot}4}\left(1 + \frac{4 \times 540f}{0{\cdot}5}\right)$$

from which $$f = 0{\cdot}00916$$

In the second case, $$H = \frac{v^2}{2g} + \frac{4flV^2}{2gD}$$

and as $$v = \frac{VA}{a}$$

$$H = \frac{V^2}{2g}\left[\left(\frac{A}{a}\right)^2 + \frac{4fl}{D}\right]$$

That is $$40 = \frac{V^2}{64{\cdot}4}\left[\left(\frac{36}{2{\cdot}25}\right)^2 + \frac{4 \times 0{\cdot}00916 \times 540}{0{\cdot}5}\right]$$

from which $$V = 2{\cdot}95 \text{ ft/sec}$$

and $$v = 47{\cdot}2 \text{ ft/sec}$$

$$\text{Horse-power of jet} = \frac{wav^3}{2g \times 550}$$

$$= \frac{62{\cdot}4 \times 0{\cdot}785 \times (\tfrac{1}{8})^2 \times 47{\cdot}2^3}{64{\cdot}4 \times 550}$$

$$= 2{\cdot}27$$

7.16. Hammerblow in Pipes. If water is flowing along a long pipe and is suddenly brought to rest by the closing of a valve, or by any similar cause, there will be a sudden rise in pressure due to the momentum of the moving water being destroyed. This will cause a pressure wave to be transmitted along the pipe which may set up noises known as knocking. The magnitude of this pressure will depend on the speed at which the valve is closed and on the length of the pipe. Knocking may often be heard in the water pipes of ordinary dwelling-houses if the tap be turned off quickly.

This sudden rise in pressure in a pipe due to the stoppage of the flow is known as the hammerblow.

Consider a long pipe of length l ft and of cross-sectional area a ft^2; let water be flowing along the pipe with a velocity of v ft/sec, and, due to the closing of a valve, let the water be brought to rest in t sec.

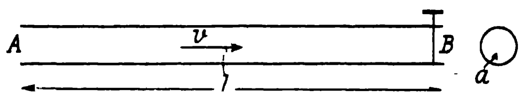


FIG. 102

Then, retardation of water $= f = \dfrac{v}{t}$ ft/sec^2

Mass of moving column of water $= \dfrac{wal}{g}$.

As force = mass × retardation,

$$\text{force on valve} = \frac{wal}{g} \times f \text{ Lb}$$

$$\text{Intensity of pressure on valve} = p = \frac{\text{force}}{\text{area}}$$

$$= \frac{wlf}{g} \text{ Lb/ft}^2$$

or

$$p = \frac{wlv}{gt} \text{ Lb/ft}^2$$

From this equation the magnitude of the pressure wave can be found.

This is the simple theory only; actually the water would compress on account of its bulk elastic modulus, and the pipe would expand laterally on account of its modulus of elasticity; these would both affect the problem.

Consider the pipe AB (Fig. 102) of length l and cross-sectional area a having water flowing in the direction AB with a velocity v. If the valve at B is suddenly closed there will be a sudden rise of pressure at B due to the inertia of the water column in the pipe. Let P be the maximum pressure at B due to this cause. The sudden rise of pressure at B causes a pressure wave to travel backwards along the pipe from B to A with a velocity v_s, which is the velocity of sound in water (§ 13.10). The time taken for the pressure wave to reach A is

$$\frac{l}{-} \text{ sec}$$

When the pressure wave reaches A the water in the pipe commences to surge backwards from B to A, thus causing the pressure to fall. The wave front of this falling pressure travels from A to B with a velocity v_s, the pressure falling to the normal pressure of the water when B is reached. Owing to the inertia of this backwards

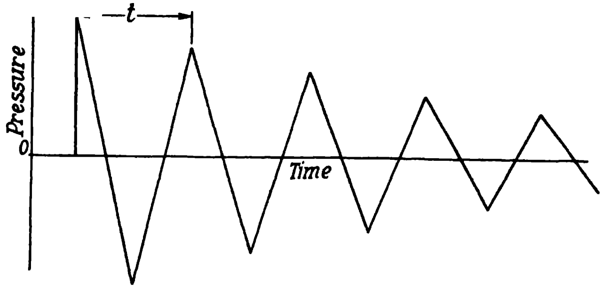


FIG. 103

flow, the pressure at B now falls to $-P$; this causes a negative pressure wave to travel from B to A with the velocity v_s. When A is reached the water commences to flow towards B, thus causing the pressure to rise. The wave front of this rising pressure then travels from A to B at a velocity v_s, the pressure in the pipe again reaching normal when the wave reaches B. Thus, a cycle is completed during which a pressure wave has travelled the length of the pipe four times.

Let t = time taken for complete operation of cycle

$= 4l/v_s$ sec

The cycle is then repeated, but, owing to frictional resistances, the maximum positive and negative pressures are continually being reduced during each repetition of the cycle. The pressure waves are thus describing a damped vibration, as shown in Fig. 103, which continues until the whole of the energy of the wave has been absorbed.

The pressure wave causes shock and noise as it vibrates backwards and forwards along the pipe. In a long pipe excessive pressure can be produced which may damage the joints. In such cases, practical methods are adopted to absorb the energy of the wave, such as stand pipes and relief valves (§ 22.10).

It will be shown in § 7.17 that, if the valve is closed within the time taken for the pressure wave to traverse the length of the pipe, the effect is the same as if the valve were closed instantaneously.

The problem of water hammer is further complicated by the elasticity of the walls of the pipe; the increase of water pressure causes the pipe to expand radially, thus absorbing part of the kinetic energy of the water column. This effect is dealt with in § 7.18.

EXAMPLE 13

A hydraulic pipe line is 2 miles long. The velocity of flow is 4 ft/sec. A valve at the lower end is closed at such a rate as to produce uniform retardation in the water column. Calculate the rise in pressure behind the valve if the latter is closed (*a*) in 20 sec, (*b*) in 1 sec. (*A.M.I.C.E.*)

(*a*) Using the equation

$$p = \frac{wlv}{gt}$$

$$= \frac{62{\cdot}4 \times 2 \times 5{,}280 \times 4}{32{\cdot}2 \times 20}$$

$$= 4{,}100 \text{ Lb/ft}^2$$

(*b*)

$$p = \frac{62{\cdot}4 \times 2 \times 5{,}280 \times 4}{32{\cdot}2 \times 1}$$

$$= 82{,}000 \text{ Lb/ft}^2$$

7.17. Pressure due to Sudden Stoppage, Neglecting Pipe Expansion. Consider the pipe of Fig. 102. Water is flowing with a velocity v and is suddenly brought to rest by the sudden closing of the valve at B. Assume the pipe to be perfectly rigid and non-elastic so that there is no radial expansion of its walls. Before the valve is closed the water has a kinetic energy due to its velocity; after the valve is closed this energy is converted to strain energy due to the compression of the water (§ 1.16).

Consider a short length dl of the water column; then,

loss of kinetic energy = gain of strain energy

Substituting eq. (15), § 1.16, for strain energy,

$$\frac{(wa\,\mathrm{d}l)v^2}{2g} = \frac{1}{2}\frac{P^2}{K}a\,\mathrm{d}l$$

from which

$$P^2 = \frac{wv^2K}{g}$$

or

$$P = v\sqrt{\frac{Kw}{g}} \qquad . \quad . \quad . \quad . \quad (27)$$

Let ρ = density of water in absolute units

$= w/g$.

Then

$$P = v\sqrt{K\rho} \qquad . \quad . \quad . \quad . \quad (28)$$

Eqs. (27) and (28) give the rise of pressure of the water in pounds per square foot due to instantaneous closing of the valve.

It can be shown that, if the time taken to close the valve is less than that required for the pressure wave to travel the length of the pipe, the pressure produced is the same as if the valve were closed

instantaneously. Assume the valve be closed in the period of time at such a rate that the water is brought to rest with a constant retardation f.

Consider the column of water in the pipe of Fig. 102.

Retarding force on end of pipe

$$= \text{mass of water} \times f$$

that is
$$Pa = \frac{wal}{g} \times f$$

from which
$$P = \frac{w}{g} lf$$
$$= \rho lf \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Let t = time taken by pressure wave to travel length of pipe $= l/v_s$.

Hence
$$l = tv_s \quad . \quad . \quad . \quad . \quad . \quad (30)$$

But, from eq. (22), § 13.10,

$$v_s = \sqrt{\frac{K}{\rho}}$$

Also
$$t = \frac{v}{f}$$

Substituting these values in eq. (30),

$$l = \frac{v}{f}\sqrt{\frac{K}{\rho}}$$

Substituting this value of l in eq. (29),

$$P = \rho \times \frac{v}{f}\sqrt{\frac{K}{\rho}} \times f$$
$$= v\sqrt{K\rho}$$

which is the same result as eq. (28). This proves that if the valve is closed within t sec the effect on the pressure rise is the same as the rise obtained by an instantaneous closing.

7.18. Effect of Pipe Elasticity on Hammerblow. The rise in pressure of the water due to a sudden stoppage of flow in a pipe causes a radial expansion of the latter; the strain energy thus stored in the pipe absorbs some of the kinetic energy lost by the water.

The increase of radial pressure of the water on the pipe causes circumferential and longitudinal stresses in the pipe walls.* If the walls of the pipe are thin compared with its diameter, it is possible to obtain a simple solution to this problem.

* See text-books on Strength of Materials for stresses in thin cylinders.

Consider a short length dl of the pipe, shown in Fig. 104.

Let P = pressure of water due to hammerblow, pounds per square foot,
r = radius of pipe in feet,
t = thickness of pipe in feet,

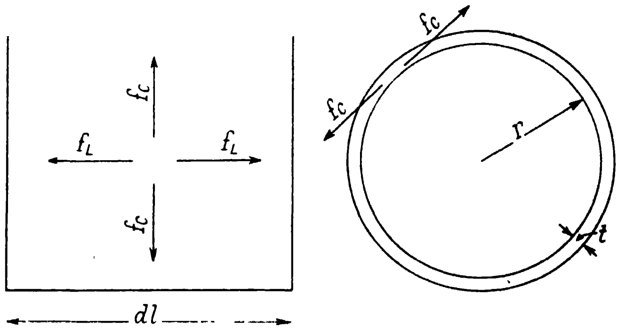


FIG. 104

a = cross-sectional area of pipe's bore in square feet,
f_c = circumferential stress in pipe, pounds per square foot $= Pr/t$,
f_L = longitudinal stress in pipe in pounds per square foot $= Pr/2t$,
E = linear elastic modulus of pipe walls, pounds per square foot,
K = bulk modulus of water, pounds per square foot,
m = Poisson's ratio.

Then volume of pipe walls $= 2\pi rt\ dl$ ft³

Working in foot units, and assuming $m = 4$,

strain energy in pipe walls

$$= \left(\frac{1}{2}\frac{f_c^2}{E} \times \text{vol.}\right) + \left(\frac{1}{2}\frac{f_L^2}{E} \times \text{vol.}\right) - \left(\frac{f_c f_L}{mE} \times \text{vol.}\right)$$

$$= \frac{1}{2E} \times 2\pi rt\ dl\left(f_c^2 + f_L^2 - \frac{2f_c f_L}{m}\right)$$

$$= \frac{\pi rt\ dl}{E}\left(\frac{P^2r^2}{t^2} + \frac{P^2r^2}{4t^2} - \frac{P^2r^2}{4t^2}\right)$$

$$= \frac{P^2\pi r^3\ dl}{Et}$$

$$= \frac{(a\ dl)rP^2}{Et} \text{ ft units} \quad . \quad . \quad . \quad . \quad . \quad (31)$$

From eq. (15), § 1.16, the strain energy stored in water of the short length of pipe considered is—

$$\frac{1}{2}\frac{P^2}{K} \times (a\,dl) \text{ ft units} \quad . \quad . \quad . \quad . \ (32)$$

Then

loss of kinetic energy of water = strain energy stored in water + strain energy stored in pipe

Substituting the values of eqs. (31) and (32),

$$\frac{w(a\,dl)v^2}{2g} = \frac{P^2(a\,dl)}{2K} + \frac{(a\,dl)rP^2}{Et}$$

that is

$$\frac{wv^2}{g} = P^2\left(\frac{1}{K} + \frac{2r}{tE}\right)$$

from which

$$P = v\sqrt{\frac{w}{g\left(\frac{1}{K} + \frac{2r}{tE}\right)}} \text{ Lb/ft}^2 \quad . \quad . \ (33)$$

Having obtained the value of P from eq. (33), the stresses f_c and f_L can then be calculated.

It will be noticed from eq. (33) that if the pipe is inelastic E is infinity; then the equation becomes the same as eq. (27).

EXAMPLE 14

Obtain a formula for the rise in pressure in a thin elastic pipe of circular section in which a flow of water is stopped by the sudden closing of a valve. Water flows in a long pipe which is 6 in. in diameter and $\frac{1}{4}$ in. thick with a velocity of 4 ft/sec, when it is suddenly brought to rest by the closing of a valve. Calculate the theoretical stress produced in the pipe near the valve taking K for water as 300,000 Lb/in.^{2} and E for the pipe material as 30×10^6 Lb/in.^{2} (*Lond. Univ.*)

In this question,

$$r = \tfrac{1}{4} \text{ ft}, \ t = \tfrac{1}{48} \text{ ft}, \ v = 4 \text{ ft/sec}$$
$$K = 300{,}000 \times 144, \ E = 30{,}000{,}000 \times 144$$

First calculate P from eq. (33).

$$\sqrt{\frac{w}{g\left(\frac{1}{K} + \frac{2r}{tE}\right)}} \text{ Lb/ft}^2$$

Substituting the above values in this equation,

$$P = 4\sqrt{\frac{62{\cdot}4}{32{\cdot}2\left(\frac{1}{300{,}000 \times 144} + \frac{2 \times \frac{1}{4} \times 48}{30{,}000{,}000 \times 144}\right)}}$$

Then $$f_c = \frac{Pr}{t} \times \tfrac{1}{144} \text{ Lb/in.}^2 = \frac{32{,}870 \times \frac{1}{4}}{\frac{1}{48} \times 144}$$

$$= 2{,}739 \text{ Lb/in.}^2$$

EXERCISES 7

1. Find the loss of head due to friction in a pipe, 1,000 ft long and 6 in. in diameter, when the quantity of water flowing is 600 gal/min. $f = 0{\cdot}01$.
Ans. 83 ft of water.

2. Using the formula $v = C\sqrt{(mi)}$, find the loss of head due to friction in a circular pipe, 100 ft long and 3 in. in diameter, when the velocity of flow is 6 ft/sec. $C = 100$. *Ans.* 5·76 ft.

3. Draw curves showing the nature of the results obtained by Froude in regard to the surface friction of planes, of varying length and of different materials, moving through water. If $f = 0{\cdot}0035$ and $n = 1{\cdot}83$, find the horse-power required to overcome the skin resistance of a ship, wetted surface, 24,000 ft², when going at 18 knots. One knot = 1·69 ft/sec. (*Lond. Univ.*)
Ans. 2,420.

4. Assuming that $R = f'AV^2$ to be the law of friction between a flow and a surface, find an expression for the work lost per second when a disc of radius r is rotated in water with a circumferential velocity v.

If the disc is surrounded by a forced vortex of double its diameter, compare the loss due to the friction of the vortex on the flat sides of the vortex chamber with the loss due to the friction on the above-mentioned disc. (*Lond. Univ.*)
Ans. $\frac{4}{5}\pi f' v^3 r^2$. 32 : 1.

5. The friction of a thin flat brass plate when towed edgewise through water at a velocity of 10 ft/sec is equal to 0·21 Lb/ft² of wetted surface, and the friction is found to vary as $V^{1\cdot 9}$.

Find how many foot-pounds of energy per minute are absorbed by the skin friction of the two surfaces of a flat circular disc, the external diameter of which is 24 in., and the internal diameter 12 in., if the disc makes 450 r.p.m. (*Lond. Univ.*) *Ans.* 27,800 ft-Lb/min.

6. What do you understand by the expression "critical" velocity of flow in a pipe?

Describe experiments on the loss of head when water flows at known velocities through a horizontal pipe of constant cross-section.

What is the law when (*a*) the velocity is less than the "critical": (*b*) the velocity exceeds the "critical"? (*Lond. Univ.*) *Ans* (*a*) kv; (*b*) kv^n.

7. Obtain an expression for the head lost in a pipe due to a sudden enlargement of area.

Comment on any assumption made.

A pipe 2 in. in diameter is 20 ft long and the velocity of the water in the pipe is 8 ft/sec. What loss of head would be saved if the central 6 ft length of pipe were replaced by a 3 in. diameter pipe, the changes of section being sudden?

Take the Darcy frictional coefficient $f = 0{\cdot}01$, and the coefficient of contraction 0·62. (*Lond. Univ.*) *Ans.* 0·52.

8. Water is discharged from a reservoir through a pipe 1 ft in diameter for 1 mile of its length, the pipe then suddenly enlarging to 2 ft diameter for the second mile.

There are two right-angled easy bends in each miles, and the difference of head between entrance and discharge ends of the pipe is 100 ft. Calculate the discharge in gallons per minute and all losses in the pipe if the Darcy coefficient is 0·008.

Draw the hydraulic gradient. (*Lond. Univ.*) *Ans.* 1,780 gal/min.

9. Two reservoirs A and B discharge through circular pipes each 2 ft in diameter and 1 mile long to a junction at D. Drom D the joint discharge is carried in a straight line with the discharge pipe from A to a third reservoir C by a 3 ft diameter pipe of negligible length. The surface level at A is 50 ft, and that of B 30 ft above that of C. Neglecting all losses other than pipe friction, find the discharge in gallons per minute from each reservoir.

Darcy's frictional coefficient $= 0{\cdot}0075$. (*Lond. Univ.*)

Ans. 7,500 and 5,800 gal/min.

10. A high-level reservoir feeds two low-service reservoirs by means of a single main 5 miles long, 30 in. in diameter, laid at a slope of 10 ft per mile. The main is then forked, and one branch, 2 miles long with a fall of 15 ft per mile, serves one reservoir, whilst the other is served by a pipe 3 miles long with a fall of 12 ft per mile. Calculate the diameters of these branch pipes so that each may deliver 4,000,000 gal per day of 24 hr. Take $f = 0{\cdot}006$. (*Lond. Univ.*) *Ans.* 1·52 ft and 1·6 ft.

11. A cylindrical tank 16 ft in diameter discharges through a pipe 300 ft long and 9 in. in diameter. Find the time taken to lower the water level in the tank from 9 ft above centre of pipe to 4 ft above centre. $f = 0{\cdot}01$.

Ans. 7·82 min.

12. Air initially at a pressure of 60 Lb/in.^{2} absolute and a temperature of 16°C flows through a 10 in. main which is 1 mile in length. Assuming that the coefficient of resistance to flow is 0·0035, calculate the discharge in cubic feet per second, assuming that the pressure at the delivery end is to be maintained at 55 Lb/in.^{2} absolute. (*Lond. Univ.*) *Ans.* 23·3 ft^3/sec.

13. Air, initially at atmospheric pressure and 60°F, flows under a pressure difference of 10 in. of water through a 12 in. main 1,000 yd long. Assuming that the coefficient of resistance to flow f is 0·004, determine the number of cubic feet of air delivered per second. (*Lond. Univ.*) *Ans.* 24·2 ft^3/sec.

14. In a water-power scheme, the total head is 503 ft, and 1,750,000 gal of water are available per hour for utilization in an impulse turbine of the Pelton type. The proposed pipe line is 2 miles long.

Determine the diameter of the pipe necessary in order that the efficiency of transmission should be 80 per cent, and also calculate the horse-power available.

If the power is supplied to the Pelton wheel through two nozzles, determine their diameter.

Neglect the losses at inlet to the pipe and at the nozzles. $f = 0{\cdot}0075$. (*Lond. Univ.*) *Ans.* 3·44 ft; 3,550; 0·555 ft.

15. In hydraulic transmission of power, state the losses which occur, and explain how they may be minimized. 100 h.p. is to be transmitted, the pressure at the inlet of the pipe being 1,000 Lb/in.^{2} If the pressure drop per mile is to be 10 Lb/in.^{2}, and if $f = 0{\cdot}006$, find the diameter of the pipe and the efficiency of transmission for 10 miles. (*Lond. Univ.*)

Ans. Efficiency = 90 per cent; 0·477 ft.

16. What is meant by "critical velocity" in fluid motion? State what factors in general have an effect on the value of this. (*A.M.I. Mech. E.*)

17. The resistance to the motion, in the direction of its plane, of a thin flat body through water is proportional to v^2, and, at 10 ft/sec, is 0·5 Lb/ft.^{2} Determine the horse-power required to rotate at 1,200 r.p.m. a submerged disc 2 ft in diameter. (*A.M.I. Mech. E.*) *Ans.* 45·4.

18. Determine the levels of the hydraulic gradient at the points B, C, and D of a pipe line discharging 12 ft^3/sec. The initial level of the gradient at A is 400 ft above datum. AB is 24 in. in diameter and 5,000 ft long; BC is 18 in. in diameter and 4,000 ft long, and CD is 20 in. in diameter and 3,000 ft long. Short taper pipes are introduced at B and C. $f = 0{\cdot}01$. (*A.M.I.C.E.*)
Ans. 377·4 ft; 300·7 ft; 266·5 ft.

19. Reservoir A at an elevation of 900 ft supplies water to reservoirs B and C at levels respectively of 600 ft and 500 ft. From A to D both supplies pass through a common pipe 12 in. in diameter and 10 miles long; the branch D to B is 9 in. in diameter and 6 miles long, and that from D to C is 6 in. in diameter and 5 miles long. How many cubic feet per second will be delivered to B and C? $f = 0{\cdot}01$. (*A.M.I.C.E.*) *Ans.* 1·00 ft^3/sec; 0·7 ft^3/sec.

20. The reservoir from which a Pelton wheel is supplied has an elevation of 1,050 ft. The pipe line is 18 in. in diameter and 9,660 ft long; it terminates at a level of 50 ft in a nozzle which gives a jet with an effective diameter of 3 in. Taking for the nozzle $C_v = 0{\cdot}97$, and for the pipe $f = 0{\cdot}006$, determine the horse-power of the jet. (*A.M.I.C.E.*) *Ans.* 1,100.

21. Two pipe lines of equal length (10,000 ft) are laid in parallel between two reservoirs whose difference of level is 50 ft. If their diameters are respectively 12 in. and 24 in., and if the frictional resistance is given by $h = flv^{1\cdot 8}/d^{1\cdot 2}$, what will be the total discharge? Take $f = 0{\cdot}005$. (*A.M.I.C.E.*)
Ans. 5·8 ft^3/sec.

22. A 6-in. pipe line 10,000 ft long is supplied with water at 1,200 Lb/in.^{2} pressure. The coefficient f in the formula $h = flv^2/2gm$ is 0·01. What is the maximum rate, in horse-power, at which energy can be delivered at the outlet from the pipe line? (*A.M.I.C.E.*) *Ans.* 355 h.p.

23. A thin flat disc enclosed in a casing containing water is to be used as a hydraulic dynamometer for absorbing and measuring the output from a petrol engine running at 1,800 r.p.m. Experiments on a similar type of surface show that its frictional resistance per square foot is equal to $0{\cdot}005v^2$ Lb, where v is the velocity in feet per second. What diameter of disc will be necessary if the engine develops 50 b.h.p.? (*A.M.I. Mech. E.*) *Ans.* 1·6 ft.

24. The loss of head in a given pipe line is proportional to v. The following are corresponding experimental values of h and of v—

h	1·5	4·5	8·0	12·0
v	2·0	3·5	4·8	6·0

What is the value of n? (*A.M.I. Mech. E.*) *Ans.* $n = 1{\cdot}97$.

25. What is meant by "critical velocity" in pipe flow. Describe how you could determine, experimentally, the value of the lower critical velocity. (*A.M.I.C.E.*)

26. Two reservoirs whose surface levels differ by 100 ft are connected by a pipe 2 ft in diameter and 10,000 ft long. The pipe line crosses a ridge whose summit is 30 ft above the level of, and 1,000 ft distant from, the higher reservoir. Find the minimum depth below the ridge at which the pipe must be laid if the absolute pressure in the pipe is not to fall below 10 ft of water, and calculate the discharge in cubic feet per second. ($f = 0{\cdot}0075$.) (*Lond. Univ.*) *Ans.* 16·66 ft; 20·5 ft³/sec.

27. Calculate the diameter of a pipe to convey gas from a holder to a power station, having given the following particulars: Gas consumption, 20,000 ft³/hr; length of pipe, 0·5 mile; delivery, 50 ft above the entrance to the pipe; pressure at holder, 4 in. of water, and at the power station 2 in.; density of gas 0·045 and of air 0·08 Lb/ft³. Take $f = 0{\cdot}005$. (*Lond. Univ.*) *Ans.* 8·25 in.

28. A chimney is 100 ft high and 5 ft in diameter. The horizontal flue is 60 ft long and of the same section. The flue gas temperature is 250°C, and its speed 20 ft/sec. Estimate the draught available at the boiler in inches of water. Take $f = 0{\cdot}01$ and weight of gas per cubic foot = 0·078 Lb at N.T.P. (*I.Mech.E.*) *Ans.* 0·0518 in. of water.

29. Compressed air is transmitted through 300 ft of 2-in. pipe. The supply pressure is 100 Lb/in.² and the flow is 80 ft³/min at the supply end. Calculate the delivery pressure assuming the temperature remains at 15°C throughout, and that $pV = 96T$ for 1 Lb of air. Prove any formula used. Take $f = 0{\cdot}005$. (*Lond. Univ.*) *Ans.* 97·2 Lb/in.²

30. Compressed air is transmitted through 5,000 ft of 2-in. diameter pipe. The inlet pressure is maintained at 100 Lb/in.² (gauge) and the temperature throughout the pipe is 15°C. Calculate the maximum flow expressed in cubic feet of free air per minute at 15°C and 14·7 Lb/in.² (absolute) if the exit pressure is not to fall below 80 Lb/in.² (gauge). Assume a friction coefficient $f = 0{\cdot}004$ and for air $pV/T = 96$.

Prove any formula used to allow for changing density. (*Lond. Univ.*)
Ans. 231 ft³/min.

31. If the water supply to a Pelton wheel is shut off by means of the spear valve in such a way that the retardation of the water in the pipe is at a uniform rate, show that, if the time of closure of the valve is the same as the time taken for sound to travel the length of the pipe in water, the rise of pressure at the valve is $p = v\sqrt{(K\rho)}$, in which v is the initial velocity, K the modulus of compressibility, and ρ the density of the water.

If the valve is closed instantaneously, prove that the rise of pressure in the pipe is given by the same formula.

Assume that the velocity of sound in water is $v_s = \sqrt{(K/\rho)}$, and the pipe is perfectly rigid. (*Lond. Univ.*)

32. Deduce an expression for the rise in pressure in a pipe due to the *sudden* closing of a valve when water is flowing through the pipe, allowing both for the compressibility of the water and the expansion of the pipe.

A steel pipe is 3 in. in diameter, and ¼ in. thick. What is the highest velocity of the water which can be suddenly stopped without stressing the pipe to more than 10,000 Lb/in.²? E for steel = 30×10^6 Lb/in.² Bulk modulus for water = 300,000 Lb/in.² (*Lond. Univ.*) *Ans.* 13·07 ft/sec

CHAPTER 8

VISCOUS FLOW OF FLUIDS

8.1. Laminar and Turbulent Flow. Exhaustive experiments on the flow of fluids in round pipes were carried out by Stanton and Pannell.* Their results were plotted with log R_e as base and $R/\rho v^2$ as ordinate, where R is the viscous resistance per square foot of wetted surface. The graph obtained is shown in Fig. 105. The portion AB of the graph represents experimental results on viscous or laminar flow, the point B being the lower critical velocity. The portion BC represents the results for the state of change from laminar flow to turbulent flow, the point C being the higher critical velocity. The portion CD of the curve represents experimental results on turbulent flow.

At the lower critical velocity, point B, it was found from the curve that

$$R_e = \frac{vd}{\nu} = 2{,}000$$

At the higher critical velocity, point C, it was found that

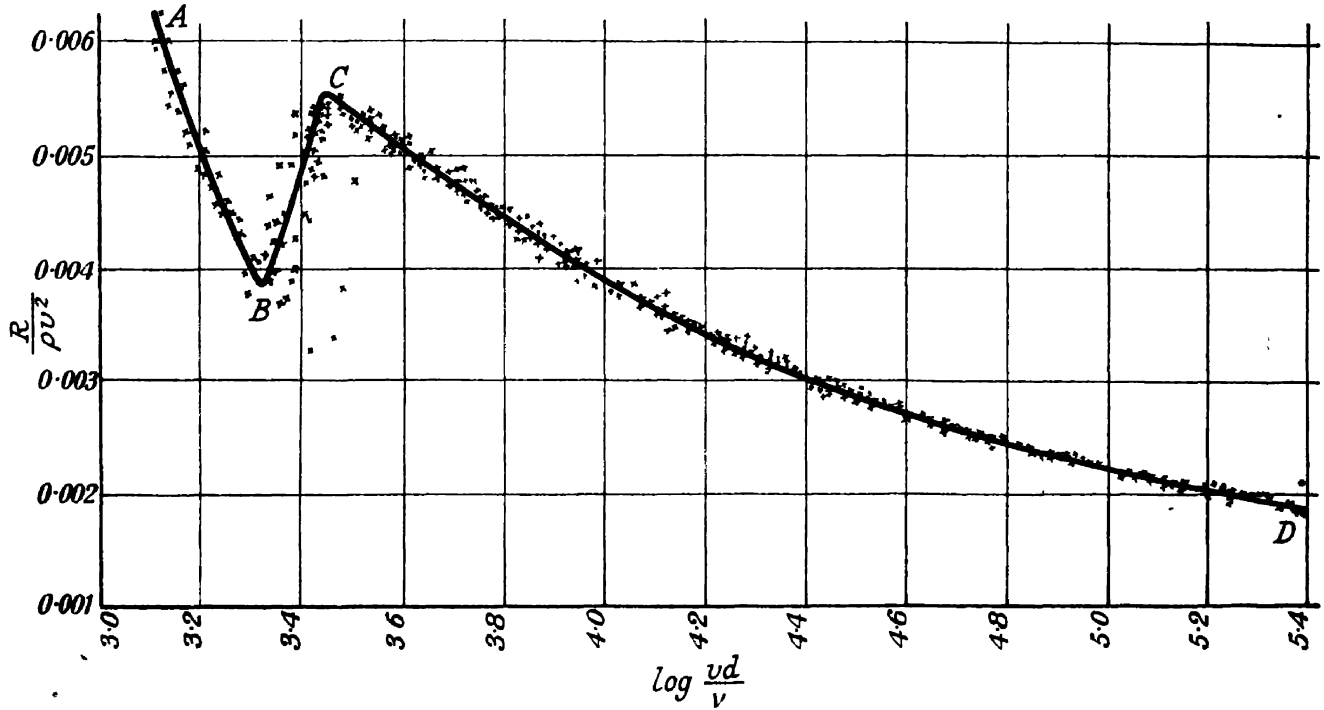
$$R_e = \frac{vd}{\nu} = 2{,}500$$

Hence it follows that, for any fluid in motion, if the value of R_e is less than 2,000 the flow is laminar; if the value of R_e is greater than 2,500 the flow is turbulent. Between these two values the fluid would be in a state of transition from one type of flow to the other. These values hold for all fluids, at all velocities and temperatures. It will be noticed that these results agree with those obtained by Reynolds (§ 7.7).

Stanton and Pannell also plotted on their curve the results of former experimenters on pipe flow and found that these results approximated to their own curve. The complete results plotted by Stanton and Pannell included experiments on the flow of water, air, and oil, through pipes varying in diameter from small capillary tubes to large water supply pipes of 18 ft diameter. It was found that, excepting for a slight deviation due to the roughness of the inside of the large pipes, they all followed the curve of Fig. 105.†

* *Phil. Trans.*, **214.**

† For latest research on pipe flow, see results of Nikuradse, Prandtl, and Von Kármán, § 9.3 and § 9.4.


A
B
C
D
0·006
0·005
0·004
0·003
0·002
0·001
$\frac{R}{\rho v^2}$
3·0
3·2
3·4
3·6
3·8
4·0
4·2
4·4
4·6
4·8
5·0
5·2
5·4
$\log \frac{vd}{\nu}$

EXAMPLE 1

The density of a fluid A is 0·8, and its coefficient of viscosity is 0·01 in c.g.s. units. The density of a second fluid B is 0·6 and its coefficient of viscosity is 0·005. Which will have the lower critical velocity under given conditions of flow? What will be the ratio of the critical velocities? (*Lond. Univ.*)

Fluid A

$$\nu = \frac{\eta}{\rho} = \frac{0{\cdot}01}{0{\cdot}8} = 0{\cdot}0125$$

For critical velocity,
$$R_e = \frac{v_c d}{\nu} = 2{,}000$$

Hence
$$v_c = \frac{0{\cdot}0125 \times 2{,}000}{d}$$
$$= \frac{25}{d} \text{ cm/sec}$$

Fluid B

$$\nu = \frac{\eta}{\rho} = \frac{0{\cdot}005}{0{\cdot}6} = 0{\cdot}00833$$

Then
$$v_c = \frac{\nu \times 2{,}000}{d}$$
$$= \frac{0{\cdot}00833 \times 2{,}000}{d}$$
$$= \frac{16{\cdot}66}{d}$$

Hence, fluid B has the lower critical velocity.

$$\text{Ratio of critical velocities} = \frac{25/d}{16{\cdot}66/d}$$
$$= 1{\cdot}5$$

8.2. Laminar Flow through Round Pipes. An equation for the viscous resistance in a round pipe may be obtained by equating the force on the fluid, due to the drop in pressure, to the viscous resistance. The definition of viscosity and viscous stress has already been dealt with in § 3.16.

Consider a fluid to be flowing along a pipe, the cross-section of which is as shown in Fig. 106, and consider a cylinder of fluid of radius y and of unit length.

Let r = radius of pipe,
v = velocity of fluid,
i = slope of hydraulic gradient,

$= $ head lost in resistance per unit length of pipe,

$p = $ pressure drop per unit length.

Viscous resistance of cylinder of radius y and unit length

$$= \text{surface area} \times \text{viscous stress}$$

$$= 2\pi y \times q$$

$$= -2\pi y \eta \frac{dv}{dy} \quad . \quad . \quad . \quad (1)$$

by substituting for q from eq. (32), § 3.16. It should be noted that dv/dy will be negative as y is now measured outwards from the centre. In § 3.16 y was measured inwards from the sides.

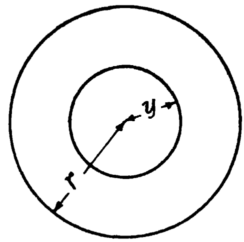


Fig. 106

Difference of total pressure on ends of cylinder

$$= \pi y^2 \times p$$

$$= \pi y^2 \rho g i \quad . \quad . \quad . \quad . \quad (2)$$

where $p = $ weight per cubic foot $\times$ loss of head per unit length

$$= \rho g \times i$$

as $\rho = \dfrac{w}{g}$ in engineers' units

As the viscous resistance of the cylinder must equal the net force on the ends of the cylinder, eq. (1) will equal eq. (2). Hence,

$$-2\pi y \eta \frac{dv}{dy} = \pi y^2 \rho g i$$

from which $\qquad dv = -\dfrac{\rho g i y \, dy}{2\eta}$

Integrating, $\qquad v = -\dfrac{\rho g i y^2}{4\eta} + c_1 \quad . \quad . \quad . \quad (3)$

where c_1 is the constant of integration.

When $y = r$, $v = 0$, as the fluid is stationary at the sides. Hence

$$0 = -\frac{\rho g i r^2}{4\eta} + c_1$$

from which $\qquad c_1 = \dfrac{\rho g i r^2}{4\eta}$

Let $v_y = $ velocity at radius y.

Then, from eq. (3),

$$v_y = \frac{\rho g i}{4\eta}(r^2 - y^2). \quad . \quad . \quad . \quad (4)$$

It will be noticed from this equation that maximum velocity occurs when $y = 0$; then

$$\text{maximum velocity} = \frac{\rho g i r^2}{4\eta}$$

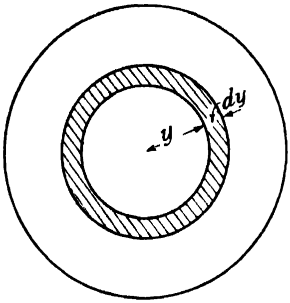


FIG. 107

Next consider a hollow cylinder of the fluid of radius y and thickness dy (Fig. 107). Let Q be the total quantity flowing through the pipe per second.

Then, quantity flowing through hollow cylinder

$$= dQ = \text{area} \times \text{velocity}$$

$$= 2\pi y \, dy \times v_y$$

Substituting for v_y from eq. (4),

$$= 2\pi y \, dy \frac{\rho g i}{4\eta}(r^2 - y^2)$$

Integrating between $y = r$ and $y = 0$,

$$Q = \frac{\pi \rho g i}{2\eta}\int_0^r (r^2 y - y^3) dy$$

$$= \frac{\pi \rho g i}{2\eta}\left[\frac{r^2 y^2}{2} - \frac{y^4}{4}\right]_0^r$$

$$= \frac{\pi \rho g i r^4}{8\eta}$$

Let $\quad v =$ mean velocity of flow in pipe

$$= \frac{Q}{\text{area of cross-section}}$$

$$= \frac{\pi \rho g i r^4}{8\eta} \div \pi r^2$$

$$= \frac{\rho g i r^2}{8\eta}$$

$$= \frac{i g r^2}{8\nu} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

as $\nu = \eta/\rho$ (§ 3.16).

It will be noticed that the mean velocity is one-half of the maximum value.

For a pipe flowing full, the hydraulic mean depth m is $r/2$; hence, substituting in eq. (5), and calling the diameter of the pipe d,

$$m i g = \frac{8\nu v}{d}$$

Dividing both sides by v^2,

$$\frac{mig}{v^2} = 8\left(\frac{vd}{\nu}\right)^{-1}$$

This may be written

$$\frac{mig}{v^2} = C\left(\frac{vd}{\nu}\right)^{n} \quad . \quad . \quad . \quad . \quad (6)$$

where C and n are constants, their values depending on whether the flow is laminar or turbulent.

Let R = viscous resistance per unit area of wetted surface.

Then, as resistance at sides is equal to net force on fluid,

$$2\pi r \times R = p \times \pi r^2$$
$$= i\rho g \times \pi r^2 \text{ (as } p = i\rho g)$$

Hence

$$R = \frac{\rho i g r}{2}$$

Putting $m = r/2$ and dividing both sides by v^2,

$$\frac{R}{\rho v^2} = \frac{mig}{v^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Combining eqs. (6) and (7),

$$\frac{R}{\rho v^2} = \frac{mig}{v^2} = C\left(\frac{vd}{\nu}\right)^{n} \quad . \quad . \quad . \quad (8)$$

which is the complete form of the equation for the flow of fluids in pipes.

It will be noticed that the term vd/ν is equal to the Reynolds number of the flow; hence, eq. (8) may be written

$$\frac{R}{\rho v^2} = \frac{mig}{v^2} = C(R_e)^n$$

If the term vd/ν is less than 2,000, the flow will be laminar and will belong to the portion AB of the curve in Fig. 105. Then by plotting log $(R/\rho v^2)$ and log (vd/ν) of this portion of the curve the values of the constants C and n for laminar flow may be obtained.

If the term vd/ν is more than 2,500, the flow will be turbulent and will be represented by the portion CD of the curve in Fig. 105. Then, by plotting log $(R/\rho v^2)$ and log (vd/ν) for this portion of the curve the values of the constants C and n for turbulent flow are obtained. From these curves it is found that—

for laminar flow, $\dfrac{vd}{\nu}$ is less than 2,000

and

$$C = 8$$
$$n = -1$$

for turbulent flow, $\frac{vd}{\nu}$ is greater than 2,500

and
$$C = 0{\cdot}032$$
$$n = -0{\cdot}23$$

In all problems on pipe flow the value of vd/ν must first be worked out; then, if less than 2,000, use the values $C = 8$ and $n = -1$, and solve from eq. (6). If the value of vd/ν is more than 2,500, use the values $C = 0{\cdot}032$ and $n = -0{\cdot}23$, and solve from eq. (6).

It is interesting to notice that eq. (6) is another form of the well-known Darcy formula (§ 7.6). By substituting in eq. (6), $m = d/4$ and $i = h_f/l$ the equation becomes—

$$h_f = \frac{4flv^2}{2gd}$$

where the coefficient
$$f = 2C\left(\frac{vd}{\nu}\right)^n$$
$$= 2CR_e^n \qquad (9)$$

It follows from this that the Darcy coefficient f is not a true constant, but varies with R_e; it is a function of the velocity, the diameter, the density, and the temperature.

The viscosity formula of eq. (6) is an alternative method for calculating the flow of fluids in pipes and gives the same results as the Darcy formula if the value of Darcy's coefficient f is calculated from eq. (9).

Using eq. (9), and inserting the above values of C and n,

$$\text{Darcy's } f = \frac{16}{R_e} \text{ for laminar flow}$$
$$= \frac{0{\cdot}064}{R_e^{0{\cdot}23}} \text{ for turbulent flow}$$

These values of f can now be substituted in the Darcy formula

$$h_f = \frac{4flv^2}{2gd}$$

This method is often more convenient than eq. (6), because it gives the head lost in terms of the velocity head.

EXAMPLE 2

Water flows along a pipe of $\frac{1}{4}$ in. diameter and 100 ft long; the pipe is running full. Find the loss of head when: (*a*) the temperature is 5°C and the velocity is 1 ft/sec, (*b*) the temperature is 70°C and the velocity is 10 ft/sec.

(*a*) From eq. (35), § 3.16,

$$\nu = \frac{0{\cdot}00001929}{1 + (0{\cdot}03368 \times 5) + (0{\cdot}000221 \times 25)}$$
$$= 0{\cdot}0000164 \text{ (ft-sec units)}$$

Next find the value of the term vd/ν.

$$R_e = \frac{vd}{\nu} = \frac{1}{0{\cdot}0000164} \times \frac{1}{4 \times 12} = 1{,}270$$

As this is less than 2,000, the flow is laminar; hence $n = -1$ and $C = 8$.

Applying eq. (6),

$$\frac{mig}{v^2} = 8\left(\frac{vd}{\nu}\right)^{-1} = 8(1{,}270)^{-1} = 0{\cdot}0063$$

from which $$i = \frac{v^2}{\frac{d}{4}g} \times 0{\cdot}0063$$

$$= \frac{1 \times 0{\cdot}0063 \times 4}{\frac{1}{4} \times \frac{1}{12} \times 32{\cdot}2} = 0{\cdot}0376 \text{ ft}$$

But $$h_f = i \times l$$

$$= 0{\cdot}0376 \times 100 = 3{\cdot}76 \text{ ft of water}$$

(*b*) From § 3.16

$$\nu = \frac{0{\cdot}00001929}{1 + (0{\cdot}03368 \times 70) + (0{\cdot}000221 \times 4{,}900)}$$

$$= 0{\cdot}00000435$$

Then $$R_e = \frac{vd}{\nu} = \frac{10}{0{\cdot}00000435 \times 4 \times 12} = 48{,}000$$

As this is more than 2,500, the flow is turbulent; hence $n = -0{\cdot}23$ and $C = 0{\cdot}032$.

Using eq. (6),

$$\frac{mig}{v^2} = 0{\cdot}032(48{,}000)^{-0{\cdot}23}$$

$$= 0{\cdot}00272$$

from which $$i = \frac{0{\cdot}00272v^2}{\frac{d}{4} \times g}$$

$$= \frac{0{\cdot}00272 \times 100 \times 4}{\frac{1}{4} \times \frac{1}{12} \times 32{\cdot}2}$$

$$= 1{\cdot}62$$

But $$h_f = i \times l$$

$$= 1{\cdot}62 \times 100$$

$$= 162 \text{ ft of water}$$

The values of ν could also have been obtained from the viscosity coefficient curve of Appendix 2.

EXAMPLE 3

Oil at a temperature of 60°F has a weight of 57·2 Lb/ft³ and a kinematic viscosity of 0·0205 ft-sec units. Find the horse-power required to pump 20 tons of this oil per hour along a pipe line 6 in. in diameter and 1,000 ft long.

$$\text{Quantity per second} = \frac{\text{weight per second}}{\text{weight per cubic foot}}$$

$$= \frac{20 \times 2{,}240}{57{\cdot}2 \times 3{,}600} = 0{\cdot}218 \text{ ft}^3$$

$$v = \frac{Q}{\text{area}} = \frac{0{\cdot}218}{\frac{\pi}{4} \times \frac{1}{4}} = 1{\cdot}11 \text{ ft/sec}$$

Then $$R_e = \frac{vd}{\nu} = \frac{1{\cdot}11 \times \frac{1}{2}}{0{\cdot}0205} = 27{\cdot}1$$

As this is less than 2,000, the flow is laminar; hence $n = -1$ and $C = 8$.

Using eq. (6),

$$\frac{mig}{v^2} = 8(27{\cdot}1)^{-1}$$

$$= 0{\cdot}296$$

from which $$i = \frac{0{\cdot}296 \times (1{\cdot}11)^2 \times 4}{\frac{1}{2} \times 32{\cdot}2}$$

$$= 0{\cdot}0907$$

But $$h_f = i \times l$$

$$= 0{\cdot}0907 \times 1{,}000$$

$$= 90{\cdot}7 \text{ ft of oil}$$

Then $$\text{horse-power required} = \frac{wh_f}{550}$$

$$= \frac{20 \times 2{,}240}{3{,}600} \times \frac{90{\cdot}7}{550}$$

$$= 2{\cdot}05$$

8.3. Laminar Flow between Flat Surfaces. The viscous, or laminar, flow of a fluid between parallel surfaces may be dealt with in a manner similar to the previous article. This type of flow occurs when leakage takes place between two surfaces, such as between the piston and the cylinder walls.

Consider two parallel surfaces at a distance of t apart through which a fluid is flowing; let b be the breadth of the surfaces and consider unit length. Fig. 108 represent a cross-sectional view of

the surfaces, perpendicular to the direction of flow. Consider a central layer of the fluid of thickness $2y$; that is, its boundaries are y from the centre. Then, the longitudinal viscous resistance of the layer will equal the longitudinal force on the ends due to the pressure drop.

Viscous resistance of layer per unit length

$$= \text{stress} \times \text{wetted area}$$

$$= -\eta \frac{\mathrm{d}v}{\mathrm{d}y} \times 2b$$

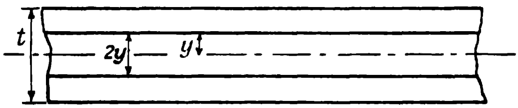


FIG. 108

This will be negative, as y in the term $\mathrm{d}v/\mathrm{d}y$ (§ 3.16) was measured from the sides.

Longitudinal force on layer per unit length

$$= \text{cross-sectional area} \times \text{pressure drop}$$

$$= 2by \times \rho gi$$

as pressure drop $= \rho gh$, and $h = i$ for unit length.

Hence, equating these two equations,

$$-\eta \frac{\mathrm{d}v}{\mathrm{d}y} 2b = 2by\rho gi$$

from which
$$\mathrm{d}v = -\frac{\rho giy\,\mathrm{d}y}{\eta}$$

Integrating,
$$v = -\frac{\rho gi}{\eta}\int y\,\mathrm{d}y$$

$$= -\frac{\rho giy^2}{2\eta} + C_1 \quad . \quad . \quad . \quad . \quad (10)$$

where C_1 is the constant of integration.

When $y = t/2$, $v = 0$, as there is no motion at the sides. Putting this limiting condition in eq. (10),

$$0 = -\frac{\rho git^2}{8\eta} + C_1$$

from which
$$C_1 = \frac{\rho git^2}{8\eta}$$

Substituting this value in eq. (10),

$$v_y = \frac{\rho gi}{2\eta}\left(\frac{t^2}{4} - y^2\right) \quad . \quad . \quad . \quad . \quad (11)$$

where v_y is the velocity at any distance y.

The maximum velocity in the whole cross-section occurs at $y = 0$. Then, from eq. (11),

$$\text{maximum velocity} = \frac{\rho g i t^2}{8\eta} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Next consider a thin layer of the fluid at y from the centre and thickness dy (Fig. 109); let dQ be the quantity flowing through layer per second. Then,

$$dQ = b\,dy \cdot v_y$$

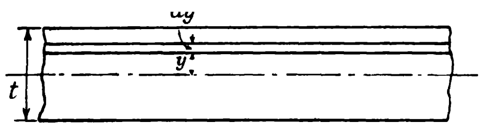


FIG. 109

Substituting for v_y from eq. (11),

$$dQ = b\,dy \times \frac{\rho g i}{2\eta}\left(\frac{t^2}{4} - y^2\right)$$

Integrating for total quantity between the surfaces,

$$Q = \frac{b\rho g i}{2\eta}\int_{-t/2}^{t/2}\left(\frac{t^2}{4} - y^2\right)dy$$

$$= \frac{b\rho g i}{2\eta}\left[\frac{t^2 y}{4} - \frac{y^3}{3}\right]_{-t/2}^{t/2}$$

$$= \frac{b\rho g i t^3}{12\eta}$$

$$\text{Mean velocity of flow} = v = \frac{Q}{\text{cross-sectional area}}$$

$$= \frac{b\rho g i t^3}{12\eta} \times \frac{1}{bt}$$

$$= \frac{\rho g i t^2}{12\eta} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

It will be noticed by comparing eqs. (12) and (13) that the mean velocity is two-thirds of the maximum velocity.

By substituting the values $m = t/2$ and $\nu = \eta/\rho$ eq. (13) becomes—

$$\frac{mig}{v^2} = 6\left(\frac{vt}{\nu}\right)^{-1} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

which is a similar form to eq. (6), which was deduced from a round pipe.

It should be noticed that the term vt/ν is a function of R_e for this type of flow.

Let R = viscous resistance per square foot of wetted surface.

Then, viscous resistance per unit length

= area of cross-section × pressure drop per unit length

Or $R \times 2b = p \times tb$

from which $R = \rho gi \times \dfrac{t}{2}$ (as $p = \rho gi$)

$$= \rho gim$$

Dividing throughout by v^2,

$$\frac{R}{\rho v^2} = \frac{mig}{v^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Combining eqs. (14) and (15),

$$\frac{R}{\rho v^2} = \frac{mig}{v^2} = C\left(\frac{vt}{\nu}\right)^n \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where C and n are constants depending on the type of flow and which are determined from experimental results. It will be noticed from eq. (14) that, for laminar flow, $C = 6$ and $n = -1$.

It will also be noticed that eq. (14) reduces to the same form as the Chezy formula, $v = C\sqrt{(mi)}$, for the flow in open channels (§ 10.2), if the Chezy constant C is written

$$\sqrt{\frac{vtg}{6\nu}}$$

But $$\sqrt{\frac{vtg}{6\nu}} = \sqrt{\frac{gR_e}{6}}$$

Hence, the Chezy constant C varies with R_e, so that

$$C = k\sqrt{R_e}$$

EXAMPLE 4

The radial clearance between a hydraulic plunger and the cylinder walls is 0·004 in.; the length of the plunger is 12 in. and the diameter 4 in. Find the velocity of leakage and the rate of leakage past the plunger at an instant when the difference of pressure between the two ends of the plunger is 30 ft of water. The temperature of the water is 10°C.

The flow through the clearance area will be the same as the flow between two parallel surfaces.

Assuming the whole of the pressure head is lost in friction,

$$i = \frac{h_f}{l} = \frac{30}{1}$$

From eq. (35), § 3.16,

$$\nu = \frac{0{\cdot}00001929}{1 + (0{\cdot}03368 \times 10) + (0{\cdot}000221 \times 100)}$$

$$\frac{0{\cdot}00001929}{1{\cdot}359} = 0{\cdot}0000141 \text{ (ft-sec units)}$$

This value could also have been obtained from the curve of Appendix 2.

From eq. (13),

$$\text{mean velocity} = v = \frac{git^2}{12\nu}$$

$$= \frac{32{\cdot}2 \times 30 \times 0{\cdot}004^2}{12 \times 0{\cdot}0000141 \times 144}$$

$$= 0{\cdot}634 \text{ ft/sec}$$

$$\text{Rate of flow} = Q = v \times \text{area of annular ring}$$

$$= v \times \pi D t$$

$$= 0{\cdot}634 \times \pi \times \frac{4}{12} \times \frac{0{\cdot}004}{12}$$

$$= 0{\cdot}000221 \text{ ft}^3\text{/sec}$$

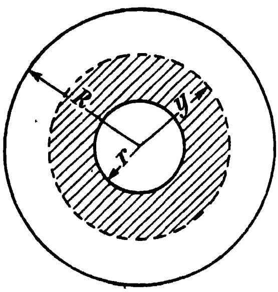


Fig. 110

8.4. Laminar Flow through Annular Space. The viscous, or laminar, flow of a fluid through the annular space between an inner and outer tube can be found in the same manner as in § 8.2 and § 8.3. Consider the annular space formed by the two tubes of Fig. 110.

Let R = radius of outer surface,
r = radius of inner surface,
q = viscous stress at inner surface per unit area.

Consider a hollow cylinder of fluid of external radius y, internal radius r, and of unit length.

Let q_y = viscous stress at radius y

$$= -\eta \frac{dv}{dy}$$

Then, viscous resistance of section considered

$$= \text{wetted area} \times \text{stress}$$

$$= 2\pi y q_y + 2\pi r q$$

$$= -2\pi y \eta \frac{dv}{dy} + 2\pi r q$$

Pressure drop on unit length $= \rho g i$ (§ 8.2)

Longitudinal force on section considered

$$= \text{cross-sectional area} \times \text{pressure drop}$$

$$= \pi(y^2 - r^2) \times \rho g i$$

Equating these equations,

$$- 2\pi y\eta \frac{dv}{dy} + 2\pi rq = \pi(y^2 - r^2)\rho gi$$

Then
$$dv = \frac{(2rq - \rho giy^2 + \rho gir^2)\,dy}{2\eta y}$$

$$= \left(\frac{rq}{\eta y} - \frac{\rho giy}{2\eta} + \frac{\rho gir^2}{2\eta y}\right) dy$$

Integrating,
$$v = \frac{rq}{\eta} \log_e y - \frac{\rho gi}{4\eta} y^2 + \frac{\rho gir^2}{2\eta} \log_e y + C \quad . \ (17)$$

where C is the constant of integration.

When $y = R, v = 0$. Hence

$$0 = \frac{rq}{\eta} \log_e R - \frac{\rho giR^2}{4\eta} + \frac{\rho gir^2}{2\eta} \log_e R + C$$

from which
$$C = \frac{\rho giR^2}{4\eta} - \frac{rq}{\eta} \log_e R - \frac{\rho gir^2}{2\eta} \log_e R$$

Substituting this value of C in eq. (17),

$$v = \frac{rq}{\eta} \log_e \frac{y}{R} + \frac{\rho gir^2}{2\eta} \log_e \frac{y}{R} + \frac{\rho gi}{4\eta} (R^2 - y^2) \quad . \ (18)$$

When $y = r$, $v = 0$. Hence

$$0 = \frac{rq}{\eta} \log_e \frac{r}{R} + \frac{\rho gir^2}{2\eta} \log_e \frac{r}{R} + \frac{\rho gi}{4\eta} (R^2 - r^2)$$

from which
$$q = - \frac{\rho gir}{2} - \frac{\rho gi}{4} \frac{(R^2 - r^2)}{r \log_e \frac{r}{R}}$$

Substituting this value of q in eq. (18),

$$v = \frac{\rho gi}{4\eta} \left[(R^2 - y^2) - (R^2 - r^2) \frac{\log_e \frac{y}{R}}{\log_e \frac{r}{R}} \right] \quad . \ (19)$$

Eq. (19) gives the variation of velocity between the radii r and R. Differentiating for a maximum,

$$\frac{dv}{dy} = 0 = - 2y + \frac{R^2 - r^2}{y \log_e \frac{R}{r}}$$

from which
$$y = \sqrt{\frac{R^2 - r^2}{2 \log_e \frac{R}{r}}} \quad . \quad . \quad . \quad . \quad . \quad . \ (20)$$

By substituting this value of y in eq. (19), the value of the maximum velocity is obtained.

The quantity of flow through the annular space can be found by considering a thin ring of radius y and thickness $\mathrm{d}y$ (Fig. 111); the velocity through this ring is given by eq. (19). Then,

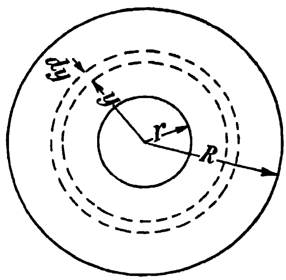


FIG. 111

$$\mathrm{d}Q = \text{area of ring} \times \text{velocity}$$

$$= 2\pi y\,\mathrm{d}y \times \frac{\rho g i}{4\eta}\left[(R^2 - y^2) - (R^2 - r^2)\frac{\log_e \dfrac{y}{R}}{\log_e \dfrac{r}{R}}\right]$$

Integrating between r and R,

$$Q = \frac{\rho g i \pi}{2\eta}\int_r^R \left[R^2 y - y^3 - (R^2 - r^2)y\frac{\log_e \dfrac{y}{R}}{\log_e \dfrac{r}{R}}\right]\mathrm{d}y$$

$$= \frac{\rho g i \pi}{2\eta}\left[\frac{R^2y^2}{2} - \frac{y^4}{4} - \frac{(R^2 - r^2)}{\log_e \dfrac{r}{R}}\left(\frac{y^2}{2}\log_e \frac{y}{R} - \frac{y^2}{4}\right)\right]_r^R$$

$$= \frac{\rho g i \pi}{2\eta}\left[\frac{R^4}{4} + \frac{R^2(R^2 - r^2)}{4\log_e \dfrac{r}{R}}\right]$$

$$- \left[\frac{R^2r^2}{2} - \frac{r^4}{4} - \frac{r^2(R^2 - r^2)}{2\log_e \dfrac{r}{R}}\left(\log_e \frac{r}{R} - \frac{1}{2}\right)\right]$$

$$= \frac{\rho g i \pi}{4\eta}(R^2 - r^2)\left[\frac{(R^2 + r^2)}{2} + \frac{(R^2 - r^2)}{2\log_e \dfrac{r}{R}}\right]$$

$$\text{Mean velocity} = \frac{Q}{\text{area of flow}}$$

$$= \frac{Q}{\pi(R^2 - r^2)}$$

$$= \frac{\rho g i}{8\eta}\left[(R^2 + r^2) + \frac{(R^2 - r^2)}{\log_e \frac{r}{R}}\right] \quad . \quad . \quad . \quad (21)$$

It will be noticed that, if $r = 0$, the problem becomes the same as the viscous flow through a round tube, and eq. (21) reduces to the same form as eq. (5).

In the special case when r is nearly of the same value as R, the annular space is very thin; consequently, eq. (21) would have to be calculated to a very fine degree of accuracy in order to obtain a reasonable result. Obviously, if the thickness of the annular ring is very small compared with R, the problem is the same as the flow between two parallel surfaces (§ 8.3) and a simpler and more accurate solution is obtained by this treatment; this method was applied to the plunger problem given in the worked-out Example (4) at the end of § 8.3.

8.5. Resistance of Oiled Bearings. A shaft revolving in an oiled bearing will be separated from the bearing by a very thin film of oil. The layer of this film adjacent to the bearing will be at rest, whilst the layer adjacent to the shaft will be revolving with the same velocity as the shaft. The resistance of the oil film will, therefore, be due to the viscosity of the oil, and the flow will be laminar on account of the low value of its R_e.

Let t = thickness of oil film,

D = dia. of shaft,

n = speed of shaft in revolutions per minute,

l = length of bearing.

Then, from § 3.16,

$$q = \eta \frac{dv}{dy}$$

where q is the resistance per unit area. In this case, as the oil film is very thin, dv will be the tangential velocity of the shaft and dy will be the average thickness of the oil film.*

Then
$$q = \eta \frac{v}{t}$$

$$= \eta \times \frac{\pi D n}{60t}$$

* Actually, the oil film will not be of uniform thickness. For the application of the principle of dimensional similarity to this problem, see § 11.8.

Tangential resistance on bearing $= q\pi Dl$

$$\text{Resisting torque} = q\pi Dl \times \frac{D}{2}$$

$$= \frac{\eta\pi^2 D^3 ln}{120t}$$

Horse-power absorbed in viscosity

$$= \frac{\text{torque in Lb-ft} \times 2\pi n}{33{,}000}$$

EXAMPLE 5

Define the terms *coefficient of viscosity* and *kinematical viscosity*. A shaft 4 in. in diameter runs in a bearing of length 8 in., the two surfaces being separated by a film of oil 0·001 in. thick. If the coefficient of viscosity of the oil is 1·53 c.g.s. units, find the torque necessary to rotate the shaft at 30 r.p.m. against the viscous resistance of the oil. (*Lond. Univ.*)

$$\eta = 1{\cdot}53 \text{ c.g.s. units}$$

$$= 1{\cdot}53 \times \frac{30{\cdot}5}{453{\cdot}6 \times 32{\cdot}2} \text{ engineers' units}$$

$$= 0{\cdot}0032 \text{ engineers' units}$$

$$\text{Viscous torque} = \frac{\eta\pi^2 D^3 ln}{120t}$$

$$= \frac{0{\cdot}0032 \times \pi^2 \times (\frac{1}{3})^3 \times \frac{2}{3} \times 30}{120 \times \frac{0{\cdot}001}{12}} \text{ Lb-ft}$$

$$= 2{\cdot}34 \text{ Lb-ft}$$

8.6. Viscous Resistance of Collar Bearing. The viscous resistance of a collar bearing can be obtained by assuming the face of the collar to be separated from the bearing surface by a thin film of oil of uniform thickness.

Two views of a collar bearing are shown in Fig. 112. Consider a thin ring of the bearing surface of radius x and thickness dx, and let v be the circumferential velocity at this radius.

Let ω = speed of shaft in radians per second,
q = viscous stress at radius x,
t = thickness of oil film,
R_1 and R_2 = external and internal radii of collar respectively.

Using the equation for viscous stress [eq. (32), § 3.16],

$$q = \eta \frac{dv}{dy}$$

In this problem, $dy = t$ and $dv = v$, as the oil film is very thin and the velocity changes from 0 to v within the thickness t. Hence, the equation may be written—

$$q = \frac{\eta v}{t}$$

$$= \frac{\eta \omega x}{t}, \text{ as } v = \omega x$$

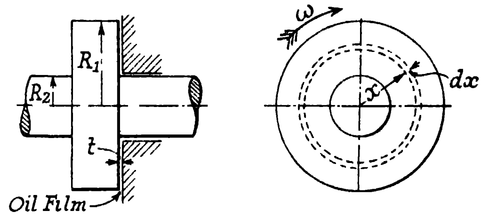


FIG. 112

Tangential viscous force on thin ring

$$= q \times 2\pi x \, dx$$

$$= \frac{2\pi\eta\omega x^2 \, dx}{t}$$

Moment of tangential force on ring

$$= \frac{2\pi\eta\omega x^3 \, dx}{t}$$

If the whole bearing surface is assumed to consist of similar concentric rings, the integration of this equation will give the total torque required to overcome the viscous resistance of the bearing.

Then, total torque required $= T = \dfrac{2\pi\eta\omega}{t}\displaystyle\int_{R_2}^{R_1} x^3 \, dx$

$$= \frac{2\pi\eta\omega}{t}\left[\frac{x^4}{4}\right]_{R_2}^{R_1}$$

$$= \frac{\pi\eta\omega}{2t}(R_1^4 - R_2^4) \quad . \quad . \quad . \quad (22)$$

If T is in Lb-ft units, then,

h.p. required to overcome viscous resistance

$$= \frac{T \times 2\pi \text{ (r.p.m.)}}{33{,}000}$$

If the bearing is of the footstep type, eq. (22) may be applied; in this case R_2 is zero and R_1 is the radius of the shaft.

In applying eq. (22), care should be taken to ensure that all dimensions are in Lb-ft units or c.g.s. units.

EXAMPLE 6

The thrust of a shaft is taken by a collar bearing provided with a forced lubrication system which maintains a film of oil of uniform thickness between the surface of the collar and the surface of the bearing. The external and internal diameters of the collar are 6 in. and 4 in. respectively. If the thickness of the film of oil which separates the surfaces is 0·01 in., and the coefficient of viscosity is 0·91 poise, show that the horse-power lost in overcoming friction when the shaft rotates at 300 r.p.m. is 0·02 very nearly. (*Lond. Univ.*)

In this problem, $\eta = \dfrac{0{\cdot}91 \times 30{\cdot}5}{32{\cdot}2 \times 453{\cdot}6}$ (§ 3.16)

$$= 0{\cdot}0019 \text{ engineers' units}$$

$$\omega = \frac{2\pi \times 300}{60}$$

$$= 10\pi \text{ rad/sec}$$

$$t = \frac{0{\cdot}01}{12} \text{ ft}$$

$$R_1 \text{ and } R_2 = 0{\cdot}25 \text{ ft and } 0{\cdot}167 \text{ ft respectively}$$

Substituting these values in eq. (22),

$$T = \frac{\pi \times 0{\cdot}0019 \times 10\pi \times 12}{2 \times 0{\cdot}01}(0{\cdot}25^4 - 0{\cdot}167^4)$$

$$= 0{\cdot}35 \text{ Lb-ft}$$

Then, $\text{h.p. absorbed} = \dfrac{0{\cdot}35 \times 2\pi \times 300}{33{,}000}$

$$= 0{\cdot}02$$

8.7. Time of Emptying Vessel by Laminar Flow through Pipe. In § 7.12 was shown the method of calculating the time taken to empty a vessel by means of a horizontal pipe, the flow being assumed turbulent. This problem will now be solved for a viscous, or laminar, discharge through the pipe.

From eq. (8), the equation for laminar flow through a round pipe can be expressed in the form

$$\frac{mig}{v^2} = \frac{8\eta}{\rho v d}$$

where $m = d/4$ and $\rho = w/g$.

Let p = drop in pressure over a length l. Then

$$h_f = \frac{p}{w}$$

and

$$i = \frac{h_f}{l} = \frac{p}{wl}$$

Substituting these values of m, ρ, and i in the above equation,

$$\frac{dpg}{w4lv^2} = \frac{8g\eta}{wvd}$$

from which
$$p = \frac{32\eta vl}{d^2} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

or
$$h_f = \frac{p}{w} = \frac{32\eta vl}{wd^2} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Let the cylindrical vessel of Fig. 113 contain a viscous liquid of height H_1 above the centre of the pipe, and let the vessel discharge through the horizontal pipe shown, the flow being viscous.

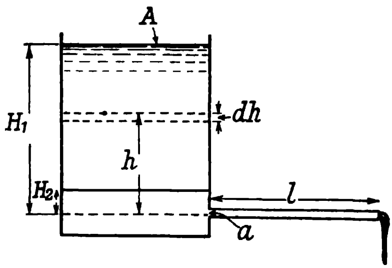


FIG. 113

Let A = area of surface of liquid in vessel,

a = area of pipe,

l = length of pipe,

C_d = coefficient of discharge at entrance to pipe.

It is required to find the time taken to lower the level of the liquid from H_1 to H_2. The final velocity head of the liquid leaving the pipe can be neglected as small.

Consider the instant when the surface of the liquid is at a height h above the centre of the pipe and let the surface fall by $-\mathrm{d}h$ in the time $\mathrm{d}t$; this represents a quantity $\mathrm{d}Q$ flowing from the vessel. Then,

$$\mathrm{d}Q = -A\,\mathrm{d}h = C_d a v\,\mathrm{d}t \quad . \quad . \quad . \quad (25)$$

From eq. (24), and assuming the head is utilized in overcoming the viscous resistance of the pipe,

$$h = h_f = \frac{32\eta vl}{wd^2}$$

from which
$$v = \frac{wd^2h}{32\eta l}$$

Substituting this value of v in eq. (25),

$$-A\,\mathrm{d}h = \frac{C_d a w d^2 h\,\mathrm{d}t}{32\eta l}$$

from which
$$\mathrm{d}t = -\frac{32A\eta l\,\mathrm{d}h}{C_d a w d^2 h}$$

Integrating between the limits H_2 and H_1,

$$\int_0^T dt = -\frac{32A\eta l}{C_d awd^2}\int_{H_1}^{H_2}\frac{dh}{h}$$

Then
$$T = -\frac{32A\eta l}{C_d awd^2}\Big[\log_e h\Big]_{H_1}^{H_2}$$

$$= \frac{32A\eta l}{C_d awd^2}\log_e\left(\frac{H_1}{H_2}\right). \quad . \quad . \quad . \quad (26)$$

EXAMPLE 7

Prove that, when a fluid of viscosity η flows through a tube of diameter d and mean velocity v, the drop in pressure per unit length of tube is $32v\eta/d^2$ for laminar flow.

The viscosity of a liquid is determined by timing the discharge of 50 cm³ of the liquid from a vessel through a horizontal capillary tube. The vessel is an upright cylinder, open at the top, 5 cm in diameter, and the capillary tube is 1 mm bore and 10 cm long. The vessel is at first filled to a height of 5 cm above the axis of the tube, and it is found that 50 cm³ are discharged in 20 min. Find the viscosity of the liquid in poises, given that its density is 0·88 g/cm³. Neglect the end effects of the tube and the velocity head at discharge. (*Lond. Univ.*)

The first part of this question is given by eq. (23), and is proved in § 8.2.

$$\frac{\pi}{4}\times 5^2 = 19{\cdot}63 \text{ cm}^2$$

$$a = \frac{\pi}{4}\times(0{\cdot}1)^2 = 0{\cdot}00785 \text{ cm}^2$$

$$d = 0{\cdot}01 \text{ cm},\ l = 10 \text{ cm},\ w = 0{\cdot}88 \text{ g/cm}^3,$$

$$T = 20\times 60 \text{ sec}$$

$$w = Mg = 0{\cdot}88\times 981 \text{ dynes}$$

$$Q = 50 = (H_1 - H_2)A$$

$$= (5 - H_2)\times 19{\cdot}63$$

from which $H_2 = 2{\cdot}45$ cm

Substituting these values in eq. (26),

$$20\times 60 = \frac{32\times 19{\cdot}63\eta\times 10}{1\times 0{\cdot}00785\times 0{\cdot}88\times 981\times(0{\cdot}1)^2}\log_e\left(\frac{5}{2{\cdot}45}\right)$$

from which
$$\eta = \frac{1{,}200\times 0{\cdot}00785\times 0{\cdot}88\times 981\times 0{\cdot}01}{32\times 19{\cdot}63\times 10\times 0{\cdot}7134}\text{ poises}$$

$$= 0{\cdot}0182 \text{ poises}$$

8.8. Determination of Coefficient of Viscosity. The coefficient of viscosity of a liquid can be found experimentally by the three following methods. The first method is also applicable to gases.

1. BY MEASUREMENT OF PRESSURE DROP DURING PIPE FLOW. It was shown in § 8.2 that the viscous resistance to the flow of a fluid along a straight uniform pipe is given by the equation

$$\frac{mig}{v^2} = C\left(\frac{\rho vd}{\eta}\right)^n$$

For a laminar flow it was proved that $C = 8$ and $n = -1$; hence, the equation becomes

$$\frac{mig}{v^2} = \frac{8\eta}{\rho vd}$$

For a circular sectioned pipe, running full,

$$m = \frac{d}{4}$$

and
$$i = \frac{h_f}{l}$$

Hence, by substituting these values in the above equation,

$$\frac{d}{4} \times \frac{h_f}{l} \times \frac{g}{v^2} = \frac{8\eta}{\rho vd}$$

from which
$$\eta = \frac{\rho g d^2 h_f}{32lv}$$

The coefficient of viscosity can be calculated from this equation if the values of h_f and v are measured for a given pipe during laminar flow. The mean velocity v is obtained by measuring the quantity of flow in a known time; the drop in pressure head h_f is measured on a known length of pipe by means of a sensitive pressure gauge. For accurate results a considerable length of pipe should be used, and the fluid must be maintained at a constant temperature. This method can be used for liquids or gases.

2. BY MEASUREMENT OF DISCHARGE THROUGH AN ORIFICE. In § 11.6 it will be shown that the rate of flow of a liquid through an orifice partly depends on the non-dimensional constant

$$\frac{\rho Dv}{\eta}$$

and, consequently, will vary with the coefficient of viscosity η. This fact is made use of in a type of viscometer which compares the viscosities of liquids by measuring the time taken for a given quantity of the liquid to discharge through a standard orifice.

This principle is used in the Redwood Viscometer (Fig. 114); it consists of a vertical cylinder containing the liquid which is allowed to discharge through an orifice situated in the centre of its base. The cylinder is surrounded by a water jacket, which can maintain the liquid under test at any required temperature by means of an

immersed electric heater. The cylinder is filled to a standard height with the liquid under test and is heated by the water jacket to the required temperature. The orifice is then opened and the time taken

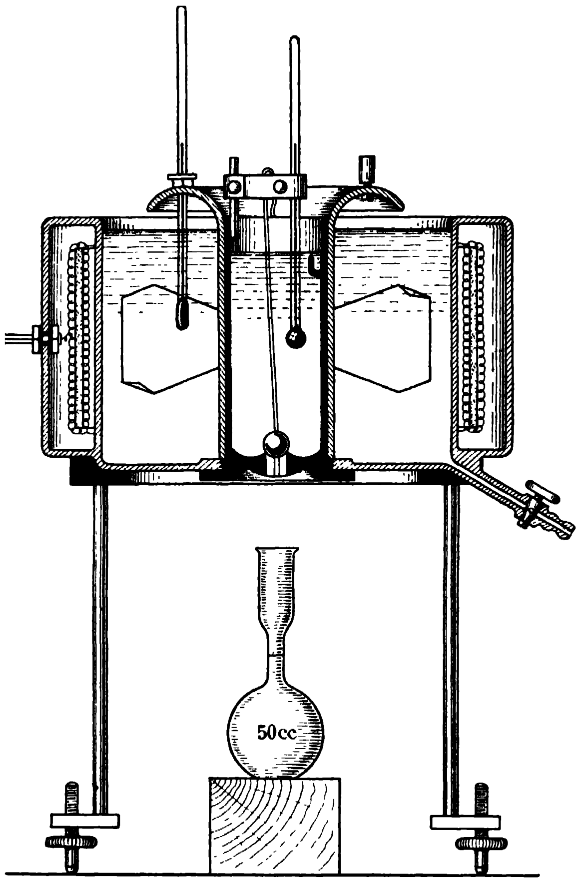


FIG. 114

for 50 cm^3 of the liquid to flow into a measuring flask is noted; the viscosity of the liquid is proportional to this time.

In one type of this viscometer the cylinder is $1\frac{7}{8}$ in. in diameter and $3\frac{1}{2}$ in. deep; the orifice is 1·7 mm in diameter and 12 mm in length.

It should be noticed that this viscometer does not give a direct measurement of η. It gives a comparison of η only; the exact value of η can be obtained by comparison with the times of discharge of liquids of known viscosity.

In this type of viscometer the whole head of the fluid, h, is utilized in overcoming the viscous resistance in the nozzle. Hence, in order to obtain good results, the fluid should flow out of the nozzle with no appreciable velocity. In these circumstances, the velocity head of the discharging fluid is extremely small compared with the static head of the fluid in the cylinder.

Using eq. (8) for the viscous resistance of a cylindrical nozzle,

$$\frac{mig}{v^2} = C\left(\frac{\rho v d}{\eta}\right)^{-1}$$

Let l = length of nozzle. As the whole head h is assumed to be lost,

$$i = \frac{h}{l}$$

Also
$$m = \frac{d}{4}$$

Substituting these values in the above equation,

$$\frac{d}{4} \times \frac{h}{l} \times \frac{g}{v^2} = C\frac{\eta}{\rho v d}$$

But d, h and l are constants for the instrument. Hence

$$\frac{\eta v}{\rho} = \text{constant}$$

$$\text{Time of discharge} = T \propto \frac{1}{v}$$

Hence
$$T = \text{a constant} \times \frac{\eta}{\rho}$$

If the fluid discharges with an appreciable velocity the above result will not hold; a correction must then be made in order to allow for the velocity head.

3. BY MEASUREMENT OF TIME OF FALL OF STEEL BALL IN LIQUID. Another method of measuring the coefficient of viscosity of a liquid is by allowing a steel sphere to fall freely through a column of the liquid, and by timing its fall through a known height. The falling sphere will at first accelerate; but the resistance to its motion increases with its velocity until it just balances the pull of gravity on the sphere. After this condition is reached the sphere falls with a constant velocity. The time of fall is measured over the constant velocity period only.

Stokes proved by a difficult mathematical analysis that the resistance to a sphere moving through a non-compressible fluid is given by the equation

$$F = 6\pi\eta r v$$

where F = resistance to slow motion (laminar flow),
r = radius of sphere,
v = velocity of sphere.

By equating this resistance to the pull of gravity on the sphere, after the condition of uniform velocity has been reached, the value of the coefficient of viscosity η can be calculated.

Let v_f = final uniform velocity of sphere,
ρ_1 = density of sphere used,
ρ_2 = density of liquid under test.

Then, pull of gravity on sphere

= weight of sphere − weight of liquid displaced

$$= \tfrac{4}{3}\pi r^3(\rho_1 - \rho_2)g$$

Now, pull of gravity = fluid resistance on sphere at velocity v_f

or $$\tfrac{4}{3}\pi r^3(\rho_1 - \rho_2)g = 6\pi\eta r v_f$$

from which $$\eta = \frac{2r^2 g(\rho_1 - \rho_2)}{9v_f} \quad . \quad . \quad . \quad . \quad (27)$$

The velocity v_f is obtained by measuring the time taken by the sphere in falling a known height through the liquid after the condition of uniform velocity has been reached. The coefficient of viscosity η can then be calculated from eq. (27).

This method of finding the coefficient of viscosity gives accurate results only if the uniform velocity v_f is low. No eddies should be caused by the falling sphere, and the cylinder containing the liquid should be of sufficient diameter to prevent its surface affecting the motion.

It will be shown in § 11.2 that the resistance of a body subject to viscous motion through a fluid is proportional to the density of the fluid, to the square of the linear dimensions, to the square of the velocity, and to the Reynolds number. Or,

$$F = k\rho \times \text{surface area} \times v^2 \times \left(\frac{\eta}{\rho v d}\right)$$

Applying this equation to the falling sphere,

$$F = k\rho\pi d^2 v^2 \left(\frac{\eta}{\rho v d}\right)$$

Hence $$F = k\pi\eta v d$$

This agrees with Stokes's analytical result if the value of k is 3.

As is explained in § 11.9, Stokes's law can be applied to the settlement of fine powders in suspension in a liquid, the fineness of the powder being graded by the time taken by a known amount of powder in settling. It is also used by engineers when dealing with silting problems.

EXAMPLE 8

When a sphere of radius r cm sinks in a liquid at a uniform velocity of v cm/sec, and η is the coefficient of viscosity of the liquid in poises, the resistance to the motion of the sphere is $R = 6\pi\eta rv$ dynes. Hence find the coefficient of viscosity of a liquid in which a sphere of diameter 0·0622 in. sinks 20 cm in 21·3 sec. The density of the liquid is 0·96 and of the sphere 7·8 g/cm.³ (*Lond. Univ.*)

Now
$$v_f = \frac{20}{21 \cdot 3}$$
$$= 0 \cdot 939 \text{ cm/sec}$$

and
$$r = \frac{0 \cdot 0622 \times 2 \cdot 54}{2}$$
$$= 0 \cdot 079 \text{ cm}$$

Using eq. (27),
$$\eta = \frac{2r^2g(\rho_1 - \rho_2)}{9v_f}$$
$$= \frac{2 \times 0 \cdot 079^2 \times 981(7 \cdot 8 - 0 \cdot 96)}{9 \times 0 \cdot 939}$$
$$= 9 \cdot 97 \text{ poise.}$$

EXERCISES 8

1. Find the kinematic viscosity of water at a temperature of 60°C. *Ans.* $5 \cdot 05 \times 10^{-6}$ ft²/sec.

2. Oil of kinematic viscosity 0·000052 ft²/sec flows through a pipe of 6 in. diameter with a velocity of 1 ft/sec. Find the value of the term vd/ν and state whether the flow is laminar or turbulent. *Ans.* 9,600; turbulent.

3. Air of kinematic viscosity of $15 \cdot 6 \times 10^{-5}$ ft²/sec flows through a pipe of 2 in. diameter. What is the maximum velocity for laminar flow? *Ans.* 1·872 ft/sec.

4. Water at 20°C flows through a pipe of 9 in. diameter and of length 2,000 ft with a velocity of 1·2 ft/sec. Find the loss of head due to viscosity. *Ans.* 1·13 ft.

5. Water at a temperature of 20°C leaks through a horizontal slot 0·01 in. deep, 4 in. in breadth, and 6 in. in length. Find the quantity of water leaking through per hour when the difference of pressure between the ends of the slot is 5 Lb/in.² *Ans.* 3·94 ft³/hr.

6. Fuel oil at a temperature of 10°C is pumped through a pipe line of 6 in. diameter and 5,000 ft in length. Find the horse-power required to pump 10 tons per hour of this oil if the weight of the oil is 57 Lb/ft³ and the kinematic viscosity at 10°C is 0·00015 ft²/sec. *Ans.* 0·0189 h.p.

7. Show that the resistance R experienced by a sphere of diameter d moving with a low velocity v through a fluid of density ρ and viscosity η can be expressed by

$$R = k\eta dv$$

It is found that a steel ball having a diameter of 0·2 cm attains a "terminal velocity" of 75 cm/sec when falling vertically through a fluid whose viscosity is 0·8 poise and density is 0·9 g/cm^3. The density of the steel ball is 7·8 g/cm^3. Determine the value for the coefficient k. (*Lond. Univ.*) *Ans.* $k = 2{\cdot}36$.

8. Define *coefficient of viscosity* and *kinematic viscosity* of a fluid. Describe any experimental means of determining the viscosity of a fluid. (*I.Mech.E.*)

9. How does the viscous leakage past a long hydraulic plunger having a very small radial clearance depend upon (1) the length of the plunger surrounded by its bush, (2) the radial clearance, (3) the diameter, (4) the difference of pressure? (*Lond. Univ.*)

10. A fine granular material of specific gravity 2·5 is in uniform suspension in still water of depth 10 ft. Regarding the particles as spheres of diameter 0·002 cm, find how long it will take for the water to clear. The resistance of a sphere in a liquid is given by the equation: $R = k\eta rv$, when the velocity v is small. Take $k = 6\pi$ and $\eta = 0{\cdot}013$ poise. (*Lond. Univ.*) *Ans.* 3·38 hr.

11. A steel ball, of diameter 0·0625 in., submerged in a liquid of specific gravity 0·91 and viscosity 1·62 poises, is allowed to fall from rest. Find the steady speed attained by the ball given that $k = 6\pi$ and that the specific gravity of steel is 7·85. (*Lond. Univ.*) *Ans.* 0·1912 ft/sec.

12. Define *coefficient of viscosity* and explain how its absolute value can be determined.

A shaft having a diameter of 2 in. rotates centrally in a bush having a diameter of 2·006 in. and length of 4 in. The annular space between the shaft and bush is filled with oil having a viscosity of 0·9 poise. Determine the horsepower required to drive the shaft when the speed of rotation is 600 r.p.m. (*Lond. Univ.*) *Ans.* 0·065 h.p.

13. Define *coefficient of viscosity*. The lower end of a vertical shaft rests in a footstep bearing. Assuming that the end of the shaft and the surface of the bearing are both flat and are separated by an oil film 0·05 cm thick, find the torque required to rotate the shaft at 750 r.p.m. Diameter of shaft, 10 cm; viscosity of oil, 1·5 c.g.s. units. (*I.Mech.E.*) *Ans.* 2,350 g-cm.

14. Find the drag in grammes weight per square centimetre on the wall of a pipe 2·5 cm in diameter, when oil whose viscosity is 2 c.g.s. units flows through with a mean speed of 100 cm/sec. What Reynolds number would this speed represent? ρ for oil $= 0{\cdot}8$ g/cm.^{3} (*I.Mech.E.*) *Ans.* 0·652 g/cm^2; $R_e = 100$.

15. Name and discuss the factors influencing pipe friction. Find the loss of head when a flow of 33 ft^3/sec takes place through 17,000 ft of 36-in. pipe, the value of $\tau/\rho v^2$ being 0·003 where τ = drag stress on wall, ρ = density, and v = speed. (*I.Mech.E.*) *Ans.* 46·3 ft.

16. Find from first principles the head lost due to laminar flow in a pipe of circular section in terms of the length l, the diameter d, the mean velocity v, and the density and viscosity of the fluid. Hence show that if the formula $4flv^2/2gd$ is used for the loss due to friction under these conditions, the value of f is $16/R_e$. Calculate the loss of head in a pipe $\frac{1}{8}$ in. in diameter and 20 ft long when water flows at half the critical velocity, if the critical velocity occurs when the Reynolds number is 2,500 and the coefficient of viscosity is 0·0101 poise. (*Lond. Univ.*) *Ans.* 2·63 ft.

17. Show that the velocity v with which a piston of diameter D and length l moves in a concentric dash-pot is given approximately by $v = (4/3\pi\eta) \times (h^3P/D^3l)$, where h is the clearance between the piston and the dash-pot, P is the load of the piston, and η is the coefficient of viscosity of the fluid. State what approximations you make. (*Lond. Univ.*)

18. Define coefficient of viscosity. Find the torque to rotate a shaft, diameter 50 mm, at 1,200 r.p.m. concentrically within a sleeve 50·17 mm in diameter and 90 mm long, flooded with oil for which $\eta = 0{\cdot}8$ g-cm^{-1}sec^{-1}. (*I.Mech.E.*) *Ans.* 10,600 g-cm.

19. Describe the two different types of flow that may occur in a pipe. The difference of pressure at the ends of a 300 ft pipe line conveying oil is maintained at 10 Lb/in.^{2} If the diameter of the pipe line is 4 in. and the viscosity and density of the oil are 3·0 c.g.s. units and 0·9 g/cm^3 respectively, determine the quantity passing. (*I.Mech.E.*) *Ans.* 0·232 ft^3/sec.

20. Oil of specific gravity 0·91 and viscosity 1·24 poises is pumped through a pipe 3 in. in diameter at a rate of 15 ft^3/min. Show that the motion is laminar, and estimate the horse-power required to pump the oil through a length of 250 ft of this pipe which rises 10 ft. (*Lond. Univ.*)
Ans. $R_e = 868$; h.p. $= 0{\cdot}89$.

21. Prove that, for slow flow through a pipe, $\tau d/\eta v = 8$, where τ = shear stress at pipe wall, d = diameter, η = viscosity, and v = mean velocity. Find the flow of oil, $\eta = 0{\cdot}5$ g-cm^{-1}-sec^{-1}, density $= 0{\cdot}8$ g-cm^{-3}, through a pipe, diameter 0·9 cm and 12 m long, when the head lost is 75 cm. (*I. Mech. E.*) *Ans.* $v = 2{\cdot}49$ cm/sec.

22. Deduce an expression for the loss of head h, when a liquid of viscosity η flows through a pipe of diameter d and length l at a velocity v.
For a liquid flowing along a similar pipe the loss of head was proportional to $v^{1\cdot75}$. Find f in the formula $h = flv^2/m2g$. (*Lond. Univ.*)

23. Derive an expression for the flow of a liquid through a tube in which the resistance is purely viscous, and apply this expression to prove that in the Redwood viscometer the time in discharging a given quantity of a liquid varies directly as the viscosity and inversely as the density of the liquid. (*Lond. Univ.*)

diagram is obtained as in Fig. 115. A straight line AB represents laminar flow whilst a curved line CD represents turbulent flow. It was shown in § 8.2 that the equations for Darcy's frictional coefficient f, obtained from these curves, were—

for laminar flow, $$f = \frac{16}{R_e} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and for turbulent flow, $$f = \frac{0{\cdot}064}{R_e{}^{0{\cdot}23}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

From these equations the correct value of f can be calculated; then the frictional head lost, in feet of fluid, can be calculated direct from Darcy's equation,

$$h_f = \frac{4flv^2}{2gd}$$

In practical engineering problems it is found the flow through pipes is a turbulent flow for gases and most liquids. But for viscous liquids, such as thick oil, the flow may be laminar. If dealing with capillary glass tubes the flow is usually laminar. In any problem on pipe flow, first calculate the Reynolds number of the flow and check whether it is less than 2,000 or more than 2,500 (§ 8.1). If the former holds, calculate f from eq. (5); if the latter holds, calculate f from eq. (6).

If the value of the velocity of flow is not known, by assuming a reasonable value the solution can be obtained by the method of successive approximations.

9.3. Nikuradse's Experiments on Rough Pipes. In order to investigate the effect of the roughness of pipe walls on the resistance to flow Nikuradse* conducted a series of experiments on pipes having diameters of 2·5 cm, 5 cm, and 10 cm. The inner surfaces of these pipes were given different degrees of roughness by coating them with grains of sand of various coarseness.

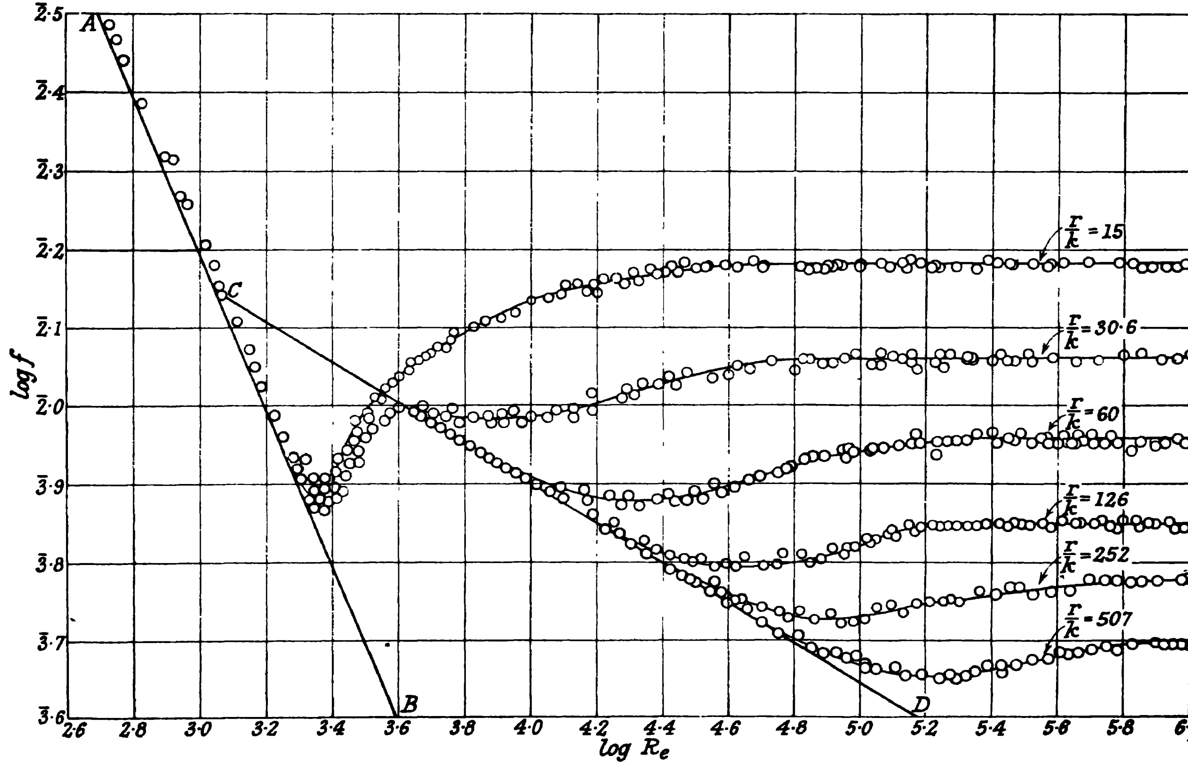
Let r = radius of pipe,

k = average height of roughness projections, or rugosities.

Then r/k = roughness factor.

The inner surfaces of the pipes were coated with sand to give six different values of r/k varying from $r/k = 15$ to $r/k = 507$. The resistance of each pipe was measured experimentally for various velocities of flow. The resistance coefficients were thus obtained for various values of the Reynolds number and for various values of r/k.

* See *Forschungsheft V.D.I.* 361 (Berlin). See also "Experiments with fluid friction in roughened pipes," by C. F. Colebrook and C. M. White, *Proc. Roy. Soc.*, **161**.



A
B
C
D
$\frac{r}{k} = 15$
$\frac{r}{k} = 30\cdot6$
$\frac{r}{k} = 60$
$\frac{r}{k} = 126$
$\frac{r}{k} = 252$
$\frac{r}{k} = 507$
log f
log R_e
$\bar{2}\cdot5$
$\bar{2}\cdot4$
$\bar{2}\cdot3$
$\bar{2}\cdot2$
$\bar{2}\cdot1$
$\bar{2}\cdot0$
$\bar{3}\cdot9$
$\bar{3}\cdot8$
$\bar{3}\cdot7$
$\bar{3}\cdot6$
2·6 2·8 3·0 3·2 3·4 3·6 3·8 4·0 4·2 4·4 4·6 4·8 5·0 5·2 5·4 5·6 5·8 6·0

FIG. 116

Let R_e = Reynolds number for the pipe flow
$= vd/\nu$,
f = frictional coefficient as used in Darcy's formula

$$h_f = \frac{4flv^2}{2gd}$$

The results of Nikuradse's experiments are shown plotted in Fig. 116; the value of $\log f$ being plotted on a base representing $\log R_e$. It will be noticed from the curves that, for small values of R_e, the flow is viscous or laminar and follows the straight line AB. During this type of flow the resistance was not affected by the roughness factor because all the results lie on the laminar flow line AB for all values of r/k. As the values of R_e increased the flow passed through a transition stage, and then became turbulent. The turbulent flow for smooth surfaces then appears to follow another straight line CD. The effect of the roughness factor can be seen by the deviation of the experimental points from this straight line. The pipes with the roughest surface cause the points to break away from the line CD at smaller values of R_e, whilst the smoother pipes cause the points to coincide with line CD up to large values of R_e.

It may be concluded from these results that a boundary sub-layer (§ 16.8) around the walls of the pipe is in existence even during turbulent flow. If k is small and the rugosities do not project beyond this boundary sub-layer, the pipe wall thus corresponds to a smooth surface. This explains why the experimental points for small values of k followed the line CD. It will be shown in Chapter 16 that the thickness of the boundary sub-layer is proportional to $\sqrt{(1/R_e)}$; hence, as the value of R_e increases the boundary sub-layer becomes thinner, and, consequently, the experimental results tend to leave the line CD for large values of R_e. Thus, the surface may be regarded as smooth if the rugosities do not penetrate beyond the boundary sub-layer; if this penetration occurs the surface may be regarded as rough.

As the experimental points for laminar flow coincided with the line AB it appears that laminar flow is not affected by the roughness of the surface.

9.4. Prandtl and Von Kármán's Equations for Pipe Flow. Prandtl and Von Kármán have shown* that the results of Nikuradse's experiments on rough pipes (§ 9.3) can be represented by an equation of the form

$$\frac{1}{\sqrt{4f}} = 2 \log \frac{r}{k} + \phi \frac{ku}{\nu} \sqrt{4f} \quad . \quad . \quad . \quad (7)$$

* See *Forschungsheft V.D.I.* 361 (Berlin).

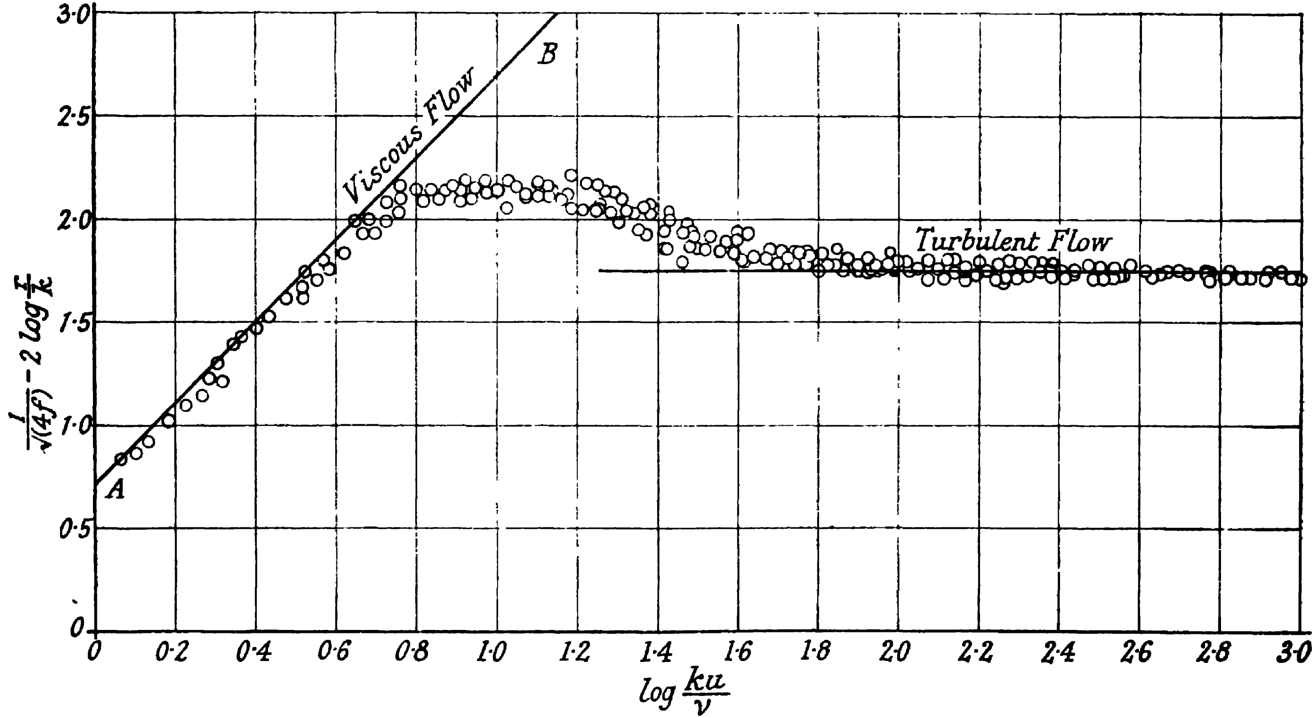

3·0
2·5
2·0
1·5
1·0
0·5
0
$\frac{1}{\sqrt{(4f)}} - 2 \log \frac{r}{k}$
Viscous Flow
B
A
Turbulent Flow
0
0·2
0·4
0·6
0·8
1·0
1·2
1·4
1·6
1·8
2·0
2·2
2·4
2·6
2·8
3·0
$\log \frac{ku}{\nu}$

FIG. 117

where f, r and k are as defined in § 9.3. They showed that if u is the local velocity near the pipe wall, the term ku/v is a local non-dimensional factor, corresponding to a local Reynolds number, on which the resistance of the surface depends. By plotting the values of

$$\frac{1}{\sqrt{4f}} - 2 \log \frac{r}{k}$$

on a base representing $\log (ku/v)$, Nikuradse's results gave the curve shown in Fig. 117. It will be noticed that the experimental points obtained from smooth surfaces tend to follow the straight line AB. As the roughness of the surface increases the points leave the line AB and, after a transition period, appear to approach a horizontal line. When this horizontal line is reached the value of the ordinate

$$\frac{1}{\sqrt{4f}} - 2 \log \frac{r}{k}$$

remains constant at 1·74. The surface is then said to be rough.

The equation to the straight line AB, representing a smooth surface laminar flow, is found to be

$$\frac{1}{\sqrt{4f}} - 2 \log \frac{r}{k} = 0{\cdot}8 + 2 \log \frac{ku}{v} \quad . \quad . \quad . \quad (8)$$

The value of the horizontal line when a completely rough surface is attained is given by the equation

$$\frac{1}{\sqrt{4f}} - 2 \log \frac{r}{k} = 1{\cdot}74 \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Prandtl has deduced the following equation for turbulent flow through smooth pipes

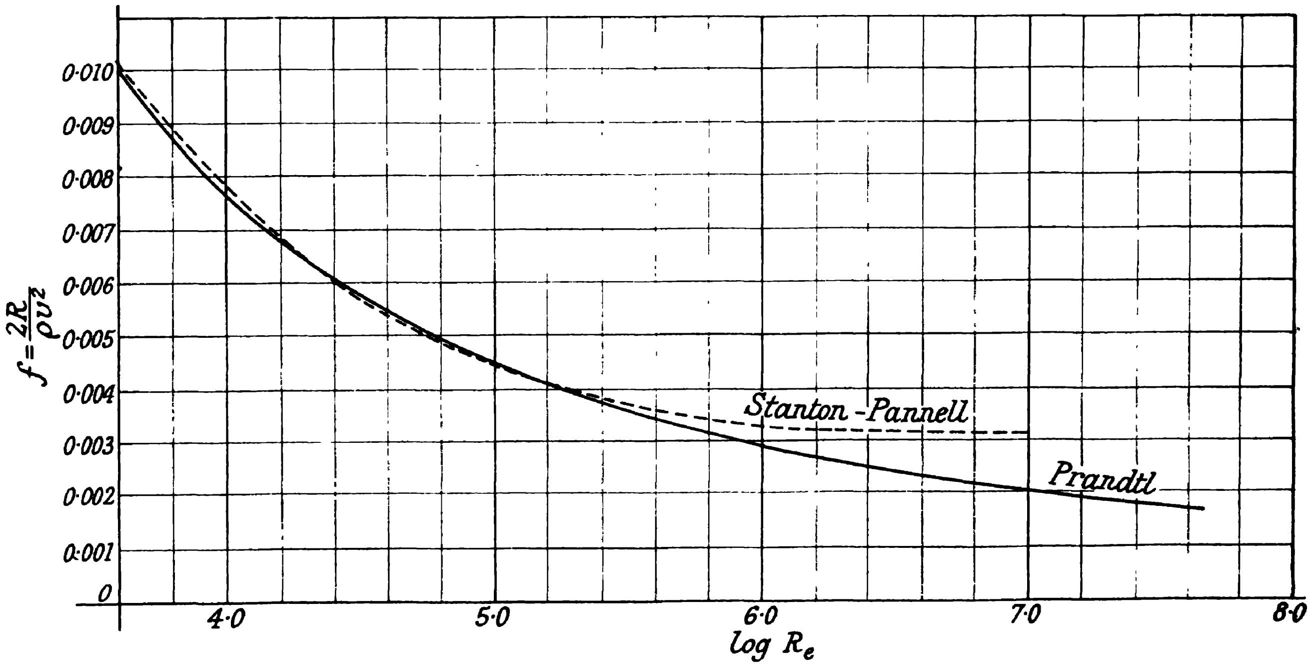
$$\frac{1}{\sqrt{4f}} = 2 \log (R_e\sqrt{4f}) - 0{\cdot}8 \quad . \quad . \quad . \quad (10)$$

where R_e is based on the mean velocity of flow.

This equation is shown plotted in Fig. 118, the ordinate representing f and the base $\log R_e$. The equation is found to agree with experimental results on turbulent flow for all values of R_e.

The dotted line in Fig. 118 represents the turbulent portion of the Stanton-Pannell curve of Fig. 105. It is found that this curve agrees with experimental results only up to a Reynolds number of logarithmic value 5·3. For larger values of R_e, experimental results agree with the Prandtl equation, which is represented by the full line of Fig. 118.

Prandtl's equation is regarded as a great scientific achievement, as experimental tests on fluid flow through pipes of all sizes, and for different types of fluids, are found to lie on this curve, even up to such a large value of Reynolds number as 40×10^6.



$f = \frac{2R}{\rho U^2}$
0·010
0·009
0·008
0·007
0·006
0·005
0·004
0·003
0·002
0·001
0
4·0
5·0
6·0
7·0
8·0
log R_e
Stanton-Pannell
Prandtl

FIG. 118

EXAMPLE 1

Find the diameter for a pipe to transmit 10,000 h.p. 20 miles with 95 per cent efficiency, when the gross head is 150 ft and the roughness of the pipe wall is equivalent to 0·1 in. sand. (*Lond. Univ.*)

Let Q = quantity of flow in cubic feet per second.

Then, $$\text{gross h.p.} = \frac{wQH}{550}$$

$$\text{eff.} = \frac{\text{h.p. transmitted}}{\text{gross h.p.}}$$

that is $$0{\cdot}95 = \frac{10{,}000 \times 550}{62{\cdot}4Q \times 150}$$

from which $$Q = 620\ \text{ft}^3/\text{sec}$$

Then $$v = \frac{Q}{a} = \frac{620}{\pi d^2/4} = \frac{790}{d^2}$$

$$h_f = \frac{4flv^2}{2gd} = 5\%\ \text{of}\ 150$$

Then $$0{\cdot}05 \times 150 = \frac{4fl\left(\dfrac{790}{d^2}\right)^2}{64{\cdot}4d}$$

Hence $$d^5 = \frac{4f \times 20 \times 5{,}280 \times 622{,}000}{0{\cdot}05 \times 150 \times 64{\cdot}4} = 5{,}450 \times 10^5 f$$

from which $$d = 55{\cdot}9 \times \sqrt[5]{f}. \qquad (11)$$

Now $$\frac{r}{k} = \frac{d}{2k} = \frac{55{\cdot}9 \times \sqrt[5]{f}}{2 \times \dfrac{0{\cdot}1}{12}} = 3{,}354\sqrt[5]{f} \qquad (12)$$

Applying eq. (9) for rough pipes,

$$\frac{1}{\sqrt{4f}} - 2\log\frac{r}{k} = 1{\cdot}74$$

Substituting for r/k from eq. (12),

$$\frac{1}{\sqrt{4f}} - 2\log\left(3{,}354\sqrt[5]{f}\right) = 1{\cdot}74$$

Solving this equation by trial or by plotting,

$$f = 0{\cdot}004$$

Substituting this value of f in eq. (11),

$$d = 55{\cdot}9 \times \sqrt[5]{0{\cdot}004}$$
$$= 55{\cdot}9 \times 0{\cdot}33$$
$$= 18{\cdot}5 \text{ ft}$$

If this value of f is substituted in eq. (12),

$$\frac{r}{k} = 1{,}118$$

As a check, using this value of r/k, the value of f can be obtained from the curves of Fig. 116 by extrapolation. From these curves,

$$\log f = \bar{3}{\cdot}6 \text{ (approximately)}$$

from which $$f = 0{\cdot}00398$$

which agrees with the value obtained from eq. (9).

9.5. Frictional Resistance of Rough Surfaces: Turbulent Flow. Prandtl and Schlichting investigated the frictional resistance of flat surfaces, of different roughness, to fluid flow and obtained the curves shown in Fig. 119, for turbulent flow. Surfaces of a known roughness were tested, the roughness coefficient being defined as the ratio of length of surface divided by the height of the roughness projections. This is a similar coefficient to that used for rough pipes (§ 9.3).

Let R = total frictional resistance of surface,
A = area of surface,
C_f = frictional coefficient,
ρ = absolute density of fluid
= w/g,
v = velocity of fluid relative to surface.

Then $$R = \frac{C_f \rho A v^2}{2} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

It will be noticed this equation is similar to the drag equation for aerofoils (§ 15.2 and § 15.3), C_f corresponding to the drag coefficient.

In the curves of Fig. 119 C_f is shown plotted on a base representing the R_e for the flow; both are plotted to log scales.

Let l = length of surface,
k = height of rugosities.

Then $$\text{roughness coefficient} = \frac{l}{k}$$

and $$R_e = \frac{\rho v l}{\eta}$$

The curve AB gives the results for surfaces which can be regarded as aerodynamically smooth; that is, when the rugosities do not penetrate beyond the laminar sub-layer. In all cases, the experimental curves at first follow the curve AB, and then after a short transition period, conform to a horizontal line. This means that C_f remains a constant value after a certain Reynolds number is reached.

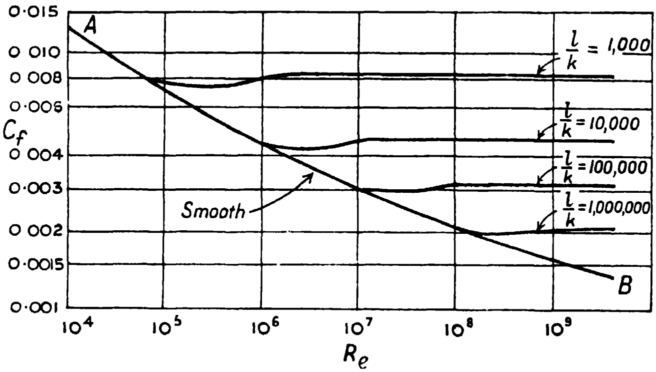


FIG. 119

It will be noticed that the rougher the surface, the earlier is the departure from the smooth curve AB.

From these curves the value of C_f can be read off, for any given turbulent flow, if the roughness coefficient of the surface is known; the total frictional resistance can then be calculated from eq. (13).

EXAMPLE 2

A ship 100 ft long has a total wetted surface of 1,600 ft² and has a speed of 10 m.p.h. in sea water of density of 64 Lb/ft³ and coefficient of viscosity of 27×10^{-6} engineers' units. If the surface of the ship has rugosities of $\frac{1}{8}$ in. in height, find the resistance of the ship at this speed and the horse-power absorbed in overcoming friction. Use the curves of Fig. 119.

$$v = \frac{10 \times 5{,}280}{3{,}600} = 14{\cdot}68 \text{ ft/sec}$$

$$\rho = \frac{w}{g} = \frac{64}{32{\cdot}2} = 1{\cdot}99 \text{ absolute units}$$

$$\frac{l}{k} = \frac{100 \times 12}{\frac{1}{8}} = 9{,}600$$

$$R_e = \frac{\rho v l}{\eta}$$

$$= \frac{1{\cdot}99 \times 14{\cdot}68 \times 100}{27 \times 10^{-6}} = 1{\cdot}082 \times 10^8$$

From the curves of Fig. 119, for the above values of R_e and l/k,

$$C_f = 0.0047$$

Using eq. (13),

$$R = \frac{C_f \rho A v^2}{2}$$

$$= \frac{0.0047 \times 1.99 \times 1{,}600 \times (14.68)^2}{2}$$

$$= 1{,}620 \text{ Lb}$$

$$\text{Frictional horse-power} = \frac{R \times v}{550}$$

$$= \frac{1{,}620 \times 14.68}{550} = 43.3$$

EXERCISES 9

1. Calculate the horse-power required to pump 30 ft³/min of acetic acid through a 3 in. diameter horizontal pipe, 240 ft long. Neglect all losses other than friction. Sp. gr. = 1·07; coefficient of viscosity, 2·5 centipoises.
Ans. 1·85 h.p.

2. A pipe 3 in. in diameter and 20 ft long is used for conveying oil at the rate of 500 gal/min. The oil has a weight of 48 Lb/ft³ and a coefficient of viscosity of 0·012 engineers' units at the temperature of the operation. Calculate the horse-power required to pump the oil through the pipe under these conditions if the overall efficiency of the pump is 75 per cent, and water weighs 10 Lb/gal.

If the oil is heated until its coefficient of viscosity is 0·0024 engineers' units, what is the horse-power now required to drive the pump?
Ans. 10·8 h.p.; 5·36 h.p.

3. It is found that 2·67 h.p. is required to pump 400 gal of brine per minute through a horizontal pipe of length 200 ft and 4 in. in diameter. If the brine has a sp. gr. of 1·18, calculate the value of its coefficient of viscosity in centipoises. *Ans.* 0·453 centipoises.

4. Sulphuric acid of sp. gr. 1·83 is pumped into a large vessel through a 1 ft diameter pipe, 80 ft long, and inclined upwards at an angle of 30° to the horizontal. Calculate the total horse-power to pump 10 ft³/sec into the vessel. Coefficient of viscosity = 40 centipoises. *Ans.* 92 h.p.

5. Thirty ft³/sec of air at a temperature of 600°R and a density of 0·091 Lb/ft³ flow isothermally through a 6 in. diameter pipe, 30 ft long. If the density of the air is assumed to remain constant at 0·091 Lb/ft³, find the pressure drop due to friction in feet of air. Coefficient of viscosity = 10×10^{-6} engineers' units. *Ans.* 558 ft of air.

6. Air, having a coefficient of viscosity of 0.375×10^{-6} engineers' units, flows isothermally through a 2 in. diameter pipe, 50 ft long; the difference of pressure between the ends of the pipe is 10 Lb/in.² The roughness coefficient of the pipe surface is 20, the temperature of the air 60°F and its mean pressure 200 Lb/in.² Find, to the second approximation, the weight of flow per minute.
Ans. 99·1 Lb/min.

7. A rough cast-iron pipe, 3 in. in diameter, was tested as a model to determine the roughness coefficient of the cast-iron surface. It was found that when the water flow through the pipe was 0·392 ft³/sec, the head lost in friction on a 10 ft length was 1·262 ft of water. Using Nikuradse's curves, find the size of the mean roughness projection of the surface. η for water $= 25 \times 10^{-6}$ engineers' units at the temperature used.

Using this value of k, find the frictional head lost in a cast-iron pipe, 6 in. in diameter and 1,200 ft long, when the water flow is 1·5 ft³/sec, the surface of the pipe being the same as the model.

Ans. $k = 0{\cdot}01613$ in.; $h_f = 56{\cdot}4$ ft of water.

8. A thin oil having a density of 50 Lb/ft³ is pumped through a 1 ft diameter pipe line 4 miles long; the coefficient of viscosity of the oil is 0·01865 poises. The roughness of the inner surface of the pipe is such that the mean height of the roughness projections is 0·03 in. The quantity of flow through the pipe is 7·85 ft³/sec. By using the roughness coefficient curves, calculate the head lost in friction in pounds per square inch and the horse-power required to drive the pump if the overall efficiency of the pump is 75 per cent. (*Lond. Univ.*)

Ans. $h_f = 296{\cdot}2$ Lb/in.²; 811 h.p.

9. Oil of specific gravity 0·88 and having a coefficient of viscosity of 1·2 poise is pumped through a 6 in. diameter pipe, 400 ft long, which is inclined upwards from the pump at an angle of 30° to the horizontal. Calculate the horse-power required to pump 65 ft³/min of the oil.

If the oil is heated until the coefficient of viscosity is 0·84 poise, find the horse-power now required to pump the oil up the inclined pipe. (*Lond. Univ.*)

Ans. 22·94 h.p. (laminar flow); 23·2 h.p. (turbulent flow).

10. Oil of specific gravity 0·8, and having a kinematic viscosity of 0·0002 ft²/sec units, is pumped through a 0·5 ft diameter pipe line of 10,000 ft in length. Find the horse-power required to pump 28,000 gal/hr.

If the oil is heated until its kinematic viscosity is 0·00002 ft²/sec units, find the horse-power now required to pump the same quantity of oil as before. (*Lond. Univ.*) *Ans.* 40·2 h.p.; 23·5 h.p.

11. Compressed air flows through a horizontal pipe from a chamber having a constant pressure of 18·6 Lb/in.² to another chamber having a constant pressure of 17·8 Lb/in.² The pipe is $\frac{1}{4}$ in. in diameter for the first 2 ft of its length and then suddenly enlarges to $\frac{1}{2}$ in. diameter; the total length of the pipe is 6 ft. If the flow is isothermal at 20°C and Darcy's f for both sections of the pipe is 0·01, calculate the weight of air flowing per minute, taking into account all losses. Neglect any change of density of the air as small. $pV = 96\,T$. (*Lond. Univ.*) *Ans.* $W = 0{\cdot}0758$ Lb/min.

12. A thick oil of density of 50 Lb/ft³, and having a coefficient of viscosity of 0·012 engineers' units, is pumped through a pipe ABC which is inclined upwards. The portion of the pipe AB is 300 ft long and has a diameter of 1 ft. The portion of the pipe BC is reduced to 6 in. diameter and is 20 ft long. The height of C above A is 120 ft. If the quantity of oil pumped per minute is 546 ft³ and the efficiency of the pump is 70 per cent, calculate the values of Darcy's f in each length of pipe and the horse-power required to drive the pump. Neglect all losses other than friction. (*Lond. Univ.*)

Ans. AB, $f = 0{\cdot}01066$ (laminar flow); BC, $f = 0{\cdot}0102$ (turbulent flow). h.p. = 235·2.

CHAPTER 10

FLOW THROUGH OPEN CHANNELS

10.1. Open Channels. The term "open channel" applies to any passage through which water is flowing when the free surface of the water is in contact with the atmosphere. The water is then under atmospheric pressure throughout. The channel may be covered in at the top or open; if covered in, it must not be running full, otherwise the pressure might rise above or fall below atmospheric. A pipe which is not running full may be classed as an open channel. An open channel may be of uniform cross-section, as a canal, sewer and aqueduct, or it may be of an irregular cross-section, such as a river.

Water flowing through an open channel is subjected to a frictional resistance at the wetted surface of the channel which obeys the same laws as stated in the previous chapters. As the pressure throughout is atmospheric, the head causing flow will be due entirely to the slope of the channel. In channels of regular cross-section the velocity of flow is constant; therefore, the head due to the slope of channel may be assumed to be lost in overcoming the frictional resistance of the sides and base.

The velocity of flow will vary at different points of the cross-section of the channel, being smaller towards the sides. All calculations on the flow through channels are based on the mean velocity of flow at any cross-section.

10.2. Formula for Flow in Open Channels. An equation for the flow of water through an open channel may be deduced in a similar manner as for the flow in pipes. In an open channel the pressure is atmospheric and may, therefore, be neglected; the head due to the slope of the pipe is assumed to be lost in friction. Hence the hydraulic gradient is equal to the slope of the channel if the latter is uniform.

Let i = slope of channel,

A = area of cross-section of channel,

m = hydraulic mean depth $= A/P$,

P = wetted perimeter,

v = mean velocity of flow,

h_f = head lost in friction,

f' = Froude's coefficient of friction between water and sides of channel for unit velocity.

Consider a section of the water of length l moving along the channel (Fig. 120). Assume slope of channel is uniform, it will therefore equal i.

Applying Froude's equation to the section,

$$\text{Frictional resistance of section} = f' \times \text{wetted area} \times (\text{velocity})^n = f'Plv^n$$

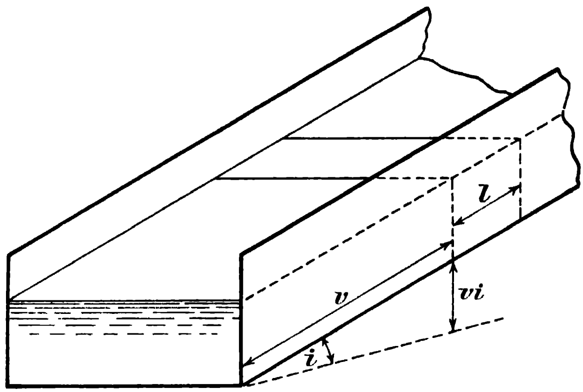


FIG. 120

Work done per second in overcoming friction

$$= f'Plv^n \times v$$

Loss of potential energy per second $=$ weight $\times$ change of altitude

$$= wAl \times vi$$

But potential energy lost $=$ work done against friction

Therefore

$$wAlvi = f'Plv^n v$$

$$wi = f'v^n \frac{P}{A}$$

Assuming $n = 2$ and substituting the hydraulic mean depth m for A/P,

$$i = \frac{f'v^2}{wm}$$

Or

$$v = \sqrt{\frac{w}{f'}\,mi}$$

$$= C\sqrt{mi} \quad . \quad . \quad . \quad . \quad (1)$$

where $C = \sqrt{\dfrac{w}{f'}}$ and is a constant depending on the shape and surface of the channel, and is determined experimentally.

Eq. (1) is known as the Chezy formula. It was deduced by him empirically and is the same form as used for the flow through pipes.

From the results of experiments on the flow of water through channels, Bazin deduced the following formula for the value of C—

$$C = \frac{157 \cdot 5}{1 + \frac{k}{\sqrt{m}}} \text{ foot units}$$

where k is a constant depending on the surface of the channel and has the following values—

Clean smooth sides of wood, brick, stone, etc.	$k = 0 \cdot 29$
Dirty sides of wood, brick, stone, etc.	$k = 0 \cdot 5$
Sides of natural earth	$k = 2 \cdot 35$

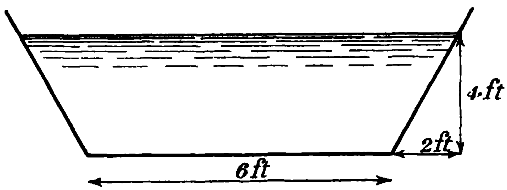


FIG. 121

Actually C will vary with the temperature, the velocity, and the size of the channel; that is, with the Reynolds number of the flow, as shown in § 8.3.

EXAMPLE 1

A trapezoidal channel, having sides of smooth stone, has a base of 6 ft and side slopes of 2 vertical to 1 horizontal. The depth of water in the channel is 4 ft. Find the quantity of water flowing if the slope of the channel is 10 ft per mile.

The section of channel is shown in Fig. 121.

$$\text{Area of section} = 32 \text{ ft}^2$$

$$\text{Wetted perimeter} = 6 + 2\sqrt{4^2 + 2^2}$$

$$= 14 \cdot 94$$

$$m = \frac{A}{P} = \frac{32}{14 \cdot 94} = 2 \cdot 14$$

Using Bazin's formula for C,

$$C = \frac{157 \cdot 5}{1 + \frac{k}{\sqrt{m}}}$$

$$\frac{157{\cdot}5}{1 + \dfrac{0{\cdot}29}{\sqrt{2{\cdot}14}}}$$

$$= 131{\cdot}5$$

Using the Chezy formula,

$$v =$$

$$= 131{\cdot}5\sqrt{2{\cdot}14 \times \frac{10}{5{,}280}}$$

$$= 8{\cdot}35 \text{ ft/sec}$$

$$\text{Quantity} = A \times v$$

$$= 32 \times 8{\cdot}35$$

$$= 268 \text{ ft}^3\text{/sec}$$

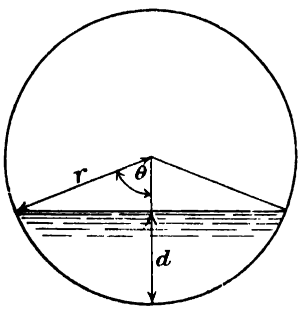


FIG. 122

EXAMPLE 2

Find the depth of flow in a circular sewer 3 ft in diameter, having a fall of 1 in 200, when the discharge is 3,500 gal/min. Take $v = 100\sqrt{mi}$, and solve by plotting. (*Lond. Univ.*)

Assume the water level to be at a height d (Fig. 122). Let r be the radius of the sewer, and θ be half the angle subtended at the centre by the water level.

From Fig. 122,

$$\cos\theta = \frac{r - d}{r}$$

from which the angle θ may be obtained in radians.

$$\text{Area of wetted cross-section} = A = \frac{r^2 2\theta}{2} - r^2 \sin\theta \cos\theta$$

$$= r^2\left(\theta - \frac{\sin 2\theta}{2}\right)$$

$$\text{Wetted perimeter} = P = r2\theta$$

$$m = \frac{\quad}{P}$$

$$v = 100\sqrt{m \times \frac{1}{200}} \text{ ft/sec}$$

$$\text{Quantity} = Q = Av \times 6{\cdot}24 \times 60 \text{ gal/min}$$

These quantities are shown tabulated in the following table for assumed values of d—

d	$\cos\theta$	θ (rad)	A	P	m	v	Q
0·5	0·666	0·841	0·775	2·523	0·307	3·92	1,137
0·9	0·4	1·157	1·778	3·471	0·512	5·06	3,360
1·0	0·333	1·225	2·04	3·675	0·555	5·26	4,020
0·95	0·3665	1·195	1·92	3·585	0·535	5·17	3,718

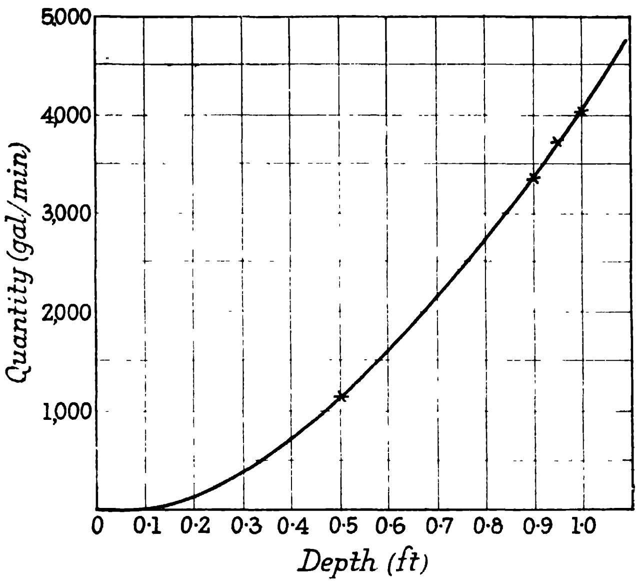


FIG. 123

d and Q are shown plotted in Fig. 123, and a curve is drawn through the points. From this curve the depth to give a discharge of 3,500 gal/min may be obtained.

$$\text{Required depth} = 0{\cdot}925 \text{ ft}$$

10.3. Rectangular Channel: Depth for Maximum Discharge. The channel with the most economical section is the one which gives the maximum discharge for a given amount of excavation. The discharge is proportional to the velocity and the area, whilst the excavation is proportional to the area. The proportions of the most economical section may be found by assuming the area to be constant, and finding the depth which gives the maximum velocity.

Consider the rectangular-sectioned channel of Fig. 124; let b be the breadth and d the depth. Assume the area of cross-section to be constant.

As
$$A = bd$$
$$b = \frac{A}{d} \tag{2}$$
$$\text{Discharge} = A \times v = A \times C\sqrt{\frac{A}{P}\,i}$$

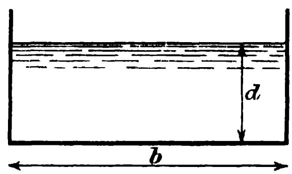


FIG. 124

As A, C, and i are constants for the channel considered, the discharge will be a maximum when P is a minimum.
$$P = b + 2d$$
Substituting for b from eq. (2),
$$P = \frac{A}{d} + 2d$$
$$= Ad^{-1} + 2d$$
Differentiating P and equating to zero for a minimum,
$$\frac{dP}{dd} = -Ad^{-2} + 2 = 0$$
Therefore
$$A = 2d^2$$
But, as $A = bd$,
$$bd = 2d^2$$
or
$$d = \frac{b}{2}$$

That is, the maximum discharge for a rectangular channel occurs when the depth of water is one-half of the breadth.

10.4. Trapezoidal Channel: Condition for most Economical Section. The most economical section for a trapezoidal channel will be when the discharge is a maximum for a given excavation. The condition for this may be found, as in the previous case, by assuming the area to be a constant.

Consider the trapezoidal channel of Fig. 125. Let b be the breadth of the base, d be the depth of water, and let the slope of the sides be $1/n$; then the horizontal projection of the wetted side is nd.

$$\text{Discharge} = A \times v = A \times C\sqrt{\frac{A}{P}i}$$

and will be a maximum when P is a minimum for the given channel.

$$A = (b + nd)d \quad . \quad . \quad . \quad . \quad . \quad (3)$$

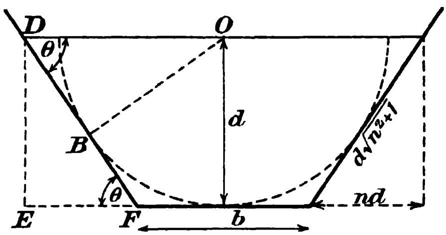


FIG. 125

Therefore $$b = \frac{A}{d} - nd \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\text{Length of sloping side} = \sqrt{n^2d^2 + d^2}$$
$$= d\sqrt{n^2 + 1}$$

Then $$P = b + 2d\sqrt{n^2 + 1}$$

Substituting for b from eq. (4),

$$P = \frac{A}{d} - nd + 2d\sqrt{n^2 + 1}$$

Differentiating P and equating to zero for a minimum,

$$\frac{dP}{dd} = -\frac{A}{d^2} - n + 2\sqrt{n^2 + 1} = 0$$

Therefore $$\frac{A}{d^2} + n = 2\sqrt{n^2 + 1}$$

Substituting for A from eq. (3),

$$\frac{b + nd}{d} + n = 2\sqrt{n^2 + 1}$$

or $$\frac{b + 2nd}{2} = d\sqrt{n^2 + 1} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Let θ be the angle of slope of the sides to the horizontal. Let O be the centre of the water surface. From O draw OB to meet a sloping side at B and perpendicular to it.

Consider triangle OBD;

$$\text{angle } ODB = \theta$$

$$\sin\theta = \frac{OB}{OD} = \frac{OB}{\frac{b}{2} + nd}$$

Consider triangle DEF;

$$\text{angle } DFE = \theta$$

$$\sin\theta = \frac{ED}{DF} = \frac{d}{d\sqrt{n^2+1}}$$

Equating these two values of $\sin\theta$,

$$\frac{OB}{\frac{b}{2} + nd} = \frac{d}{d\sqrt{n^2+1}}$$

It will be seen from eq. (5) that these two denominators are equal. Therefore, $OB = d$

That is, if a semicircle is drawn with centre at O and of radius d, the three sides of the section will be tangential to it.

Therefore, the most economical trapezoidal section is when the three sides are tangential to a semicircle described on the water line.

As triangle ODB is similar to triangle DFE, it follows that

$$OD = DF$$

Now

$$m = \frac{A}{P} = \frac{\left(OD + \frac{b}{2}\right)d}{2DF + b}$$

$$= \frac{\left(OD + \frac{b}{2}\right)d}{(2OD + b)}$$

$$= \frac{d}{2}$$

This is another condition for maximum discharge which will be found useful for the solution of problems of this nature.

EXAMPLE 3

A trapezoidal channel is to be designed for conveying 10,000 ft³ of water per minute. Determine the cross-sectional dimensions of the channel from the following data—

Slope 1 in 1,600; sides inclined 45°; cross-section to be a minimum; $v = 90\sqrt{(mi)}$. (*Lond. Univ.*)

Using eq. (5) and putting $n = 1$,

$$\frac{b + 2d}{2} = d\sqrt{1+1}$$

from which $b = 0{\cdot}828d$

$$\text{Area of section} = A = (b + d)d$$

$$= (0{\cdot}828d + d)d$$

$$= 1{\cdot}828d^2$$

$$\text{Wetted perimeter} = P = b + 2d\sqrt{2}$$
$$= 0{\cdot}828d + 2{\cdot}828d$$
$$= 3{\cdot}656d$$
$$m = \frac{A}{P} = \frac{1{\cdot}828d^2}{3{\cdot}656d}$$
$$= 0{\cdot}5d$$
$$\text{Quantity per second} = A \times v$$
$$= A \times 90\sqrt{mi}$$

Therefore
$$\frac{10{,}000}{60} = 1{\cdot}828d^2 \times 90\sqrt{0{\cdot}5d \times \frac{1}{1{,}600}}$$

Squaring both sides and simplifying,
$$1{\cdot}026 = d^4 \times 0{\cdot}0003125d$$

Hence
$$d^5 = 3{,}280$$

Then
$$d = 5{\cdot}04 \text{ ft}$$
$$b = 0{\cdot}828d$$
$$= 4{\cdot}17 \text{ ft}$$

10.5. Circular Section: Depth for Maximum Velocity. The velocity of flow in a given circular channel will depend upon the depth of the water. As the velocity is proportional to the hydraulic mean depth, its maximum value may be obtained by differentiating A/P and equating to zero.

Consider the circular channel of Fig. 122.

Let d = depth of water for maximum velocity,

θ = ½ angle subtended at centre by water line, in radians,

and r = radius of channel section.

Then,

$$\text{area of wetted section} = A = \text{area of sector} - \text{area of triangle}$$
$$= r^2\theta - r^2\frac{\sin 2\theta}{2}$$
$$= r^2\left(\theta - \frac{\sin 2\theta}{2}\right)$$
$$\text{Wetted perimeter} = P = 2r\theta$$

and
$$m = \frac{A}{P}$$

As
$$v = C\sqrt{mi}$$

v = maximum when m is a maximum, that is, when A/P is a maximum.

Differentiating A/P and equating to zero,

$$\frac{\mathrm{d}\left(\frac{A}{P}\right)}{\mathrm{d}\theta} = P\frac{\mathrm{d}A}{\mathrm{d}\theta} - A\frac{\mathrm{d}P}{\mathrm{d}\theta} = 0$$

from which

$$Pr^2(1 - \cos 2\theta) - A2r = 0$$

Substituting for P and A,

$$2r^3\theta(1 - \cos 2\theta) = 2r^3\left(\theta - \frac{\sin 2\theta}{2}\right)$$

Therefore $\qquad 2\theta = \tan 2\theta$

The solution of which is

$$2\theta = 257\tfrac{1}{2}^\circ$$

From Fig. 122, $\qquad d = r - r\cos\theta$

For maximum velocity, $\qquad d = r - r\cos\frac{257\frac{1}{2}^\circ}{2}$

$$= r(1 + 0{\cdot}62)$$

$$= 1{\cdot}62r$$

Or, depth for maximum velocity $= 0{\cdot}81 \times$ diameter of channel

10.6. Circular Section: Depth for Maximum Discharge. Referring to Fig. 122, and using the same notation as in § 10.5,

$$\text{discharge} = A \times C\sqrt{mi}$$

$$= A \times C\sqrt{\frac{A}{P}i}$$

$$= C\sqrt{\frac{A^3}{P}i}$$

Therefore, discharge is a maximum when A^3/P is a maximum. Differentiating and equating to zero for a maximum,

$$\frac{\mathrm{d}\left(\frac{A^3}{P}\right)}{\mathrm{d}\theta} = \left(P \times 3A^2\frac{\mathrm{d}A}{\mathrm{d}\theta} - A^3\frac{\mathrm{d}P}{\mathrm{d}\theta}\right) \times \frac{1}{P^2} = 0$$

from which $\qquad 3P\frac{\mathrm{d}A}{\mathrm{d}\theta} - A\frac{\mathrm{d}P}{\mathrm{d}\theta} = 0$

Substituting for A and P from § 10.5,

$$6r^3\theta(1 - \cos 2\theta) - 2r^3\left(\theta - \frac{\sin 2\theta}{2}\right) = 0$$

from which $\qquad 3\theta(1 - \cos 2\theta) = \theta - \frac{\sin 2\theta}{2}$

Or $\qquad 4\theta - 6\theta\cos 2\theta = -\sin 2\theta$

The solution of this equation is $\theta = 154°$

Then, for maximum discharge, $d = r - r\cos\theta$

$$= r(1 + 0{\cdot}9)$$
$$= 1{\cdot}9r$$

Or, depth for maximum discharge $= 0{\cdot}95 \times$ diameter

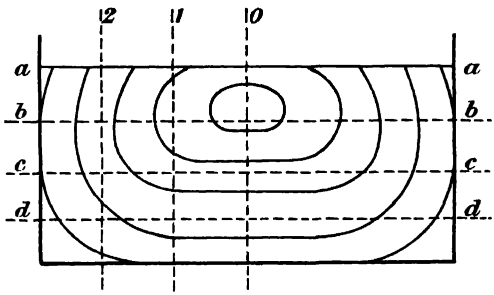


FIG. 126

EXAMPLE 4

Find, either graphically or by calculation, the depth for the maximum discharge for a circular culvert.

Find the depth of water for maximum velocity along a 6 ft diameter culvert. (*Lond. Univ.*)

The depth for maximum discharge may be found by the above method; or, it may be found by plotting as was done in Example 2.

Depth for maximum velocity may be found from the equation of § 10.5—

$$\text{depth} = 0{\cdot}81 \times \text{diameter}$$
$$= 0{\cdot}81 \times 6 = 4{\cdot}86 \text{ ft}$$

10.7. Variation of Velocity over Cross-section of a Channel. The velocity of flow varies at different points of the cross-section of the channel. The frictional resistance of the sides causes the water to slow down towards the sides of the channel, and the frictional resistance between the water surface and the atmosphere causes a slight reduction of velocity at the free surface. The maximum velocity will be on the vertical centre-line of the channel at a point a little below the free surface.

The variation of velocity over the cross-section of a rectangular channel is shown in Fig. 126. The curves shown are lines of equal velocity; they have the greatest value at the centré, just below the water surface, and decrease towards the sides and base. In Fig. 127 are shown the variations of velocity on horizontal section lines taken at different depths. The velocities at different points of the section lines a, b, c, and d (Fig. 126) are plotted on a base representing the width of the channel.

Fig. 128 shows the variation of velocity on the vertical section lines 0, 1, and 2 (Fig. 126). The horizontal axis represents the velocity and the vertical axis the depth.

The mean velocity on any vertical section occurs at a depth of approximately 0·6 of the total depth; it varies with the type of channel and with the nature of the sides. The discharge of the whole channel may be obtained by dividing the section into vertical rectangles and finding the mean velocity of each rectangle. Using

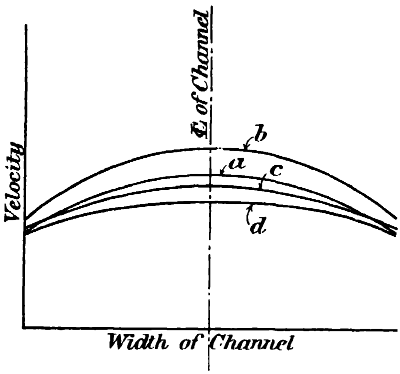


FIG. 127

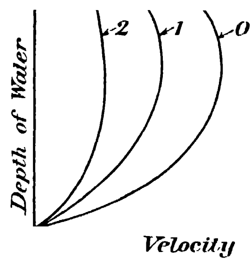


FIG. 128

this mean velocity, the discharge through each rectangle may be obtained. The sum of all these discharges will be the total discharge of the channel.

The mean velocity of each rectangular strip may be taken, approximately, as the velocity at a depth of 0·6 of the total depth.

10.8. Measurement of Flow of Irregular Channels. By the term "irregular channels" is included large rivers and small streams. The quantity of flow of a small stream may be obtained by fitting a notch or weir across the stream; the discharge may then be calculated by measuring the head over the notch. This method could not be used for a large river on account of the expense and of the obstruction to navigation it would cause. In this case it is necessary to measure the cross-section of the river, and to measure the velocity of flow at various points of this cross-section.

Let Fig. 129 represent the cross-section of the river at the point chosen. This should be on a straight uniform portion of the river. The cross-section is then divided into vertical rectangles as shown, and the mean velocity of each rectangle found. This can be obtained approximately by assuming the mean velocity to occur at a depth of 0·6 of the total depth and measuring the velocity at that point, or it may be found more accurately by measuring the velocity at several

depths and calculating the mean from these measurements. The discharge through each rectangle may then be obtained by multiplying the area by the mean velocity. Then, by adding together the discharge of each rectangle, the total flow of the river is obtained.

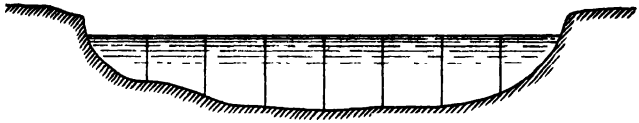


FIG. 129

The velocity of flow may be measured by the following methods—

1. PITOT TUBE. The Pitot tube is held with the orifice facing upstream at the depth at which the velocity is required. The velocity is then obtained by the method given in § 3.15.

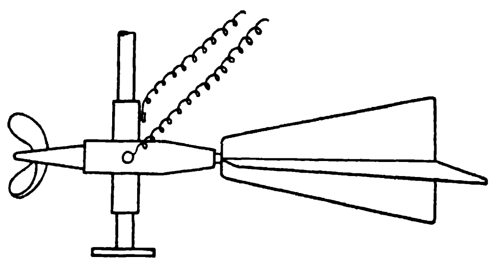


FIG. 130

2. CURRENT METER.* A type of current meter is shown in Fig. 130. It consists of a wheel containing blades or cups, which are rotated by the flowing water; these are headed towards the current by means of a tail on which vanes or fins are fixed. An electric current is passed to the wheel from a battery above the water by means of wires, and a commutator is fixed to the shaft of the revolving blades which makes and breaks the electric circuit each revolution. A revolution counter above the water is worked by this electric current. The meter is lowered into the water to the required depth, and the velocity obtained from the revolution counter.

The Amsler Current Meter is shown in Figs. 131 and 132. This is a universal instrument suitable for both measurements in very slow running waters and yet strong enough for use in great velocities. The propeller is of a strictly helical shape; it is made of one single piece of hard and very resistant aluminium. There is no friction between the axis of the propeller and the electric contact. The

* For a description of the Aerofoil flow recorder, see § 15.11.

shaft of the propeller runs on the one side in a ball bearing and on the other in a sapphire bearing, ensuring smooth running and consequently great accuracy of results, even when using the current meter for slow speeds. The current meter commences rotating at a speed of 1 in. per second.

The contact for transmitting the rotation of the propeller to the electric bell inside the casing of the current meter is arranged in a

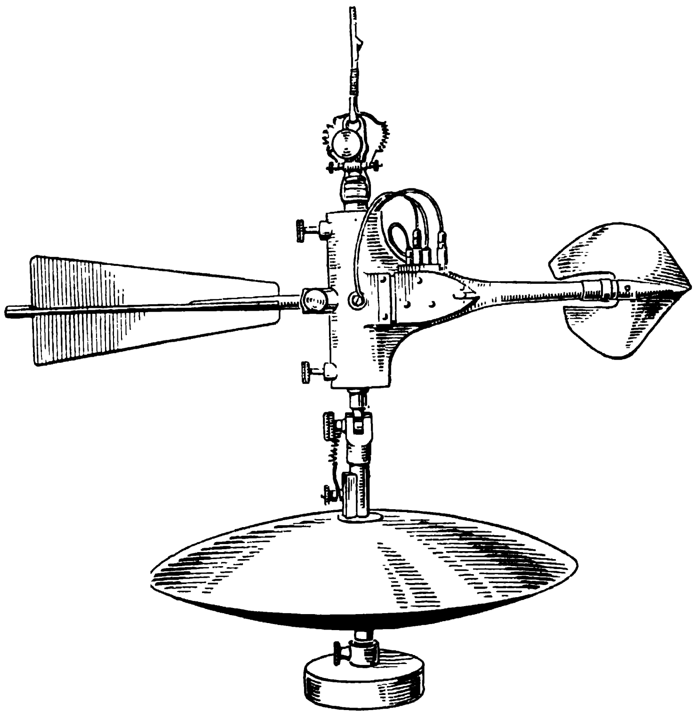


FIG. 131. AMSLER CURRENT METER WITH UNIVERSAL JOINT, WEIGHT, AND GROUND SINKER

(Courtesy of Amsler and Co., Ltd.)

watertight chamber, thus preventing any corrosive and electrolytic influence of the water, especially sea water or acidulated water. The instrument can be taken to pieces quickly and conveniently without any tools. The ball bearings remain fast to their axis and cannot therefore be lost.

This current meter can also be provided with an additional contact for single revolutions of the propeller and with an observation telephone. At every revolution of the propeller a crack is then heard in the telephone. If the propeller turns backward, as may be

the case in whirlpools or backwater, a double crack is heard at every revolution. By means of this telephone it is possible to ascertain whether the propeller turns regularly, forwards or backwards, and also to count directly the number of revolutions of propeller for a certain interval of time if the water flows very slowly. The instrument makes contact at every fifty revolutions of the propeller. The two terminals on the instrument casing are connected, by means of

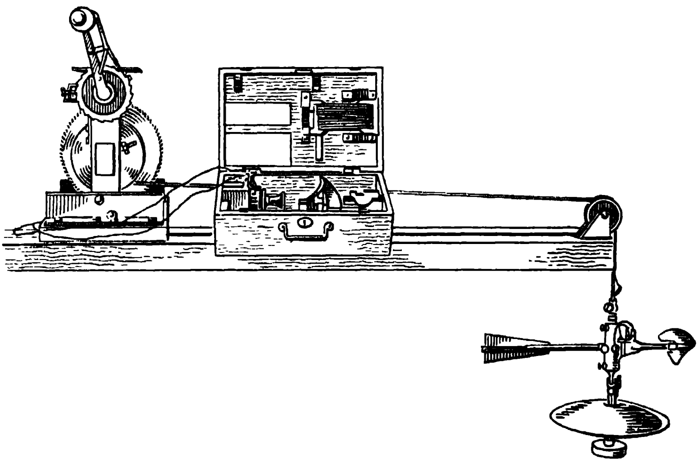


FIG. 132. AMSLER CURRENT METER WITH UNIVERSAL JOINT AND SINGLE WINDLASS
(*Courtesy of Amsler and Co., Ltd.*)

a double wire, with the electric bell and the battery, which are located in the instrument case.

3. FLOATS. A simple way of measuring the velocity of flow of a river is by means of floats. The surface velocity at any section may be obtained by a single float. The time taken for the float to traverse a known distance is measured and the velocity calculated. A single float gives the surface velocity only, and is affected by the wind and air resistance.

A better method is to use double floats. A double float consists of a surface float on to which is attached a hollow metal sphere, heavier than water, and suspended from it by a cord of known length (Fig. 133). The depth of the lower float may be regulated by the length of the cord. The velocity is then obtained by timing the top float over a known distance. This gives the mean between the velocity of the surface and the velocity of the layer traversed by the lower float.

The best type is the rod float. This consists of a vertical wooden rod which is weighted at the bottom to keep it vertical. The rod will travel with a velocity equal to the mean velocity of the section.

It should be as long as the depth of the river will permit, and the top should be made conspicuous by painting it white. Some types of rod are made telescopic, so that the length may be adjusted to suit any depth.

Weeds at the bottom of a river will interfere with the use of a rod or double float. If possible, a section of the river which is free from weeds should be chosen.

4. CHEMICAL METHOD. Another method for finding the discharge of an irregular channel is by inserting a chemical solution of known

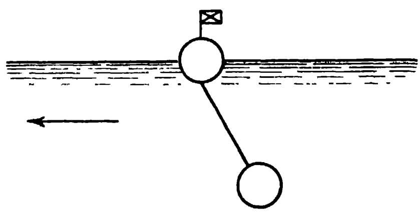


FIG. 133

weight and strength uniformly at a certain section. Then, by finding the strength of the solution at another section lower down the stream, the discharge may be calculated. A solution must be chosen which readily mixes with water; for this reason, and on account of cheapness, common salt is generally used. Great care must be taken with this method, a uniform stretch of channel should be used, and the solution should be inserted at several places over the cross-section.

At a section of the stream, sufficiently below the inserting section for the solution to have mixed evenly with the stream, samples of the stream are taken at various points, from which the weight of salt per cubic foot of water is measured.

Let Q = discharge of stream in cubic feet per second,
q = quantity of solution injected in cubic feet per second,
W = weight of salt per cubic foot of stream water at lower section,
w = weight of salt per cubic foot of solution injected.

As the weight of salt injected per second must equal the weight per second passing the lower section of the stream, then

$$qw = QW$$

from which
$$Q = \frac{qw}{W}$$

This method is very unreliable unless the water is well mixed before reaching the lower section at which the samples are taken. The average results from all the samples must be used.

10.9. River Bends. It is known from experience that a river flowing round a bend scours the bank on the outside of the bend, and material is deposited on the inside. This means that the bend is continually increasing, and eventually the river breaks through the narrow neck thus formed and makes an island of the land which previously formed the inside of the bend. After a time the main water course will be through the breach and the bent channel will be partly silted up, until finally it becomes a horse-shoe lake. These horse-shoe lakes are frequently found at the sides of rivers.

The scouring of the outside of a river bend is mainly due to the impact of the water as it strikes the bank. Another explanation is given by Prof. James Thomson, who accounts for it by the action

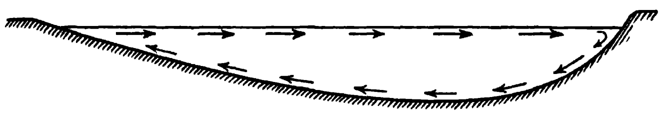


Fig. 134

of a transverse current which flows along the bottom of the river from the outside of the bend to the inside, as shown in Fig. 134. Owing to the centrifugal force the pressure of the water on the outside of the bend will increase; but as the water near the surface has a greater velocity than that near the bottom, the pressure at the surface will be greater than that at the bottom. This will cause the water to flow downwards and form a cross current which will transport material from the outside of the bend to the inside. This explanation may be the cause of some of the silting, but the main quantity is probably due to impact on the outside of the bend and to still water at the inside.

10.10. Water Supply and Rainfall. The water supply for a district is usually obtained by building a dam across certain water-courses, such as mountain streams. As this stops the flow of the stream, the inhabitants of the land below the dam, who were formerly supplied with water by the stream, must be compensated by a daily supply of water from the dam. Such water is known as compensation water, and the quantity is fixed by law to be one-third of the total amount collected by the dam. As the greater part of this compensation water must be supplied in the day time, it is usual to have a special reservoir for it, so that the claimants may use it as they wish.

In supplying the population of a district with water, the following considerations are necessary—

1. Rainfall.
2. Amount lost by evaporation, absorption, and percolation.
3. Maximum period in which available supply falls short of demand.

The rainfall of a district is measured with a rain gauge, such as shown in Fig. 135, and is averaged over several years. The following list gives the average rainfall of a few districts—

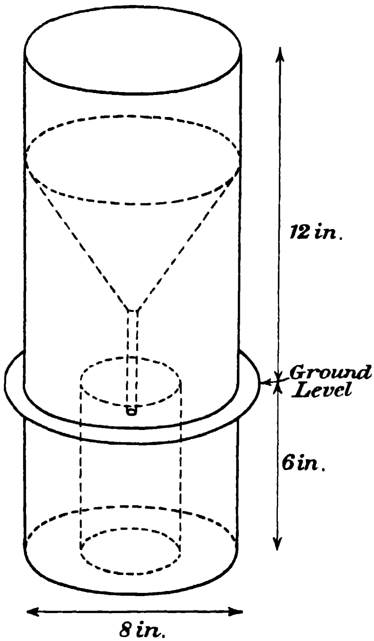


FIG. 135

Seathwaite	.	136 in. per year
Keswick .	.	60
Buxton .	.	54
Midlands .	.	25
London .	.	25
Manchester	.	37·4

The water supply is reckoned on the three consecutive driest years, and the following rules are in use—

1. The wettest year has a rainfall of $1\frac{1}{2}$ times the mean rainfall.
2. The driest year has $\frac{2}{3}$ the mean rainfall.
3. The driest two consecutive years have $\frac{3}{4}$ of mean rainfall per year.
4. The driest three consecutive years have $\frac{4}{5}$ of mean rainfall per year.

It is usual to work on the last of these rules.

Some of the rainfall is lost by evaporation, absorption and percolation, the amount depending on the nature of the ground on which the rain falls. The following table will give an idea of the amounts lost by these causes—

Rainfall	Percolation		Evaporation		
	Soil	Sand	Soil	Sand	Wate
in.	in.	in.	in.	in.	in.
25·7	7·6	21·4	18·1	4·3	20·6

The average loss from these causes may be taken as from 10 to 20 in. of the annual rainfall.

Reservoirs are built in which the water is stored. The amount stored should be equal to about 150 days' supply. In towns, small reservoirs, known as service reservoirs, are built for district supply.

The total amount of water required to be stored for the whole water supply is given by the following empirical rule—

$$N = \frac{1{,}000}{\sqrt{h}}$$

where N = number of days' supply to be stored,
and h = inches of rainfall in three consecutive dry years.

The demand for the water is not regular, and it is found that one-half of the amount used daily will be drawn off in 6 hours. That is, the maximum rate of flow is 100 per cent more than the average rate. This must be taken into account in designing the supply pipes.

The amount of water consumed varies in different districts. The following figures are an average of the amounts supplied—

Domestic Supply

17 gallons per day per head in towns.
12 gallons per day per head in rural districts.

Trade Supply

5 to 20 gallons per day per head.

The following gives the quantities of water supplied per day for several large towns—

Philadelphia	215 gallons per head
Glasgow	52
Perth	50
Manchester	27
Liverpool	25

These figures include the consumption for trade, domestic, municipal, and leakage. The large variation is probably due to trade consumption and leakage. For an average, a total consumption of 30 gallons per head per day may be used.

EXAMPLE 5

The average rainfall over a catchment area of 1,680 acres, as determined for a period of 35 years, is 36·6 in. per annum. Assuming that there is a possibility of three consecutive dry years, during which the average rainfall is only 80 per cent of the above average, and assuming that the evaporation loss in such dry years is equivalent to 15 in. of rainfall per annum, determine in gallons the minimum annual yield from this catchment area.

If one-third of this yield has to be supplied for compensation water, what population could be supplied from this catchment area, if the daily supply is 48 gallons per head? (*Lond. Univ.*)

$$\text{Minimum average rainfall} = 36{\cdot}6 \times \frac{80}{100} = 29{\cdot}25 \text{ in.}$$

Deducting loss due to evaporation,

$$\text{collectable rainfall} = 29{\cdot}25 - 15{\cdot}0 = 14{\cdot}25 \text{ in.}$$

Volume of rain collected per annum

$$= \text{area} \times \text{depth}$$
$$= 1{,}680 \times 4{,}840 \times 9 \times \frac{14{\cdot}25}{12}$$
$$= 86{,}800{,}000 \text{ ft}^3$$
$$= 542{,}000{,}000 \text{ gal}$$

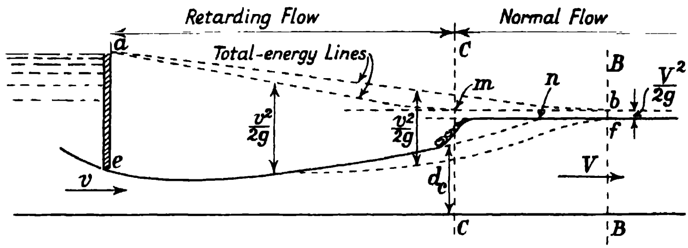


FIG. 136. FORMATION OF STANDING WAVE

Deducting one-third for compensation water,

actual amount available per annum

$$= 542{,}000{,}000 \times \tfrac{2}{3}$$
$$= 361{,}000{,}000 \text{ gal}$$
$$\text{Population supplied} = \frac{361{,}000{,}000}{48 \times 365}$$
$$= 20{,}600$$

10.11. Hydraulic Jump. Fig. 136 shows water issuing from a sluice, at a high velocity, into a channel of uniform width. Owing to the frictional resistance of the sides of the channel, there will be a loss of energy as the water flows along the channel; this will reduce the velocity. At the vertical section B the normal type of channel flow is proceeding; at this section the slope of the channel bed is just sufficient to overcome the frictional resistance of the sides, and the flow proceeds with a normal velocity V, in agreement with the Chezy formula. The total-energy line for this portion of the channel is the dotted line mb, which is at a height of $V^2/2g$ above the water surface. If the dotted line ab represents the total-energy line of the water as it flows from the sluice, the normal channel flow commences at b, where it meets the normal-energy line mb. The distance between the total-energy line and the water surface represents the velocity head.

The water flowing from the sluice at e has its high velocity continuously reduced by the friction of the channel; this causes the depth to increase in proportion to the reduction of velocity. Let the

line *ef* represent the water surface. Now f must be on the same vertical section as the point b for the flow to remain stable; in which case the rate of loss of total energy has been in proportion to the rate of increase in depth, in such a way that both reach their normal values on the same vertical section.

Now suppose the loss of total energy is greater than the previous supposition and follows the dotted line *am*; let this total-energy line meet the normal-energy line at m on the vertical section C. The increase of depth will now be at a different rate; let the water surface now lie on the line *en*, meeting the normal water surface at n. This new condition will not be stable because the energies on the vertical section C do not balance. For a stable condition at C the velocity head should be $V^2/2g$, whereas it is actually $v^2/2g$. At this section there will now occur a sudden heaping up of the water surface until it reaches the normal channel level at m. This sudden increase of depth will reduce the velocity from v to V, and the energies of the water will now balance. The sudden heaping up of the water at section C is known as the *hydraulic jump* or *standing wave*. It is caused by the fact that whilst the energy loss is proportional to v^2, the increase in depth is only proportional to v. The phenomenon is liable to occur at the foot of waterfalls, in the exit channels of sluices, and in the vicinity of under-water obstructions. The tidal bore, which flows up the tidal portion of many rivers, is a fast moving hydraulic jump due to the choking of the river flow by the incoming tide at the mouth.

It will be noticed that the hydraulic jump occurs at the section where the total-energy line from the sluice meets the normal total-energy line of the channel.* Actually, there is a small loss of energy at the jump which causes the jump to advance upstream to the left of section C. This causes a sudden drop in the total-energy line *am* at the commencement of the jump.

For any given problem the total-energy line *am* and the water surface *en* can be plotted to scale if the following assumptions are made—

1. The velocity of the water in the channel is constant over the cross-section. Actually, the velocity varies considerably as shown in Figs. 126 to 129.
2. The value of the frictional coefficient f is constant at all velocities. This is not strictly true as f varies with the velocity (§ 8.2 and § 8.3).

10.12. Specific Energy of a Channel's Cross-section. Consider a constant quantity of water to be flowing through the cross-section of the rectangular channel shown in Fig. 137. As the quantity is assumed

* For further information on hydraulic jump see *The Standing Wave or Hydraulic Jump*, Central Board of Irrigation, Publication No. 7 (Simla, India).

constant, the velocity of the stream will depend on the depth. If the depth is small, the velocity will be large, as

$$v = \frac{Q}{bd}$$

On the other hand, if the depth is large the velocity is correspondingly small.

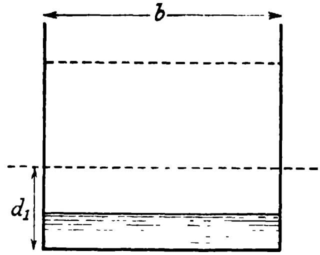


FIG. 137

The specific energy of the stream at this cross-section consists of the static energy due to its depth, plus the kinetic energy. Or,

$$\text{specific energy} = E = d + \frac{v^2}{2g} \tag{6}$$

This is not the total energy of the water, as no reference has been made to any datum height; consequently, the slope of the channel is not included in the term specific energy.

In Fig. 138 the static energy, kinetic energy and specific energy for this cross-section have been plotted for a fixed quantity of flow and for various depths of the stream. The static energy is represented by the straight line ox; the corresponding kinetic energy by the curve yz. By adding the horizontal ordinates of these two curves the specific energy line abc is obtained. It will be noticed that the specific energy at first becomes less as the depth increases. At the point b the specific energy has its minimum value; beyond this point there is an increase of specific energy as the depth increases. The depth at the point b is the depth at minimum energy, and is called the *critical depth*. For each value of specific energy to the right of b, there are two depths for the given quantity of flow considered; both of these depths produce the same specific energy.

If a horizontal line is drawn through b, the area above this line is known as the area of tranquil flow; the area below is known as the area of rapid or streaming flow.

Consider the condition represented by the line ABC. If the total energy conditions are favourable for a hydraulic jump (§ 10.11) to

occur at this section, before the jump occurs the flow is rapid, and the depth of the channel is CB; after the jump occurs the flow is tranquil and the depth is CA. Thus, the depth of the stream has increased from CB to CA whilst the energy remains constant, except for losses due to turbulence. It will be noticed that, if a hydraulic jump occurs, it must do so before the depth reaches the

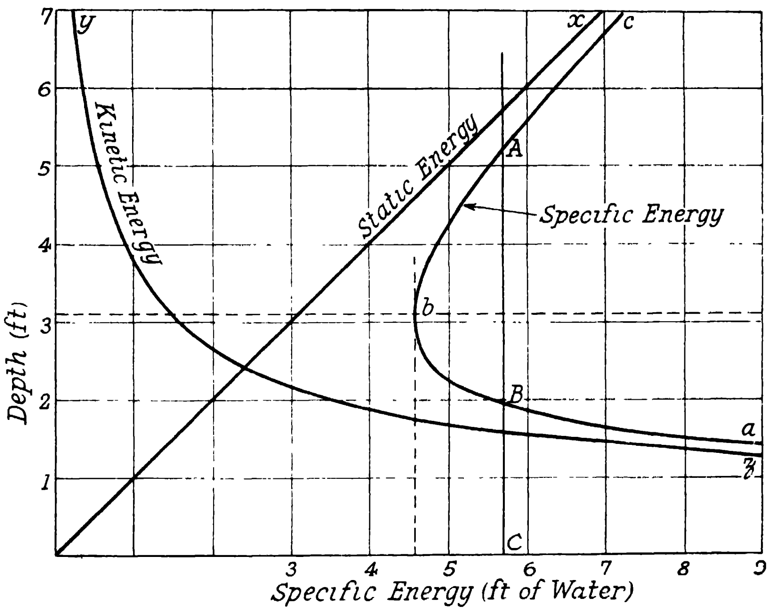


FIG. 138

critical depth, because there are two depths for equilibrium only before the critical depth is reached.

The depth at minimum specific energy, or critical depth, can be obtained by differentiating eq. (6) and equating to zero; or,

$$\frac{dE}{d(d)} = 0$$

Consider unit width of channel and let Q be the quantity of flow through unit width per second. Then,

$$v = \frac{Q}{d}$$

It should be noticed that this is an approximation only as it assumes the velocity to be uniform throughout the section considered.

Substituting for v in eq. (6),

$$E = d + \frac{Q^2 d^{-2}}{2g}$$

Then $$\frac{\mathrm{d}E}{\mathrm{d}(d)} = 1 - \frac{2Q^2 d^{-3}}{2g} = 0$$

from which $$\left(\frac{Q}{d}\right)^2 = gd$$

or $$v^2 = gd \qquad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence, depth for minimum specific energy $= v^2/g$.

From eq. (7) it will be seen that $\frac{v}{\sqrt{gd}} = 1$. The term $\frac{v}{\sqrt{gd}}$ is called the *Froude number* (§ 3.17); hence, minimum specific energy occurs when the Froude number is unity.

Example 6

Water enters a horizontal channel of uniform width, with a velocity of 20 ft/sec. The depth of the water at entrance is 2 ft. Calculate the critical depth of the water.

Using eq. (7), $$d = \frac{v^2}{32{\cdot}2}$$

Also, quantity per second per foot width

$$= 20 \times 2 = v \times d$$

Hence $$d = \frac{40}{v} = \frac{v^2}{32{\cdot}2}$$

from which $$v^3 = 40 \times 32{\cdot}2$$

Therefore $$v = 10{\cdot}9 \text{ ft/sec}$$

and $$d = \frac{40}{10{\cdot}9}$$
$$= 3{\cdot}66 \text{ ft}$$

10.13. Depth for Maximum Flow at Given Specific Energy. In § 10.12 the critical depth of a channel was defined as the depth producing minimum specific energy for a given quantity of flow Q. The critical depth may also be defined as the depth producing a maximum flow for a given specific energy. The critical depth satisfies both of these conditions; hence, either definition may be used.

Using the notation of § 10.12, consider the channel of Fig. 137, but this time assume the specific energy E remains constant whilst the flow Q is varied. Then,

$$E = d + \frac{v^2}{2g} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

from which $$v = \sqrt{2g(E - d)}$$

Now, $$\begin{aligned} Q &= Av \\ &= bd\sqrt{2g(E - d)} \\ &= b\sqrt{2g(Ed^2 - d^3)} \end{aligned}$$

Hence, Q is a maximum when the term $(Ed^2 - d^3)$ is a maximum. Differentiating this term and equating to zero for a maximum,

$$\frac{dQ}{d(d)} = 2Ed - 3d^2 = 0$$

from which $$E = \tfrac{3}{2}d$$

Substituting this value of E in eq. (8),

$$\frac{3}{2}d = d + \frac{v^2}{2g}$$

Hence $$d = \frac{v^2}{}$$

or $$\frac{v}{\sqrt{gd}} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

which is the same result as that obtained in § 10.12. Hence, the depth of a channel for maximum flow, for a given specific energy, is the critical depth.

It will be proved in § 10.18 that the velocity given by eq. (9) is also the velocity of surface waves in shallow water; that is, when the Froude number equals unity. It is found from observation that ripples, or surface wave formation, occur on the surface of the water when the critical depth is reached; the flow of these shallow water surface waves is dealt with in § 10.18.

10.14. Depth at Hydraulic Jump. The depth of the water in the channel after the hydraulic jump has occurred can be calculated by equating the force on a section of the water to its rate of change of momentum. Let Fig. 139 represent the water in the vicinity of the hydraulic jump. Consider the equilibrium of the mass of water between two vertical sections (1) and (2), one before the jump and the other after. Consider unit width of channel, dimensions being in feet.

Let p_1, v_1 and d_1 apply to water at section (1),

p_2, v_2 and d_2 apply to water at section (2),

q = quantity of water flowing per second, per unit width $= v_1 d_1 = v_2 d_2$.

Now $$\text{average } p_1 = \frac{wd_1}{2}$$

and $$\text{average } p_2 = \frac{wd_2}{2}$$

$$\text{Horizontal force on section (2)} = p_2 \times \text{cross-sectional area}$$
$$= p_2 d_2$$
$$= \frac{wd_2^2}{2}$$

$$\text{Horizontal force on section (1)} = \frac{wd_1^2}{2}$$

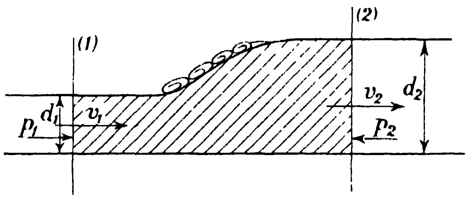


FIG. 139. DEPTH AT HYDRAULIC JUMP

Now $$\text{force} = \text{mass per second} \times \text{change of velocity}$$

that is $$\frac{wd_2^2}{2} - \frac{wd_1^2}{2} = \frac{wq}{g}(v_1 - v_2) \tag{10}$$

But $$v_1 = \frac{q}{d_1} \text{ and } v_2 = \frac{q}{d_2}$$

Hence, substituting in eq. (10),

$$d_2^2 - d_1^2 = \frac{2q}{g}\left(\frac{q}{d_1} - \frac{q}{d_2}\right)$$

or $$(d_2 - d_1)(d_2 + d_1) = \frac{2q^2}{g}\left(\frac{d_2 - d_1}{d_2 d_1}\right)$$

Hence $$d_2 + d_1 = \frac{2q^2}{gd_2 d_1}$$

that is $$d_2^2 + d_2 d_1 = \frac{2q^2}{gd_1}$$

Solving this quadratic for d_2,

$$d_2 = -\frac{d_1}{2} + \sqrt{\frac{2q^2}{gd_1} + \frac{d_1^2}{4}} \tag{11}$$

Substituting for $q = v_1 d_1$,

$$d_2 = -\frac{d_1}{2} + \sqrt{\frac{2v_1^2 d_1}{g} + \frac{d_1^2}{4}} \tag{12}$$

From this equation the depth of the channel on the downstream side of the jump can be calculated.

Loss of energy due to jump

$$= \left(\frac{v_1^2}{2g} + d_1\right) - \left(\frac{v_2^2}{2g} + d_2\right) \quad . \quad . (13)$$

EXAMPLE 7

Water is flowing along a channel of uniform width, the quantity of flow being 40 ft³/sec per foot width of channel, causing a standing wave to occur. If the depth of the water on the upstream side of the standing wave is 3 ft, find the height of the wave.

Calculating on 1 ft width of channel,

$$q = 40 \text{ ft}^3/\text{sec}$$
$$d_1 = 3 \text{ ft}$$

Using eq. (11),

$$d_2 = -\frac{3}{2} + \sqrt{\frac{2 \times 40^2}{32{\cdot}2 \times 3} + \frac{3^2}{4}}$$
$$= -1{\cdot}5 + \sqrt{33{\cdot}2 + 2{\cdot}25}$$
$$= -1{\cdot}5 + 5{\cdot}94 = 4{\cdot}44 \text{ ft}$$
$$\text{Height of wave} = d_2 - d_1$$
$$= 4{\cdot}44 - 3{\cdot}0$$
$$= 1{\cdot}44 \text{ ft}$$

10.15. Non-uniform Flow in Channels. Fig. 140 represents a longitudinal section of a channel in which the velocity of the water is not constant. Owing to the frictional resistance of the sides and base of the channel, the water is slowing down; this causes an increase in the depth, as the quantity of flow is constant.

Consider two vertical sections at a distance dl apart.

Let v = velocity of flow at left-hand section,
d = depth at left-hand section.

Let the velocity change by dv and the depth by d(d) over the length dl. Actually dv is negative for the case shown in Fig. 140.

Let α = slope of total-energy line,
i = slope of base of channel.

Applying Bernoulli's equation to the two vertical sections, and assuming the base of the channel at the right-hand section to be the datum line,

$$i\,\mathrm{d}l + d + \frac{v^2}{2g} = d + \mathrm{d}(d) + \frac{(v + \mathrm{d}v)^2}{2g} + \alpha\,\mathrm{d}l$$

from which
$$i\,\mathrm{d}l = \mathrm{d}(d) + \frac{v\,\mathrm{d}v}{g} + \alpha\,\mathrm{d}l$$

if small quantities of the second order are ignored.

Hence $$\frac{d(d)}{dl} = i - \frac{v\,dv}{g\,dl} - \alpha \quad . \quad . \qquad (14)$$

But, quantity of flow per foot width of channel

$$= Q = v \times d = \text{a constant}$$

Hence $$\frac{dQ}{dl} = 0$$

or $$\frac{d(vd)}{dl} = 0$$

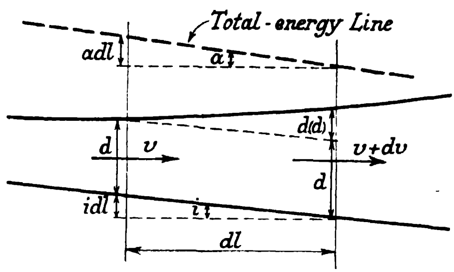


FIG. 140

Differentiating, and treating both v and d as variables,

$$v\frac{d(d)}{dl} + d\frac{dv}{dl} = 0$$

from which $$\frac{dv}{dl} = -\frac{v}{d} \times \frac{d(d)}{dl} \qquad (15)$$

Substituting eq. (15) in eq. (14),

$$\frac{d(d)}{dl} = i + \frac{v^2}{dg}\frac{d(d)}{dl} - \alpha$$

that is $$\frac{d(d)}{dl}\left(1 - \frac{v^2}{gd}\right) = i - \alpha$$

or $$\frac{d(d)}{dl} = \frac{i - \alpha}{\left(1 - \frac{v^2}{gd}\right)} \quad . \qquad (16)$$

This equation represents non-uniform flow in a channel; it will be noticed that $d(d)/dl$ is the slope of the water surface.

From eq. (16) it will be seen that, if $\alpha = i$, then

$$\frac{d(d)}{dl} = 0$$

and the flow is then at uniform depth.

It was shown in § 10.12 that when

$$\frac{v^2}{gd} = 1$$

the critical depth for the channel is reached. If this condition is inserted in eq. (16), it will be noticed that $d(d)/dl$ becomes equal to

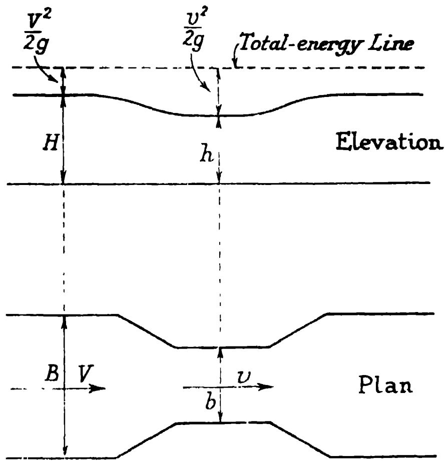


FIG. 141

infinity; the water surface would then be vertical. This condition is never reached in practice as eq. (16) would not hold for this extreme limit, as small changes only have been assumed. Either steady flow would have been reached, or a hydraulic jump would have occurred, before this condition of critical depth could be attained (§ 10.11 and § 10.12).

It will also be noticed that $\alpha = h_f/dl$ where h_f is the total head lost in friction and eddies, but $h_f \propto f$ where f is a frictional coefficient.

Actually f will vary with the velocity and with the surface of the channel sides; it will not be constant, as has been assumed in the above solution; therefore α is not constant throughout the length of channel. From this it appears that eq. (16) has no practical value.

10.16. Venturi Flume in Channel. The quantity of water flowing along a channel can be measured by restricting the width as shown in Fig. 141. This is known as a Venturi flume and corresponds to the throat of a Venturi meter, used for the measurement of pipe flow.

Let B, V, and H represent the normal breadth, velocity, and depth of the channel at the entrance of the flume. Let b, v, and h

represent the breadth, velocity, and depth at the narrowest part of the flume.

As the quantity of flow past all sections is the same,

$$Q = VBH = vbh \quad . \quad . \quad . \quad . \quad . \quad (17)$$

It will be noticed that v is larger than V on account of the restriction of the cross-section; hence, the water level may fall in the narrow portion of the flume, as the total energy remains practically constant. This is shown in the sectional elevation of Fig. 141.

Let H_1 = total energy at any section.

Applying Bernoulli's equation to the inlet and throat, and neglecting all losses,

$$H_1 = H + \frac{V^2}{2g} = h + \frac{v^2}{2g} \quad . \quad . \quad . \quad . \quad (18)$$

as the depth of water represents its pressure head plus datum head.

From eq. (18),

$$v = \sqrt{2g}(H_1 - h)^{1/2}$$

Also $$Q = bhv$$

Hence $$Q = b\sqrt{2g}h(H_1 - h)^{1/2} \quad . \quad . \quad . \quad . \quad (19)$$

$$= k(H_1h^2 - h^3)^{1/2}$$

where k is a constant for a given flume, which includes all losses. It will be noticed from this equation that Q is a maximum when $(H_1h^2 - h^3)$ is a maximum. Hence, differentiating,

$$\frac{dQ}{dh} = 2H_1h - 3h^2 = 0$$

Then $$2H_1 - 3h = 0$$

or $$h = \tfrac{2}{3}H_1 \quad . \quad . \quad . \quad . \quad (20)$$

This proves that the flow through the flume is a maximum when the depth at the throat is two-thirds of the total energy of flow.

Substituting in eq. (18) the value of H_1 from eq. (20)

$$\frac{v}{\sqrt{gh}} = 1$$

or, the maximum flow occurs when the Froude number is unity. As this is the condition for the critical depth, it follows that the maximum flow occurs at the critical depth (§ 10.13).

Substituting this value of h in eq. (19),

$$\text{maximum } Q = b\sqrt{2g} \,.\, \tfrac{2}{3}H_1(H_1 - \tfrac{2}{3}H_1)^{1/2}$$

$$= 3{\cdot}09bH_1^{3/2} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

For any flow through the channel, the quantity of flow can be calculated by measuring the depths at the entrance and throat of

the flume. The equation for the quantity is the same as the equation for a pipe Venturi meter deduced in § 3.6. Or

$$Q = \frac{Aa\sqrt{2g}}{\sqrt{A^2 - a^2}} \sqrt{(H - h)} \text{ ft}^3/\text{sec}$$

where A = area of channel at entrance in square feet and a = area of flume at throat in square feet.

Then $$Q = \frac{BH \times bh\sqrt{2g}}{\sqrt{B^2H^2 - b^2h^2}} \sqrt{(H - h)} \quad . \quad . \quad . \ (22)$$

This should be multiplied by a coefficient, found experimentally, in order to allow for any losses in the flume.*

It is possible for a hydraulic jump to occur in the downstream portion of the flume if the conditions for the formation of the jump are favourable (§ 10.11).

EXAMPLE 8

In a Venturi flume the floor is horizontal and the throat is rectangular.

The width of the throat is 1 ft, and the width at inlet is 1·5 ft. On the downstream side a "jump" occurs so that the conditions for maximum flow exist at the throat.

If the rate of flow is 1·5 ft³/sec, find the depth of water at the throat and at inlet.

Assume that the coefficient of discharge is unity. (*Lond. Univ.*)

Using eq. (21) for maximum flow,

$$Q = 3{\cdot}09bH_1^{3/2}$$

that is $$1{\cdot}5 = 3{\cdot}09 \times 1 \times H_1^{3/2}$$

from which $$H_1 = \left(\frac{1{\cdot}5}{3{\cdot}09}\right)^{2/3}$$

$$= 0{\cdot}618 \text{ ft}$$

From eq. (20),

$$h = \tfrac{2}{3}H_1$$

$$= \tfrac{2}{3} \times 0{\cdot}618$$

$$= 0{\cdot}412 \text{ ft}$$

Using eq. (18),

$$\frac{v^2}{2g} = \frac{H_1}{3}$$

Then $$v = \sqrt{2g \times \frac{0{\cdot}618}{3}}$$

* For practical information on hydraulic flumes, see *Fluming*, by A. M. R. Montagu, Central Board of Irrigation Publication No. 6 (Simla, India).

$$= 3 \cdot 64 \text{ ft/sec}$$

Also $$H + \frac{V^2}{2g} = H_1$$

and, from eq. (17), $$H = \frac{Q}{BV}$$

Hence $$\frac{Q}{BV} + \frac{V^2}{2g} = H_1$$

that is $$\frac{1 \cdot 5}{1 \cdot 5V} + \frac{V^2}{2g} = 0 \cdot 618$$

or $$64 \cdot 4 + V^3 = 39 \cdot 8V$$

Solving this equation by trial or by plotting,

$$V = 1 \cdot 76 \text{ ft/sec}$$

Then $$H = H_1 - \frac{V^2}{2g}$$

$$= 0 \cdot 618 - \frac{(1 \cdot 76)^2}{2g}$$

$$= 0 \cdot 57 \text{ ft}$$

10.17. Deep Water Surface Waves. The transmission of a surface wave over deep water is brought about by a local circulation of the surface water, as shown in Fig. 142. The restoring forces acting on

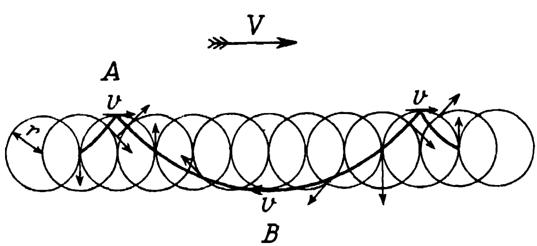


FIG. 142

the wave are mainly due to gravity, but surface tension also tends to resist wave motion, especially in small waves. In most waves the effect of surface tension is small and may be neglected.

Consider the deep water wave shown in Fig. 142. Each particle of water on the surface describes circles as shown in the figure. When the particle is at the crest A it has a velocity v in the same direction as the wave. By the time the trough of the wave reaches this section, the velocity of the particle is now in the opposite direction to that of the wave. Thus, this local circulation causes the water to flow upwards to form the crest of the wave and downwards to cause the trough. The particles of water below the surface

also flow in circles, but the diameter of these circles decreases with the depth until, at a certain depth, the local motion ceases. As the total energy at the surface remains constant, the decrease of kinetic energy between the trough and the crest must equal the increase of potential energy between these sections.

Let V = velocity of wave,
r = radius of circle described by particle,
T = period of wave in seconds,
λ = wavelength,
v = velocity of particle.

Then,
$$v = \frac{2\pi r}{T}$$

and
$$T = \frac{\lambda}{V} \qquad . \quad . \quad . \quad . \quad . \quad (23)$$

Bring the wave to rest by giving the whole mass of water a velocity of V backwards. Then—

$$\text{velocity of particle at crest } A = -V + v$$
$$= -V + \frac{2\pi r}{T}$$
$$\text{velocity of particle at trough } B = -V - v$$
$$= -V - \frac{2\pi r}{T}$$

Consider unit mass of water at crest A and trough B;

decrease of kinetic energy between B and A
= decrease of potential energy between A and B

or
$$\frac{1}{2g}\left[\left(-V - \frac{2\pi r}{T}\right)^2 - \left(-V + \frac{2\pi r}{T}\right)^2\right] = 1 \times 2r$$

from which
$$V = \frac{gT}{2\pi}$$

Substituting for T from eq. (23),
$$V = \sqrt{\frac{g\lambda}{2\pi}} \qquad . \quad . \quad . \quad . \quad . \quad (24)$$

It will be noticed that this equation is independent of the density of the liquid. It will also be noticed that eq. (24) can be written

$$\frac{\sqrt{g\lambda}}{V} = \sqrt{2\pi}$$

where $\frac{\sqrt{g\lambda}}{V}$ represents the Froude number of the wave motion.

10.18. Shallow Water Surface Waves. If the depth of the liquid is considerably less than the wavelength λ, the equation for the wave velocity given in § 10.17 will not apply. Under these circumstances the liquid may be regarded as shallow; the velocity of the wave is affected by the action of the bed in destroying the local circulation immediately above.

Consider the shallow water wave shown in Fig. 143; the wave is transmitted by means of a local circulation near the surface.

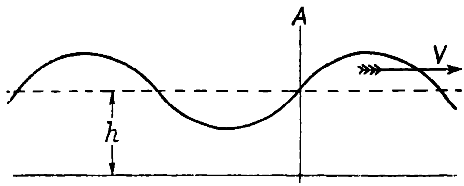


FIG. 143

Let V = velocity of wave,
h = mean depth of water.

Bring the wave to rest by giving the whole mass of water a velocity V in the opposite direction. Let Q be the flow through a section of unit width measured perpendicular to the direction of flow. Consider a vertical section A (Fig. 143) at which the depth is the mean depth h.

Then $$V = \frac{Q}{h}$$

At any vertical section the total energy of the water is constant, because the increase of kinetic energy between the trough of the wave and its crest must equal the loss of potential energy between these sections.

At section A,

$$\text{total energy of water} = E = h + \frac{V^2}{2g}$$

$$= h + \frac{Q^2}{2gh^2} \quad . \quad . \quad . \quad . \quad (25)$$

As this is the same at all vertical sections, it follows that $\mathrm{d}E/\mathrm{d}h = 0$; hence, differentiating eq. (25),

$$\frac{\mathrm{d}E}{\mathrm{d}h} = 1 - \frac{2Q^2}{2gh^3} \quad 0$$

that is $$1 - \frac{V^2}{gh} = 0, \text{ as } \frac{Q}{h} = V$$

Hence $$V = \sqrt{gh} \quad . \quad (26)$$

It will be noticed that this equation is proportional to the Froude number (§ 3.17) and the velocity is the critical velocity because the value of the Froude number is unity; hence, it follows that waves are formed when water flows in a channel with this velocity and depth.

EXERCISES 10

1. Using Bazin's formula $C = \dfrac{157 \cdot 5}{1 + \dfrac{k}{\sqrt{m}}}$, find the value of C for a broad shallow river of 2 ft depth. (Take wetted perimeter as breadth of river.)
Ans. 59·2.

2. A channel 10 ft wide at the bottom and with sides sloping 1 to 1, has a slope of 3 ft per mile. What would be the discharge if the water is 4 ft deep in the channel and $C = 95$ in the equation $v = C\sqrt{(mi)}$? (*Lond. Univ.*)
Ans. 205·5 ft³/sec.

3. Find the maximum discharge for least excavation of a rectangular channel 10 ft wide, when $C = 105$ and slope $= 1$ in 1,000.
Ans. 262·5 ft³/sec.

4. Explain what is meant by the "best cross-section" for a channel, and how it is determined.
A channel with side slopes at 45° is to have a cross-section of 120 ft². Determine the dimensions for the best section. (*Lond. Univ.*)
Ans. Depth = 8·1 ft. Base = 6·7 ft.

5. Deduce the formula for the depth of water in a circular conduit for maximum discharge.
Find the depth for maximum discharge in a circular brick sewer 4 ft in diameter. (*Lond. Univ.*) *Ans.* 3·8 ft.

6. A district has a drainage area of 2,500 acres, with a population of 20 persons per acre. The daily water supply to the district is equal to 40 gal per head. During dry weather it is found that 7 per cent of the daily dry weather flow passes along the sewer between the hours of 12 noon and 1 p.m.
Assuming a maximum rainfall of 1 in. in 24 hr over the whole area, determine the diameter of a circular sewer, having a slope of 1 in 3,000, which will take the maximum dry weather flow and the rainfall, without the sewer becoming more than half full. [Assume $C = 130$.] (*Lond. Univ.*)
Ans. 8·93 ft.

7. The bed of a stream has a slope of 1 in 1,000, and the depth of the water is 3 ft. A dam is to be built across the stream and provided with a sluice gate. Find the height of the dam so that the rise in level of the water, when the sluice gate is closed, may be limited to 7·5 ft. Take $C = 65$ in the formula $v = C\sqrt{(mi)}$, the coefficient of discharge of the dam, as a weir, 0·56, and assume, in calculating m, that the breadth of the stream is large in comparison with the depth. (*Lond. Univ.*) *Ans.* 8·17 ft above bed of stream.

8. A rectangular channel is 5 ft deep and 10 ft wide. If the value of C in Chezy's formula is 100, determine the discharge if the gradient is 1 in 1,000. (*A.M.I.Mech.E.*) *Ans.* 250 ft³/sec.

9. Show how the basic formula for steady flow in channels of constant slope and section is derived.

The depth of water in a circular brick-lined conduit, 6 ft in diameter, is to be 5 ft, and its capacity 50 million gallons a day. The water surface subtends an angle of 96° 20′ at the axis of the conduit. What must the gradient be? $C = 123$. (*A.M.I.C.E.*) *Ans.* 1/2,030.

10. You are required to ascertain the discharge of a river by means of current meter observations. Describe with diagrams the procedure at the site, and explain carefully how you would arrive at the discharge from your meter readings. The meter may be assumed calibrated and ready for use. (*A.M.I.C.E.*)

11. A concrete-lined channel has a bottom width of 10 ft, side slopes of 1 horizontal to 3 vertical, and a gradient of 1 in 800. When flowing 3 ft deep, it is found to have a capacity of 220 ft³/sec. What is the value of C? (*A.M.I.C.E.*) *Ans.* 133.

12. A stream is 40 ft wide at water level. At horizontal intervals of 5 ft the following results are obtained by current meter—

Distance from bank (ft)	0	2·5	7·5	12·5	17·5	22·5	27·5	32·5	37·5	40
Depth of water (ft)	0	1·0	2·2	3·2	4·4	5·0	3·4	2·2	1·2	0
Mean velocity on vertical (ft/sec)	—	1·5	1·9	2·2	2·9	3·3	2·7	1·8	1·4	—

What is the discharge in cubic feet per second? (*A.M.I.Mech.E.*)
Ans. 283·8 ft³/sec.

13. Deduce the Chezy formula for uniform flow in channels. An irrigation channel has a gradient of 1 in 2,000, a bottom width of 16 ft, and side slopes of 1 vertical to 2 horizontal. If the depth of water is 4 ft and the value of C is 90, what is the mean velocity and the capacity in cubic feet per second? (*A.M.I.Mech.E.*) *Ans.* $v = 3{\cdot}38$ ft/sec; $Q = 324$ ft³/sec.

14. A brick-lined sewer has a semicircular bottom and vertical side walls 2 ft apart. If the slope is 1 in 1,000 determine the discharge when the maximum depth of water is 3 ft. Take C in Chezy's formula as 90. (*A.M.I.C.E.*)
Ans. 14·05 ft³/sec.

15. Find the downstream height of a hydraulic jump occurring on a level bed when the upstream depth is 3 ft and velocity 35 ft/sec.

What conditions are necessary for a jump to occur? (*Lond. Univ.*)
Ans. 14·7 ft.

16. Explain the meaning of the term *critical depth* used in connexion with non-uniform flow along open channels. Show that in the case of non-uniform flow along a rectangular channel having a constant width of B ft the critical depth is given by

$$D_c = \sqrt[3]{\frac{Q^2}{gB^2}}$$

where Q is the quantity of water flowing along the channel in cubic feet per second. (*Lond. Univ.*)

17. A sluice spans a channel of rectangular section 60 ft wide, has an opening $2\frac{1}{2}$ ft deep, and discharges 1,720 ft³/sec of water. If a standing wave is formed on the downstream side of the sluice, determine the probable height of the crest above the upper edge of the sluice. (*Lond. Univ.*) *Ans.* 0·92 ft.

18. A Venturi flume is now frequently employed for the purpose of measuring the quantity of water flowing along an open channel. Explain how the venturi flume operates and describe the method of calculating the quantity from the experimental observations. (*Lond. Univ.*)

19. Show that the slope of the water surface at any point along a horizontal rectangular open channel is given by

$$\frac{\mathrm{d}D}{\mathrm{d}L} = \frac{fDQ^2}{2m(Q^2 - gB^2D^3)}$$

where Q is the quantity in cubic feet per second; D is the depth, B is the width and m is the hydraulic mean depth; f is the coefficient of friction. (*Lond. Univ.*)

20. Derive a formula for the change in level produced at a hydraulic jump. Show that a jump is impossible if the depth exceeds the critical depth. Water flows in a horizontal conduit of rectangular section with a velocity of 10 ft/sec where the depth is 2 ft, at which section a jump occurs. Calculate the depth after the jump and the energy lost per pound of water. (*Lond. Univ.*)
Ans. 2·67 ft; 0·014 ft-Lb.

21. Show that the discharge of a Venturi flume with a horizontal bed can be expressed as $Q = kbH^{3/2}$ where b is the breadth at the throat and H is the "still-water" head measured upstream. Assume that a standing wave will occur after the throat, and neglect friction. Calculate the discharge where the throat is 6 in. wide, the contraction of width is $1\frac{1}{2} : 1$, and the depth in the 9 in. section is 8 in., using a method of successive approximation. (*Lond. Univ.*) *Ans.* 2·03 ft³/sec.

22. A Venturi flume is to be installed in a channel conveying water with the object of raising the level of water upstream. The channel is rectangular in section and is 40 ft wide, has a gradient of 1 in 6,400 and a depth of water 5 ft. The width of the throat section of the flume is 20 ft. If the bed of the flume at the throat is a streamlined hump, find the necessary height of the hump in order that the depth of water on the upstream side shall be 6 ft. Take $v = 140\sqrt{(mi)}$ for the channel, ignore hydraulic losses in the flume, and assume that a standing wave is formed on the downstream side of the hump. (*Lond. Univ.*) *Ans.* 2 ft.

23. Prove that the velocity of a wave of low elevation travelling over the surface of shallow water is given by $C = \sqrt{(gh)}$, in which h is the depth. Hence, explain the significance of the term "critical depth" as it is used in connexion with the flow of water in a channel. (*Lond. Univ.*)

24. Prove that the critical depth of flow in a channel occurs when the Froude number is unity.

Water flows under a sluice gate, with streaming flow, into a channel. After traversing a short length of the channel a hydraulic jump occurs, after which the flow becomes tranquil and has a depth of 5 ft and a velocity of 6 ft/sec. Calculate the depth of water on the upstream, or streaming flow, side of the jump, and the loss of energy per pound of water at the jump. (*Lond. Univ*).
Ans. $d_1 = 1{\cdot}68$ ft; loss of head $= 1{\cdot}08$ ft of water.

CHAPTER 11

DIMENSIONAL ANALYSIS

11.1. Introduction. Dimensional analysis is a mathematical method of obtaining the equations governing certain natural phenomena by balancing the fundamental dimensions, mass, length and time, of the problem considered. Such a method is known as the principle of dimensional similarity. The method can be applied to all types of fluid resistance, fluid flow through orifices, pipes and over weirs, and many other problems in fluid mechanics and thermodynamics.

The equations produced by this method also give the non-dimensional constant which governs the problem under consideration. This non-dimensional constant is very important in engineering practice, as it enables the behaviour of problems of the same type to be predicted providing the linear dimensions are geometrically similar.

The experimental coefficients governing the problem, which can be found only from tests, are a function of the problem's non-dimensional factor. If the coefficients for a given phenomenon are plotted on a base representing its non-dimensional constant, the points will lie on a smooth curve (Figs. 144, 145 and 146), from which the correct coefficient can be read off for any similar problem.

Some authorities hold the opinion that dimensional analysis is not a fundamental solution of a problem because a knowledge of all the variables of the equation must be known beforehand. But if the solution is commenced with some wrong variables, or missing variables, the correct non-dimensional constant will not be obtained. If the experimental values of the coefficient are now plotted on a base representing this false non-dimensional constant, it will be found that the points are irregular and that it is impossible to draw a smooth curve through them. This shows that wrong or missing variables have been assumed and another attempt must be made. Thus it is possible to arrive at the correct solution by trial and error, providing enough experimental values of the coefficient have been obtained for the check.

The solution of a problem by dimensional analysis consists in writing down the equation governing the problem in terms of all the variables on which the problem depends. It is assumed that the indices of the variables are not known; these are now represented by the symbols a, b, c, etc. Then, by balancing the fundamental dimensions, M, L and T, of each side of the equation, the values of the indices a, b, c, etc. are found, together with the non-dimensional constant.

The following fundamental dimensions are required in this chapter—

mass = M
linear dimension = L
time = T
area = L^2
volume = L^3
velocity = LT^{-1}
acceleration = LT^{-2}
force = mass × acceleration = MLT^{-2}
$\eta = ML^{-1}T^{-1}$ (§ 3.16)
ρ = mass/volume = ML^{-3}
$\nu = L^2T^{-1}$ (§ 3.16)
$g = LT^{-2}$
volume per second = $Q = L^3T^{-1}$
surface tension = $\sigma = MT^{-2}$ = force per unit length
bulk modulus = $K = ML^{-1}T^{-2}$ = stress per unit strain

11.2. Frictional or Viscous Resistance. It is known that the governing factor of this problem is the coefficient of viscosity η. Hence, obtain equations of all unknown indices in terms of the index of η.

It is known that the viscous resistance of a fluid depends on the wetted area, the velocity, the density, and the coefficient of viscosity. Assuming the actual law of variation to be unknown, the equation for the resistance may be written

$$R = k_1\rho^a\eta^b l^c v^d$$

or
$$R \propto \rho^a\eta^b l^c v^d \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where a, b, c and d are unknown indices, R is the total resistance, and k_1 is a constant to be determined experimentally. Now the fundamental units for each side of this equation must balance; hence, by putting the total resistance R as a force, and by substituting the fundamental units for ρ, η, and v, the equation becomes

$$MLT^{-2} = k(ML^{-3})^a(ML^{-1}T^{-1})^bL^c(LT^{-1})^d$$

that is
$$MLT^{-2} = kM^aL^{-3a}M^bL^{-b}T^{-b}L^cL^dT^{-d}$$

As η is the governing variable, obtain all indices in terms of the index of η, which in this case is b.

Now, as the indices of M on each side of the equation are equal,

$$1 = a + b$$

from which
$$a = 1 - b \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Also, as the indices of L on each side of the equation are equal,

$$1 = -3a - b + c + d \quad . \quad . \quad . \quad . \quad (3)$$

Also, as the indices of T on each side of the equation are equal,

$$-2 = -b - d$$

from which $\quad d = 2 - b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$

Substituting the values of eqs. (2) and (4) in eq. (3),

$$1 = -3(1 \quad b) - b + c + (2 - b)$$

from which $\quad c = 2 - b \quad (5)$

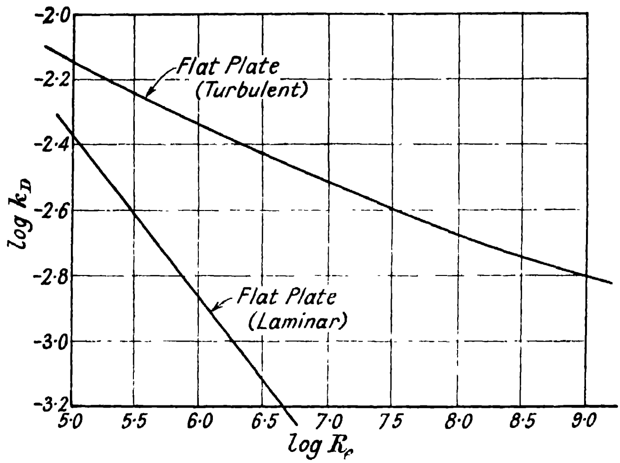


FIG. 144. RELATION BETWEEN DRAG COEFFICIENT AND REYNOLDS NUMBER

Substituting the values of a, c, and d in eq. (1),

$$R \propto \rho^{1-b}\eta^{b}l^{2-b}v^{2-b}$$

Re-arranging and placing all terms containing index b within the brackets,

$$R \propto \rho l^2 v^2 \left(\frac{\eta}{\rho l v}\right)^b \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

It will be noticed that the term inside the brackets is the inverse of the Reynolds number (§ 3.17); hence R is a function of R_e.

Eq. (6) can be written

$$R = k_1 \rho l^2 v^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where k_1 is a frictional coefficient and is an unknown function of R_e; it is usually termed the *frictional drag coefficient* and represented by k_D.

This solution has proved that the frictional coefficient k_D is a function of the non-dimensional constant $\rho vl/\eta$ which is the Reynolds

number. Any experimental results obtained for the value of k_D, if plotted on a base of R_e, will give a smooth curve. The logs of these ordinates are shown plotted in Fig. 144, separate graphs being obtained for laminar flow and turbulent flow. These results were obtained from experiments on flat plates and confirm the above conclusion that k_D is a function of the non-dimensional constant R_e.

It will be noticed from eq. (7) that the term l^2 is a function of the surface area.

Let A = surface area in contact with fluid.

Then, eq. (7) may be written

$$R = (k_1\rho)Av^2$$

This is the same as Froude's equation if $f' = k_1\rho$.

Eq. (7) may also be written

$$R = \frac{C_f\rho Av^2}{2}$$

This is the same as the frictional drag equation given in § 9.1 (eq. (1)). In this case, $C_f = 2k_1$.

11.3. Surface Wave Resistance of Ships. This resistance is due to the formation of surface waves by a ship. The pressure in the water due to the ship's motion cannot be maintained at the water surface where the pressure must be atmospheric. Hence, the water is lifted against gravity by the water pressure; thus the governing factor in this case is g.

The total resistance will depend on the density of the liquid, the value of g, the linear dimensions and the velocity. The equation may be written,

$$R \propto \rho^a g^b l^c v^d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

As g is the governing variable, obtain all indices in terms of the index of g, which in this case is b.

Substituting the fundamental dimensions in the variables,

$$\begin{aligned} \mathrm{MLT^{-2}} &= (\mathrm{ML^{-3}})^a(\mathrm{LT^{-2}})^b\mathrm{L}^c(\mathrm{LT^{-1}})^d \\ &= \mathrm{M}^a\mathrm{L}^{-3a}\mathrm{L}^b\mathrm{T}^{-2b}\mathrm{L}^c\mathrm{L}^d\mathrm{T}^{-d} \end{aligned}$$

Equating indices of M,

$$1 = a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Equating indices of T,

$$-2 = -2b - d$$

Hence $$d = 2 - 2b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Equating indices of L,

$$1 = -3a + b + c + d$$

Substituting for a and d from eqs. (9) and (10),

$$1 = -3 + b + c + 2 - 2b$$
$$= -1 - b + c$$

Hence $\qquad c = 2 + b \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$

Substituting eqs. (9), (10) and (11) in eq. (8), and putting all terms containing b inside the brackets,

$$R \propto \rho l^2 v^2 \left(\frac{gl}{v^2}\right)^b \quad . \quad . \quad . \quad . \quad (12)$$

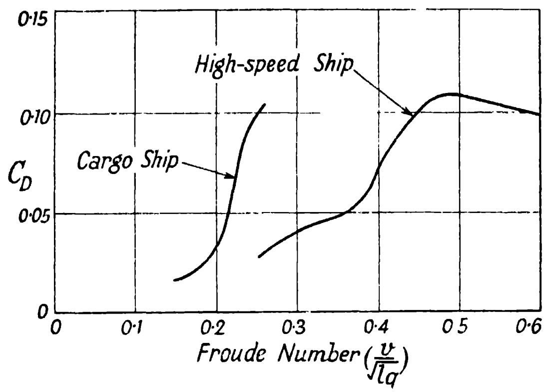


FIG. 145

It will be noticed that the term inside the brackets is the square of the inverse of the Froude number, F_r (§ 3.17). This proves that in this case the non-dimensional constant is the Froude number.

Eq. (12) may be written

$$R = k_2 \rho l^2 v^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where the resistance coefficient k_2 is a function of F_r.

Eq. (13) can also be written in the form

$$D = \frac{C_D \rho A v^2}{2}$$

where $C_D \propto 2k_2$,

A is a function of l^2 and is the immersed cross-sectional area amidships,

D is the drag due to surface wave formation,

C_D is the drag coefficient and is a function of $2k_2$.

In Fig. 145 is shown plotted the value of C_D on a base of F_r for two types of ships.* It will be noticed the results give one curve

* From "Experimental fluid dynamics applied to engineering practice," by G. A. Hankins, *Engineering*, **157** (25th February and 3rd March, 1944).

for a slow-moving cargo ship, and a separate curve for a high-speed ship. The two curves do not coincide because the two ships were not geometrically similar.

11.4. Non-dimensional Factor for Compression Wave Resistance. Compression waves are formed by the relative movement of a body completely submerged in a fluid; they also occur in the transmission of sound and will depend on the bulk elastic modulus K (§ 13.10). Let R be the resistance to the body caused by the formation of the pressure wave. Assume,

$$R = k_3 \rho^a l^b v^c K^d$$

then

$$R \propto \rho^a l^b v^c K^d. \qquad (14)$$

where k_3 is an experimental coefficient, l the linear dimension, v the relative velocity between the body and fluid, and a, b, c, d are unknown indices of which the values are to be found.

In this problem the governing variable is K; hence, obtain all indices in terms of the index of K, which in this case is d.

Substituting the fundamental dimensions in eq. (14),

$$\mathrm{MLT^{-2}} = (\mathrm{ML^{-3}})^a \mathrm{L}^b (\mathrm{LT^{-1}})^c (\mathrm{ML^{-1}\,T^{-2}})^d$$

Then

$$\mathrm{MLT^{-2}} = (\mathrm{M}^a\mathrm{L}^{-3a})(\mathrm{L}^b)(\mathrm{L}^c\mathrm{T}^{-c})(\mathrm{M}^d\mathrm{L}^{-d}\mathrm{T}^{-2d})$$

Equating the indices of M,

$$1 = a + d$$

from which

$$a = 1 - d$$

Equating the indices of T,

$$-2 = -c - 2d$$

from which

$$c = 2 - 2d$$

Equating the indices of L,

$$1 = -3a + b + c - d$$

Substituting for the values of a and c,

$$1 = -3 + 3d + b + (2 - 2d) - d$$

Hence

$$b = 2$$

Substituting these values of a, b, and c in eq. (14),

$$R \propto \rho l^2 v^2 \left(\frac{K}{\rho v^2}\right)^d$$

$$= k_3 \rho l^2 v^2 \qquad (15)$$

Hence, the constant k_3 is a function of $(K/\rho v^2)$, which is the non-dimensional factor governing this type of resistance.

It will be shown in § 13.10 that the velocity of sound in a fluid is given by the equation

$$v_s = \sqrt{\frac{K}{\rho}}$$

where v_s is the velocity of sound in the fluid under consideration.

Substituting this value in the non-dimensional factor,

$$k_3 \text{ is a function of } \left(\frac{v_s^2}{v^2}\right)$$

or it may be written as a function of v/v_s.

This non-dimensional factor is known as the *Mach number*, M_a, (§ 3.17), and is used as a criterion when dealing with bodies having velocities in the vicinity of the velocity of sound (§ 17.1).

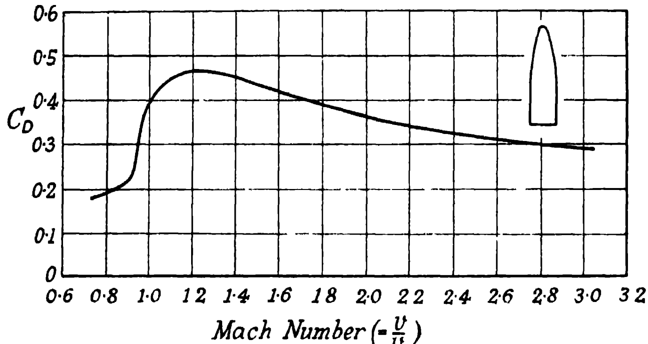


FIG. 146

It is evident from eq. (15) that the experimental coefficient k_3 is not a true constant for all fluids, but varies with the elastic constant, the density and the velocity. For any given fluid, k_3 will vary with the velocity.

Eq. (15) may be written

$$D = \frac{C_D \rho A v^2}{2}$$

where $C_D = 2k_3$.

The variation of C_D with the Mach number during a test on a projectile is shown in Fig. 146. It will be noticed that a continuous curve is obtained. thus proving that C_D is proportional to M_a. The portion of the curve before $M_a = 1$, represents a subsonic speed, and the portion beyond $M_a = 1$, represents a supersonic speed. At the point where M_a is unity the speed is sonic.

11.5. Non-dimensional Factor for Weirs. The experimental coefficients used in problems on rectangular and triangular weirs or notches (§ 5.2–§ 5.6) are not true constants for all heads, but can be shown to vary with such quantities as η, ρ, H, g and the surface tension σ. It will be assumed that the weirs or notches considered are geometrically similar. Then, for a rectangular weir or notch, the breadth B is proportional to H. For a triangular notch, the angle θ is constant (§ 5.3).

Assume the equation for the discharge is in the form

$$Q = kH^a g^b \eta^c \rho^d \sigma^e$$

Then $$Q \propto H^a g^b \eta^c \rho^d \sigma^e \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Obtain all indices in terms of c and e, because η and σ are the governing variables.

Substituting the fundamental dimensions in eq. (16),

$$\mathrm{L^3T^{-1}} = \mathrm{L}^a(\mathrm{L}^b\mathrm{T}^{-2b})(\mathrm{M}^c\mathrm{L}^{-c}\mathrm{T}^{-c})(\mathrm{M}^d\mathrm{L}^{-3d})(\mathrm{M}^e\mathrm{T}^{-2e})$$

Equating the indices of M,

$$0 = c + d + e$$

from which $$d = -c - e$$

Equating the indices of T,

$$-1 = -2b - c - 2c$$

from which $$b = \tfrac{1}{2} - \tfrac{1}{2}c - e$$

Equating the indices of L,

$$3 = a + b - c - 3d$$

Substituting for b and d,

$$3 = a + (\tfrac{1}{2} - \tfrac{1}{2}c - e) - c + (3c + 3e)$$

from which $$a = 2\tfrac{1}{2} - 1\tfrac{1}{2}c - 2c$$

Substituting these values of a, b, and d in eq. (16),

$$Q \propto H^{5/2}g^{1/2} \times \left(\frac{\eta}{H^{3/2}g^{1/2}\rho}\right)^c \left(\frac{\sigma}{H^2 g\rho}\right)^e \quad . \quad . \quad (17)$$

Hence k is a function of $\left(\frac{\eta}{H^{3/2}g^{1/2}\rho}\right)\left(\frac{\sigma}{H^2 g\rho}\right)$.

Thus, the coefficient of discharge of a rectangular or triangular notch, or weir, will vary with the head, the viscosity, the density and the surface tension. It is not, therefore, a constant as was assumed in § 5.2–§ 5.6. The actual variation of C_d is small, and it is of sufficient accuracy for practical purposes to assume C_d to be a constant within reasonable limits of head.

As the breadth B has been assumed to be proportional to H in a rectangular weir, eq. (17) when applied to a rectangular weir becomes

$$Q = kBH^{3/2}g^{1/2}\left(\frac{\eta}{H^{3/2}g^{1/2}\rho}\right)^c \left(\frac{\sigma}{H^2 g\rho}\right)^e$$

There are thus two non-dimensional constants for a weir—

$\frac{\eta}{H^{3/2}g^{1/2}\rho}$ is a constant for both weirs

and $\qquad \dfrac{\sigma}{H^2 g\rho}$ is a constant for both weirs

It will be noticed that these two conditions cannot occur simultaneously. The first of these is the Reynolds number, as $v \propto \sqrt{(2gH)}$; the second is the surface tension constant.

From the results of the two weir experiments shown in Figs. 69 and 71 it will be seen from the deviation of the points from the straight line that C_d is not actually a constant, but is varying slightly with the head. This confirms the above analytical results.

EXAMPLE 1

Calculate the values of the two non-dimensional constants of a rectangular weir, given in eq. (17). The water flowing over the weir has a head of 4 ft, a kinematic viscosity of $8{\cdot}42 \times 10^{-6}$ ft-sec units, and a surface tension of $5{\cdot}04 \times 10^{-3}$ Lb/ft.

From eq. (17), the two non-dimensional constants are

(1) $$\frac{\eta}{H^{3/2}g^{1/2}\rho} \text{ or } \frac{\nu}{H^{3/2}g^{1/2}}$$

and (2) $$\frac{\sigma}{H^2 g\rho}$$

Hence, the value of (1) is

$$\frac{\nu}{H^{3/2}g^{1/2}} = \frac{8{\cdot}42 \times 10^{-6}}{(4)^{3/2} \times \sqrt{32{\cdot}2}} = 1{\cdot}86 \times 10^{-7}$$

The value of (2) is

$$\frac{\sigma}{H^2 g\rho} = \frac{5{\cdot}04 \times 10^{-3}}{16 \times 32{\cdot}2 \times \dfrac{62{\cdot}4}{32{\cdot}2}} = 5{\cdot}05 \times 10^{-6}$$

11.6. Non-dimensional Factor for Small Orifice. It was shown in § 4.4 that the discharge through a small orifice under a constant head H (Fig. 147) is given by the equation

$$Q = C_d A\sqrt{2gH}$$

where A is the area of the orifice in square feet and C_d is the coefficient of discharge. This equation may be written

$$Q = kD^2\sqrt{gH} \qquad . \quad . \quad . \quad . \quad (18)$$

where k is a constant and D the diameter of the orifice.

If the principle of dimensional similarity be applied to this problem, it is found that k is not a true constant, but varies with

such factors as the head, viscosity, density, gravity and surface tension.

Let p = mean pressure at orifice.

Then $\quad p = wH$

$\quad\quad = \rho g H$

Applying the principle of dimensional similarity to this problem, and ignoring the effect of surface tension as small,

$$Q = kp^a D^b g^c \rho^d \eta^e$$

Then $\quad Q \propto p^a D^b g^c \rho^d \eta^e$

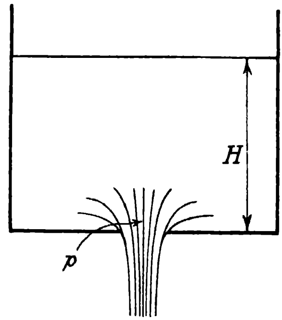


FIG. 147

Inserting the fundamental dimensions,

$$L^3T^{-1} = (M^aT^{-2a}L^{-a})L^b(L^cT^{-2c})(M^dL^{-3d})(M^eL^{-e}T^{-e})$$

Obtain all indices in terms of c and e.

Equating the indices of M,

$$0 = a + d + e$$

Hence $\quad a = -d - e$

Equating the indices of T,

$$-1 = -2a - 2c - e$$

Substituting for a,

$$-1 = (+2d + 2e) - 2c - e$$

from which $\quad d = -\frac{1}{2} - \frac{1}{2}e + c$

Hence $\quad a = \frac{1}{2} - \frac{1}{2}e - c$

Equating the indices of L,

$$3 = -a + b + c - 3d - e$$

Substituting for a and d,

$$3 = (-\tfrac{1}{2} + \tfrac{1}{2}e + c) + b + c + (\tfrac{3}{2} + \tfrac{3}{2}e - 3c) - e$$

from which $\quad b = 2 + c - e$

Hence, substituting for a, b, and c in the fundamental equation, and separating all terms containing e and c,

$$Q = kp^{1/2}D^2\rho^{-1/2}\left(\frac{\eta}{D\rho^{1/2}p^{1/2}}\right)^e\left(\frac{D\rho g}{p}\right)^c$$

Substituting $p = \rho g H$,

$$= k\sqrt{\frac{\rho g H}{\rho}}\,D^2\left(\frac{\eta}{D\rho^{1/2}\rho^{1/2}g^{1/2}H^{1/2}}\right)^e\left(\frac{D\rho}{H\rho}\right)^c$$

$$= k\sqrt{gH}\,D^2\left(\frac{\eta}{D\rho\sqrt{2gH}}\right)^e\left(\frac{D}{H}\right)^c$$

Hence, the coefficient C_d is a function of

$$\left(\frac{\eta}{D\rho\sqrt{2gH}}\right)\left(\frac{D}{H}\right)$$

The second term shows that for true geometrical similarity $D \propto H$; then, if the head and orifice diameter are varied to suit this condition, the value of C_d should be a function of

$$\left(\frac{\eta}{D\rho\sqrt{2gH}}\right)$$

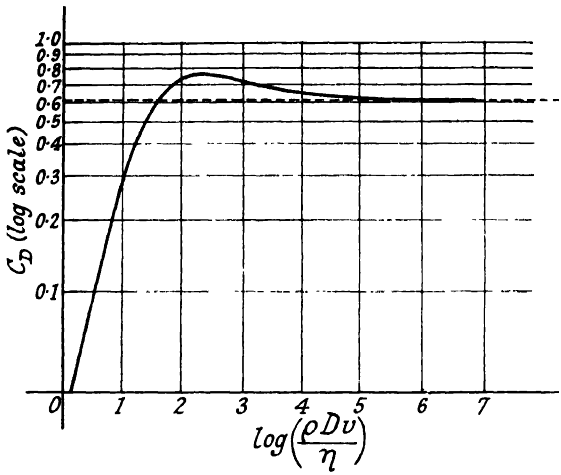


FIG. 148

Substituting for $v = \sqrt{(2gH)}$, this constant may be written

$$\frac{\rho Dv}{\eta}$$

It will be noticed from this form of the non-dimensional constant that it is the same as the Reynolds number.

Fig. 148 shows the results of experimental measurements of C_d for a certain orifice. In this figure C_d is plotted against log $(\rho Dv/\eta)$.

It will be noticed that the first portion of the curve of Fig. 148, representing low Reynolds numbers, denotes a laminar type of flow, and that the final horizontal portion of the curve represents a full turbulent flow. The curve connecting these two curves is a transition curve which denotes the gradual development of turbulence.

This application of the principle of similarity to an orifice shows that the coefficient of discharge is not a constant, but varies with the head, the density, the viscosity and consequently with the temperature.

In Fig. 149 the author has plotted, on a base representing R_e, the results of Hamilton-Smith's experiments on round orifices. It will be noticed that the points lie on a smooth curve and that the value of C_d falls as R_e increases, becoming stabilized at a value of 0·597. This curve corresponds to the portion of curve of Fig. 148 situated to the right of $\log R_e = 4$.

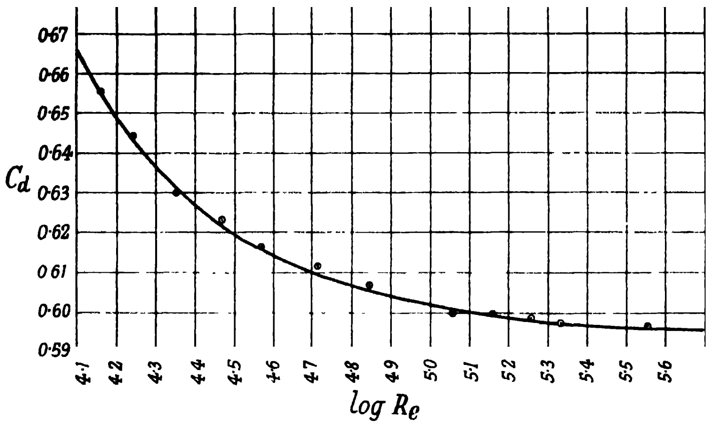


FIG. 149

EXAMPLE 2

One of the non-dimensional constants for the discharge through a small orifice was found to be $\dfrac{\eta}{D\rho\sqrt{2gH}}$. Calculate the value of this constant for water, having a coefficient of viscosity of 0·01 c.g.s. units, discharging through a 1 in. diameter orifice under a head of 2·0 ft.

From § 3.16,

$$\eta = \frac{0{\cdot}01 \times 30{\cdot}5}{453{\cdot}6 \times 32{\cdot}2} \text{ engineers' units}$$

and

$$\rho = \frac{62{\cdot}4}{32{\cdot}2}$$

Hence

$$\nu = \frac{\eta}{\rho} = \frac{0{\cdot}01 \times 30{\cdot}5}{453{\cdot}6 \times 32{\cdot}2} \times \frac{32{\cdot}2}{62{\cdot}4}$$
$$= 10{\cdot}79 \times 10^{-6} \text{ ft units}$$

Then, the value of the non-dimensional constant is

$$\frac{\nu}{D\sqrt{2gH}} = \frac{10{\cdot}79 \times 10^{-6}}{\frac{1}{12}\sqrt{64{\cdot}4 \times 2}}$$
$$= 11{\cdot}42 \times 10^{-6}$$

EXAMPLE 3

Using the curve of Fig. 148 find the value of C_d for a circular orifice of 1 in. diameter through which water is discharging under a head of 2 ft. The viscosity of the water is $3{\cdot}18 \times 10^{-5}$ engineers' units.

The non-dimensional constant for an orifice was found to be of the form

$$\frac{\rho D v}{\eta}$$

where $\rho = w/g$.

Now
$$v = \sqrt{2gH}$$
$$= \sqrt{64{\cdot}4 \times 2}$$
$$= 11{\cdot}33 \text{ ft/sec}$$

Then
$$\frac{\rho D v}{\eta} = \frac{62{\cdot}4 \times \frac{1}{12} \times 11{\cdot}33}{32{\cdot}2 \times 3{\cdot}18 \times 10^{-5}}$$
$$= 5{\cdot}74 \times 10^4$$

Hence
$$\log \frac{\rho D v}{\eta} = 4{\cdot}759$$

Then, from the curve of Fig. 148,
$$C_d = 0{\cdot}62$$

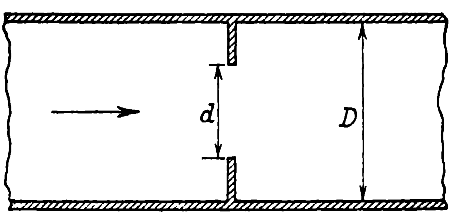


FIG. 150

11.7. Non-dimensional Factor for Orifice in Pipe. Consider a pipe of diameter D containing an orifice of diameter d fitted in a diaphragm as shown in Fig. 150.

Let p = difference of pressure between the two sides of the orifice, due to loss of head.

This problem produces non-dimensional factors which can be found by applying the principle of dimensional similarity. Assume

$$Q = kp^a d^b D^c \rho^d \eta^e \quad . \quad . \quad . \quad . \quad (19)$$

Then
$$Q \propto p^a d^b D^c \rho^d \eta^e$$

Substituting the fundamental dimensions in this equation,

$$L^3T^{-1} = k(M^aL^{-a}T^{-2a})L^bL^c(M^dL^{-3d})(M^eL^{-e}T^{-e})$$

Obtain all indices in terms of c and e.

Equating the indices of M,

$$0 = a + d + e$$

Hence $\qquad d = -a - e \qquad \ldots \ldots \qquad (20)$

Equating the indices of T,

$$-1 = -2a - e$$

Hence $\qquad a = \frac{1}{2} - \frac{1}{2}e$

Substituting this value of a in eq. (20),

$$d = -\tfrac{1}{2} - \tfrac{1}{2}e$$

Equating the indices of L,

$$3 = -a + b + c - 3d - e$$

Substituting for a and d,

$$3 = (-\tfrac{1}{2} + \tfrac{1}{2}e) + b + c + (\tfrac{3}{2} + \tfrac{3}{2}e) - e$$

from which $\qquad b = 2 - e - c$

Substituting these values of a, b, and d in eq. (19),

$$Q \propto p^{1/2}d^2\rho^{-1/2}\left(\frac{\eta}{dp^{1/2}\rho^{1/2}}\right)^e\left(\frac{D}{d}\right)^c \qquad \ldots \qquad (21)$$

Hence, the constant k is a function of $(\eta/dp^{1/2}\rho^{1/2})$, which corresponds to the Reynolds number, and is also a function of (D/d).

From eq. (21),

$$Q = kd^2\sqrt{\frac{p}{\rho}}$$

11.8. Non-dimensional Factor for Resistance of Oiled Bearings. The viscous resistance of an oiled bearing depends on the linear dimensions of the bearing, the coefficient of viscosity of the oil, the speed of rotation, and on the pressure on the bearing. This resistance causes the frictional torque or moment on the bearing. A non-dimensional constant for oiled bearings having true dimensional similarity can be obtained by equating the fundamental dimensions.

Let R = frictional resisting torque on bearing,
N = speed of shaft in r.p.m.,
D = linear dimension of bearing,
p = pressure per unit area on bearing.

Then $\qquad R \propto \eta N D p$

Let $\qquad R = k\eta^a N^b D^c p^d \qquad \ldots \ldots \qquad (22)$

Inserting the fundamental dimensions, M, L, and T,

$$ML^2T^{-2} = k(M^aL^{-a}T^{-a})(T^{-b})(L^c)(M^dT^{-2d}L^{-d})$$

Obtain all indices in terms of d.

Equating the indices of M,

$$1 = a + d$$

Hence $$a = 1 - d$$

Equating the indices of L,

$$2 = -a + c - d$$
$$= -(1 - d) + c - d$$

Hence $$c = 3$$

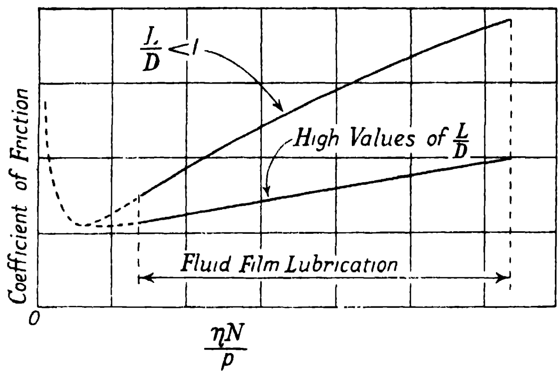


FIG. 151

Equating the indices of T,

$$-2 = -a - b - 2d$$
$$= -(1 - d) - b - 2d$$

Hence $$b = 1 - d$$

Substituting these values of a, b and c in eq. (22),

$$R \propto \eta^{1-d} N^{1-d} D^3 p^d$$

Hence $$R \propto \eta N D^3 \left(\frac{p}{\eta N}\right)^d$$

or $$R = k\eta N D^3 \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where k is a function of $(p/\eta N)$.

It will be seen from this that the non-dimensional factor for geometrical similar oiled bearings is

$$\frac{\eta N}{p}$$

and the frictional coefficient for the oiled bearing will be a function of this parameter.

Fig. 151 shows the variation of the coefficient of friction with the non-dimensional factor, taken from tests*; two curves are shown,

* From "Experiment fluid dynamics applied to engineering practice," by G. A. Hankins, *Engineering*, **157**.

one for bearings having an L/D ratio less than unity, the other for bearings having high values of L/D. It will be noticed that in each case the value of the frictional coefficient is a function of the non-dimensional factor.*

11.9. Resistance of Sphere Moving in Fluid. The motion of a sphere moving through a fluid depends on the density and viscosity of the fluid and on the radius and velocity of the sphere. Hence, the resistance to motion R will be an equation of the form

$$R \propto \rho^a \eta^b r^c v^d \qquad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where r is the radius of the sphere and v its linear velocity. Substituting the fundamental dimensions in the above equation and obtaining all indices in terms of d,

$$\mathrm{MLT^{-2}} = (\mathrm{ML^{-3}})^a(\mathrm{ML^{-1}T^{-1}})^b\mathrm{L}^c(\mathrm{LT^{-1}})^d$$

Equating the indices of M,

$$1 = a + b$$

from which $\qquad a = 1 - b \qquad . \quad . \quad . \quad . \quad . \quad . \quad (25)$

Equating the indices of T,

$$-2 = -b - d$$

from which $\qquad b = 2 - d \qquad . \quad . \quad . \quad . \quad . \quad . \quad (26)$

Substituting this value of b in eq. (25),

$$a = 1 - 2 + d$$
$$= -1 + d \qquad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Equating the indices of L,

$$1 = -3a - b + c + d$$

Substituting the values of b and a from eqs. (26) and (27),

$$1 = 3 - 3d - 2 + d + c + d$$

Hence $\qquad c = d \qquad . \quad . \quad . \quad . \quad . \quad . \quad (28)$

Substituting in eq. (24) the values of a, b, and c from eqs. (26), (27), and (28),

$$R \propto \rho^{-1+d}\eta^{2-d}r^d v^d$$

Then $\qquad R \propto \dfrac{\eta^2}{\rho}\phi\left(\dfrac{\rho r v}{\eta}\right)^d \qquad . \quad . \quad . \quad . \quad . \quad . \quad (29)$

or $\qquad R = k_1 \dfrac{\eta^2}{\rho}$

where k_1 is a function of $\rho r v/\eta$, which is the Reynolds number.

* For further information, see "The film lubrication of the journal bearing," by R. O. Boswall and J. C. Brierley, *Proc. Instn. Mech. Engrs.* **122** (1932), p. 423.

If the velocity of the sphere is very small, the motion is of a laminar type of flow, in which case $R \propto v$ [§ 8.2, eq. (8)]. Applying this condition to eq. (24), it follows that the suffix d is unity. Substituting this value of d in eq. (29),

$$R = k\frac{\eta^2}{\rho} \times \frac{\rho r v}{\eta}$$

where k is a constant.

Hence $$R = k\eta r v \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

An equation of the same form as eq. (30) was deduced analytically by Stokes for the resistance of a sphere moving in a fluid with a uniform velocity. He calculated the value of k to be 6π. Substituting this value in eq. (30),

$$R = 6\pi\eta r v \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Eq. (31) is known as Stokes' law. This law is used in the grading of fine powders, the particles of which are too small to be measured by direct means. The powder is mixed in suspension in a suitable liquid and the time taken for the powder to settle is measured. If the powder is assumed to consist of spherical-shaped particles it should approximately obey Stokes' law in settling to the bottom of the container. The time taken by the powder in settling is thus a function of the diameter of its particles; this provides a method of grading. It will be noticed that the finer powders take a longer period in being completely deposited.

Stokes' law is also used to calculate the amount of silting which may occur in problems dealing with channel flow and weirs. The silting is due to the depositing of fine particles of sand or earth which is in suspension in the flowing water. Whilst the water has a high velocity, these particles remain in suspension, but when the flow becomes small or stationary the particles are deposited on the stream bed at a rate given by Stokes' law. This is noticeable at the inside edge of river bends, in backwaters, and on the stagnant side of dams and weirs.

11.10. Non-dimensional Factor for Propellers. The principle of dimensional similarity can be applied to screw propellers, which are geometrically similar, by considering the thrust of the propeller T, the diameter of the propeller d, the speed of advance v, and its speed of rotation n. Let ρ and η apply to the fluid in which the propeller is immersed. Assume T is proportional to ρ, d, v, n, and η; then

$$T \propto \rho^a d^b v^c n^e \eta^f \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Inserting the fundamental dimensions for these factors,

$$\mathrm{MLT^{-2}} = (\mathrm{ML^{-3}})^a \mathrm{L}^b (\mathrm{LT^{-1}})^c (\mathrm{T^{-1}})^e (\mathrm{ML^{-1}T^{-1}})^f$$

Then $$\mathrm{MLT^{-2}} = \mathrm{M}^a \mathrm{L}^{-3a} \mathrm{L}^b \mathrm{L}^c \mathrm{T}^{-c} \mathrm{T}^{-e} \mathrm{M}^f \mathrm{L}^{-f} \mathrm{T}^{-f}$$

Obtain all indices in terms of e and f.

Equating the indices of M,

$$1 = a + f$$

Hence $\qquad a = 1 - f \qquad \ldots \ldots \ldots (33)$

Equating the indices of T,

$$-2 = -c - e - f$$

Then $\qquad c = 2 - e - f \qquad \ldots \ldots \ldots (34)$

Equating the indices of L,

$$1 = -3a + b + c - f$$

Substituting the values of a and c from eqs. (33) and (34),

$$1 = -3 + 3f + b + 2 - e - f - f$$

from which $\qquad b = 2 - f + e \qquad \ldots \ldots \ldots (35)$

Substituting in eq. (32) the values of a, b and c from eqs. (33), (34) and (35),

$$T \propto \rho^{1-f} d^{2-f+e} v^{2-e-f} n^e \eta^f$$

Hence $\qquad T \propto \rho d^2 v^2 \left(\frac{\eta}{\rho d v}\right)^f . \left(\frac{dn}{v}\right)^e$

or $\qquad T = k d^2 v^2 \qquad \ldots \ldots \ldots (36)$

where k is a function of $(\eta/\rho d v)$ and (dn/v).

11.11. Non-dimensional Constants by Group Method. The method of application of the principle of dimensional similarity demonstrated in § 11.5–§ 11.10 can be simplified by dealing with the variables in groups of four. This group method considerably reduces the labour when a large number of variables are being considered. In applying the group method the following rules should be observed—

1. Choose three variables, preferably those which occur in the fundamental equation, and which contain all three fundamental dimensions M, L and T.
2. Form groups containing all the above three plus each of the other variables, in turn.
3. Give indices a, b and c to the three variables of (1) only.

As an example, apply the method to the orifice of § 11.6 and Fig. 147. Consider the variables to be Q, p, D, g, ρ, η and surface tension σ. Now,

$$0 = \phi(Q p D g \rho \eta \sigma)$$

Let Q, p and D be the three variables chosen to satisfy Rule (1).

Then, $\qquad 0 = \phi(Q^a p^b D^c g \rho \eta \sigma)$

Group 1. $\qquad 0 = \phi(Q^a p^b D^c g) \qquad \ldots \ldots \ldots (37)$

Inserting the fundamental dimensions,

$$0 = (L^{3a}T^{-a})(M^b T^{-2b} L^{-b}) L^c (LT^{-2})$$

Equating the indices of M,

$$0 = b$$

Equating the indices of T,

$$0 = -a - 2b - 2$$

Hence $\quad a = -2$

Equating the indices of L,

$$0 = 3a - b + c + 1$$

Hence $\quad c = 5$

Inserting these values in eq. (37) the non-dimensional factor for this group becomes

$$\left(\frac{D^5 g}{Q^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (38)$$

But, from eq. (18), $\quad Q \propto D^2\sqrt{gH}$

Substituting this value in eq. (38), the non-dimensional factor becomes

$$\left(\frac{D}{H}\right)$$

GROUP 2. $\quad 0 = \phi(Q^a p^b D^c \rho) \quad . \quad . \quad . \quad . \quad . \quad (39)$

Inserting the fundamental dimensions,

$$0 = (L^{3a}T^{-a})(M^b T^{-2b} L^{-b})L^c(ML^{-3})$$

Equating the indices of M,

$$0 = b + 1$$

Hence $\quad b = -1$

Equating the indices of T,

$$0 = -a - 2b$$

Substituting for b, $\quad 0 = -a + 2$

Hence $\quad a = 2$

Equating the indices of L,

$$0 = 3a - b + c - 3$$

Substituting for a and b,

$$0 = 6 + 1 + c - 3$$

Hence $\quad c = -4$

Inserting these values of a, b and c in eq. (39), the non-dimensional factor for this group becomes

$$\left(\frac{Q^2 \rho}{p D^4}\right) \quad . \quad . \quad . \quad . \quad . \quad (40)$$

But as $Q \propto D^2\sqrt{(p/\rho)}$ (§ 11.6) this factor becomes

$$\frac{D^4 p\rho}{\rho p D^4} = 1$$

This proves that the coefficient k is independent of the density ρ.

GROUP 3. $$0 = \phi(Q^a p^b D^c \eta) \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Inserting the fundamental dimensions,

$$0 = (L^{3a}T^{-a})(M^b T^{-2b} L^{-b})L^c(ML^{-1}T^{-1})$$

Equating the indices of M,

$$0 = b + 1$$

Hence $b = -1$

Equating the indices of T,

$$0 = -a - 2b - 1$$

Substituting for b, $0 = -a + 2 - 1$

from which $a = 1$

Equating the indices of L,

$$0 = 3a - b + c - 1$$

Substituting for a and b,

$$0 = 3 + 1 + c - 1$$

Hence $c = -3$

Inserting these values of a, b, and c in eq. (41), the non-dimensional factor for this group becomes

$$\left(\frac{Q\eta}{pD^3}\right)$$

But, from § 11.6, $Q \propto D^2\sqrt{(p/\rho)}$ and $p = \rho g H$; hence this factor becomes

$$\frac{D^2\sqrt{p}\,\eta}{\sqrt{\rho}\,pD^3}$$

or

$$\frac{\eta}{\sqrt{gH}\,\rho D}$$

which is equal to R_e.

GROUP 4. $$0 = \phi(Q^a p^b D^c \sigma) \quad . \quad . \quad . \quad . \quad . \quad (42)$$

Inserting the fundamental dimensions,

$$0 = (L^{3a}T^{-a})(M^b T^{-2b} L^{-b})L^c(MT^{-2})$$

Equating the indices of M,

$$0 = b + 1$$

Hence $b = -1$

Equating the indices of T,

$$0 = -a - 2b - 2$$

Substituting for b, $\quad 0 = -a + 2 - 2$

Hence $\quad a = 0$

Equating the indices of L,

$$0 = 3a - b + c$$

Substituting for a and b,

$$0 = 0 + 1 + c$$

Hence $\quad c = -1$

Inserting these values of a, b, and c in eq. (42), the non-dimensional factor for this group becomes

$$\left(\frac{\sigma}{pD}\right)$$

Substituting $p = \rho gH$, the factor becomes

$$\left(\frac{\sigma}{\rho gHD}\right)$$

But it was proved by Group 1 that $D \propto H$; hence this factor may be written

$$\left(\frac{\sigma}{\rho gH^2}\right)$$

From the results of Group 2, eq. (40) may be written

$$Q = k\sqrt{\frac{p}{\rho}}\,D^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

As $\quad$ Group 2 $= \phi$(Group 1, Group 3, Group 4)

then
$$\frac{Q}{\sqrt{\frac{p}{\rho}}\,D^2} = \phi\left[\left(\frac{D}{H}\right)\left(\frac{\eta}{\sqrt{gH}\rho D}\right)\left(\frac{\sigma}{\rho gH^2}\right)\right]$$

Hence
$$Q = \sqrt{\frac{p}{\rho}}\,D^2\phi\left[\left(\frac{D}{H}\right)\left(\frac{\eta}{\sqrt{gH}\rho D}\right)\left(\frac{\sigma}{\rho gH^2}\right)\right]$$

This proves that the coefficient of discharge of a small orifice depends on the three non-dimensional functions contained inside the square brackets. It may also depend on other factors which have not been included in this solution.

It will be noticed that this group method of solution gives the same results as the method demonstrated in § 11.6 and provides a simpler solution.

EXERCISES 11

1. Discuss the Froude and Reynolds numbers, giving illustrations of their significance.

Give a rule for the formation of dimensionless groups in problems involving more than four variables. Turbine models are tested under conditions giving the same specific speed as the prototype, yet the efficiency of the model is usually lower. Explain this. (*Lond. Univ.*)

2. The quantity of fluid flowing along a pipe is determined from the pressure drop P across a diaphragm having a central circular orifice. Show, by application of the principles of geometrical and dynamical similarity, that the volume flowing per second can be expressed by

$$Q = C.A\sqrt{\frac{P}{\rho}}$$

where ρ is the density of the fluid, A is the area of the orifice, and C is a coefficient which depends upon the pipe and orifice dimensions and the Reynolds number. State the units that must be used for the quantities concerned in both the English and metric systems. (*Lond. Univ.*)

3. Show that a rational formula for the resistance to the motion of partially submerged similar bodies through a liquid in which the formation of surface waves is the important factor, viscosity being negligible, is

$$R = \rho l^2 v^2 \,.\, F(gl/v^2)$$

in which ρ is the density of the liquid, l the length, and v the speed of the body. (*Lond. Univ.*)

4. In two geometrically similar shaft bearings, in which the same lubricant is used, the speeds of rotation are the same and viscous flow occurs. Make use of the method of dimensions and prove that if the loads carried per unit area are the same for each, the moments of the frictional resistances are proportional to the cubes of the linear dimensions. (*Lond. Univ.*)

5. In geometrically similar shaft bearings in which the motion of the lubricant is purely viscous, show that a rational expression for the frictional resistances should be of the form

$$R = \eta N D^3 \phi\left(\frac{p}{\eta N}\right)$$

in which R is the moment of the frictional resistances, η the viscosity of the lubricant, N the speed of rotation, D the diameter of the shaft, and p the load carried per unit area of bearing surface. Hence, show that in similar bearings the frictional resistances at corresponding speeds vary as the product pD^3. What is the ratio of the corresponding rotational speeds in two given similar bearings? (*Lond. Univ.*)

Ans. $\dfrac{p_2\eta_1}{p_1\eta_2}$

6. Show that the flow over a 90 degree vee notch for a fluid of kinematic viscosity ν will be

$$Q = H^{5/2} g^{1/2} \phi\left[\frac{H^{3/2} g^{1/2}}{\nu}\right]$$

where H is the head and g the acceleration due to gravity. (*Lond. Univ.*)

7. If the resistance to the motion of a sphere through a fluid is a function of the density ρ and the viscosity η of the fluid, and the radius r and the velocity v of the sphere, show that the resistance $R = (\eta^2/\rho) \,.\, \phi(vr\rho/\eta)$. Hence, show that, if at very low velocities $R \propto v$, $R = k\eta r v$, when k is a dimensionless constant. (*Lond. Univ.*)

8. A model to one-tenth scale of a broad-crested weir gave these values—

Head (ft) . . .	0·1	0·3	0·7
Flow (ft³/sec) . .	0·23	1·22	5·7

Find the corresponding full-scale values. Discuss the likelihood of discrepancies between such a model and the full-scale weir. (*I.Mech.E.*)

Ans. 72·8; 386; 1,805 ft³/sec.

9. Show, by the use of the method of dimensions, that the resistance to the motion of a sphere of radius r falling through a viscous fluid at a slow velocity v is given by $R = k\eta rv$, in which η is the viscosity of the fluid and k is a dimensionless constant. (*Lond. Univ.*)

10. For the flow of a fluid through similar pipes at speeds above the critical velocity prove that the drop in pressure per unit length is given by $p/l = \rho v^2/d \,.\, \phi(R_e)$, in which ρ is the density and v the velocity of the fluid, d is the diameter of the pipe, and R_e is the Reynolds number.

Hence, show that the frictional coefficient f in the formula $4flv^2/2gd$ for the frictional loss is a function of the Reynolds number. (*Lond. Univ.*)

11. Show that the resistance to motion of a body deeply submerged in a fluid is given by $R = \rho l^2 v^2 \phi(vl/\nu)$ where l is some one specified dimension of the body and where ρ and ν are respectively the density and kinematic viscosity of the fluid. (*Lond. Univ.*)

12. A jet-propelled aeroplane has a speed of 600 m.p.h. and a propulsion efficiency of 60 per cent. If the total resistance of the plane at this speed is 2,500 Lb, calculate (i) the weight of gases discharged per second; (ii) the necessary diameter of the jet orifice at outlet in feet, if the pressure and temperature at discharge are 10 Lb/in.² and 600°F respectively; (iii) the Mach number of the plane's flight if the pressure and temperature of the atmosphere at the altitude of the flight are 10 Lb/in.² and 40°F. The sonic velocity is

$$\sqrt{\frac{\gamma p}{\rho}}, \text{ where } \gamma = 1{\cdot}4 \text{ and } pV = 53T. \quad (\textit{Lond. Univ.})$$

Ans. (i) 68·6 Lb/sec; (ii) 1·29 ft; (iii) 0·808.

CHAPTER 12

DYNAMICAL SIMILARITY AND MODEL TESTING

12.1. Introduction. The full implications of the Reynolds number were never realized by Reynolds who considered the ratio merely as a criterion for the critical velocity in pipe flow. Lord Rayleigh* has shown that it is a non-dimensional factor which governs all problems on fluid flow frictional resistance, and that similar non-dimensional constants exist for many other natural phenomena, as shown in Chapter 11.

It is the practice in modern engineering design that, when a large object, such as a ship, aeroplane, propeller or pipe line, etc., is to be made, a scale model is constructed and tested so that the performance of the large object can be calculated from the test results of the scale model. Lord Rayleigh showed that the scale-model tests give comparable results only when the non-dimensional factor of the model test is equal to that of the large object, when working under its design conditions. By equating the non-dimensional factor of the large object to that of the model, the test speed of the model is obtained. This is known as the *corresponding speed*, and the comparison of the two conditions between the large object and the test results of a scale model, at its corresponding speed, is known as the principle of dynamical similarity.

As an example of the method, suppose it is required to estimate the fluid frictional resistance of a large object. A geometrically similar model is constructed and its resistance is measured when the test speed is its corresponding speed.

Let suffix m refer to the model. Then, applying eq. (6) in § 11.2, to object and model,

$$\frac{R_m}{R} = \frac{\rho_m l_m{}^2 v_m{}^2 \phi R_{e_m}}{\rho l^2 v^2 \phi R_e} \qquad . \quad . \quad . \quad . \quad (1)$$

where ϕ means "a function of."

If $R_{e_m} = R_e$ the unknown functions of eq. (1) will cancel. Then

$$\frac{R_m}{R} = \frac{\rho_m l_m{}^2 v_m{}^2}{\rho l^2 v^2} \qquad . \quad . \quad . \quad . \quad . \quad (2)$$

There is then true dynamical similarity between the large object and the scale model during the test. As

$$R_{e_m} = R_e$$

$$\frac{\rho_m l_m v_m}{\eta_m} = \frac{\rho l v}{\eta}$$

* *Scientific Papers*, **6**, Art. 392.

Hence
$$v_m = \frac{\rho}{\rho_m} \times \frac{l}{l_m} \times \frac{\eta_m}{\eta} \times v \quad . \quad . \quad . \quad (3)$$
$$= \text{corresponding speed of test}$$

If the model test takes place with the same fluid whilst at the same temperature as that of the large object under its design conditions,

$$\rho_m = \rho \text{ and } \eta_m = \eta$$

Then eq. (3) becomes

$$v_m = \frac{l}{l_m} v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Thus, the relation between the corresponding speed and the speed of the large object is the same as their linear dimension ratio, during a test for frictional resistance.

It should be noticed that, if their Reynolds numbers are equal, their frictional coefficients k_1 (§ 11.2) must also be equal, as k_1 is a function of R_e. This fact provides an alternative solution, in place of applying eq. (2), for calculating R from the test measurement of R_m.

12.2. Model Testing for Frictional Resistance

1. Model of Ship or Aeroplane. In these cases the linear dimensions of the scale model are about one-hundredth of those of the large craft. A large liner may be 800 ft long; hence a model of 8 ft length gives a linear dimension ratio of 1/100. About the same scale would be required for a large air-liner of 150 ft wing span; the scale model, which is tested in a wind tunnel, would be about 18 in. in wing span.

If this large linear dimension ratio be applied to eq. (4), it will be seen that the corresponding speed for a friction resistance test would be an impossible large amount. The model of the liner with a cruising speed of 30 knots would require a corresponding speed of 3,000 knots. The air-liner with a cruising speed of 400 m.p.h. would require a model testing speed of 40,000 m.p.h. for dynamical similarity during a friction test. It will be seen from these results that it is impossible to perform a scale model test for frictional resistance in the same fluid as that used by the large craft. For sea-going ships the difficulty is overcome by using the method explained in § 12.3. For aeroplanes, the model is tested by one of the methods described in § 12.4.

It is sometimes convenient to test the model in a different fluid to that used by the large craft; then eq. (3) is used for obtaining the corresponding speed. By this method the variation in values of the density and coefficient of viscosity are used to reduce the corresponding speed.

2. Model Testing for Pipe Flow. The frictional resistance of a large pipe line can be estimated by testing the flow through a small pipe as a scale model. In this case the corresponding speed is given by eq. (3). The linear dimensional ratio is based on the pipe diameters, and a different fluid may be used. The method is shown in the following worked-out example.

Example 1

The loss of head in a pipe 1 in. in diameter and 100 ft long, through which water is flowing at 10 ft/sec, was found to be 70 ft of water. Calculate the loss of head in a 3 in. pipe 60 ft long through which air is flowing at the corresponding speed. For water, $\rho = 62{\cdot}4$ Lb/ft³ and $\eta = 0{\cdot}01$ c.g.s. units. For air, $\rho = 0{\cdot}075$ Lb/ft³ and $\eta = 0{\cdot}00015$ c.g.s. units.

As this is a frictional resistance, the corresponding speed will be when the Reynolds numbers are equal. Then

$$\frac{\eta}{\rho d v} \text{ for water} = \frac{\eta}{\rho d v} \text{ for air}$$

That is
$$\frac{0{\cdot}01}{62{\cdot}4 \times 1 \times 10} = \frac{0{\cdot}00015}{0{\cdot}075 \times 3 \times v}$$

from which
$$v = \frac{62{\cdot}4 \times 1 \times 10 \times 0{\cdot}00015}{0{\cdot}075 \times 3 \times 0{\cdot}01}$$
$$= 41{\cdot}6 \text{ ft/sec}$$

Now, resistance to flow $= R = \rho l^2 v^2 \phi\left(\frac{\eta}{\rho d v}\right)$

$$= \rho \times \text{wetted area} \times v^2 \phi\left(\frac{\eta}{\rho d v}\right)$$

Hence
$$\frac{R \text{ for air}}{R \text{ for water}} = \frac{(\rho \times \pi d l v^2) \text{ for air}}{(\rho \times \pi d l v^2) \text{ for water}} \quad . \quad (5)$$

as the term $\phi(\eta/\rho d v)$ will cancel at the corresponding speed. But

$$R = \text{pressure} \times \text{cross-sectional area}$$
$$= \rho h \times \frac{\pi}{4} d^2 \quad . \quad . \quad . \quad (6)$$

as $p = \rho h$. Hence, equating eqs. (5) and (6),

$$\frac{(\rho h \times \frac{\pi}{4} d^2) \text{ for air}}{(\rho h \times \frac{\pi}{4} d^2) \text{ for water}} = \frac{(\rho \times \pi d l v^2) \text{ for air}}{(\rho \times \pi d l v^2) \text{ for water}}$$

Hence
$$\frac{h \text{ for air}}{h \text{ for water}} = \frac{\frac{1}{d}(l v^2) \text{ for air}}{\frac{1}{d}(l v^2) \text{ for water}}$$

That is $$\frac{h \text{ for air}}{70} = \frac{60 \times 41{\cdot}6^2 \times 1}{3 \times 100 \times 100}$$

Hence, loss of head for air $= 242$ ft of air

$$= \frac{242 \times 0{\cdot}075}{62{\cdot}4}$$

$$= 0{\cdot}291 \text{ ft of water}$$

It is not necessary in a question of this type to bring all terms to engineers' units, as the factors required to do this would cancel.

Alternative Solution. Use suffix 1 for water and suffix 2 for air.

As $R_{e_1} = R_{e_2}$, then $f_1 = f_2$ (where f is the Darcy coefficient).

$$h_{f_1} = \frac{4f_1 l_1 v_1^2}{2gd_1}$$

that is $$70 = \frac{4f_1 \times 100 \times 10^2}{64{\cdot}4 \times \frac{1}{12}}$$

Hence $$f_1 = 0{\cdot}0094$$
$$= f_2$$

Now $$h_{f_2} = \frac{4f_2 l_2 v_2^2}{2gd_2}$$

where $v_2 = 41{\cdot}6$ ft/sec, as before.

Then $$h_{f_2} = \frac{4 \times 0{\cdot}0094 \times 60 \times 41{\cdot}6^2}{64{\cdot}4 \times \frac{1}{4}} = 242 \text{ ft of air}$$

The result is thus the same as obtained by the first method.

12.3. Surface Wave Resistance. It was shown in § 7.3 that the resistance of a surface ship was partly due to surface friction, or viscosity, and partly due to surface wave formation, or gravity.

For true dynamical similarity between ship and model both of the following conditions must hold—

(1) $\dfrac{\nu}{lv} = \dfrac{\nu_m}{l_m v_m}$ (§ 11.2) as the Reynolds numbers are equal,

(2) $\dfrac{lg}{v^2} = \dfrac{l_m g}{v_m{}^2}$ (§ 11.3) as the Froude numbers are equal,

case (1) being the condition for viscous resistance and case (2) the condition for wave resistance. If both model and ship are floating in the same fluid, the corresponding speed for case (1) is when

$$lv = l_m v_m$$

or $$v_m = v\frac{l}{l_m}$$

For case (2), the corresponding speed is when

$$\frac{l}{v^2} = \frac{l_m}{v_m{}^2}$$

or
$$v_m = v\sqrt{\frac{l_m}{l}}$$

Hence, the corresponding speed varies in each case; it would, therefore, be impossible to test the model for the total resistance. To overcome this difficulty it is usual to calculate the frictional, or viscous, resistance of the ship and model from the coefficient of friction and surface area as explained in § 7.3. The total resistance of the model is then measured experimentally at the corresponding speed for wave resistance; that is, at a corresponding speed proportional to $\sqrt{(l_m/l)}$. By subtracting the frictional resistance from the total resistance of the model the wave resistance is obtained. The wave resistance of the ship is then obtained by proportion; this, added to the frictional resistance of the ship, will give the total resistance.

Let R_w = wave resistance of ship,
R_f = frictional resistance of ship,
r_w = wave resistance of model,
r_f = frictional resistance of model.

Then
$$R = R_w + R_f \quad . \quad . \quad . \quad . \quad (7)$$
and
$$R_m = r_w + r_f \quad . \quad . \quad . \quad . \quad (8)$$

For wave resistance only,

$$\frac{R_w}{r_w} = \frac{\rho l^2 v^2 \phi\left(\dfrac{lg}{v^2}\right)}{\rho_m l_m{}^2 v_m{}^2 \phi\left(\dfrac{l_m g}{v_m{}^2}\right)}$$

Then, for corresponding speed, in order that the last term of each will cancel,

$$\frac{v_m}{v} = \sqrt{\frac{l_m}{l}}$$

Then
$$\frac{R_w}{r_w} = \frac{\rho l^2 v^2}{\rho_m l_m{}^2 \left(v\sqrt{\dfrac{l_m}{l}}\right)^2}$$

$$= \frac{\rho l^3}{\rho_m l_m{}^3}$$

If the same fluid be used, $\rho = \rho_m$. Hence

$$\frac{R_w}{r_w} = \left(\frac{l}{l_m}\right)^3$$

Substituting from eqs. (7) and (8),

$$\frac{R - R_f}{R_m - r_f} = \left(\frac{l}{l_m}\right)^3$$

from which the total resistance of the ship is obtained.

EXAMPLE 2

The resistance of a hydroplane may be assumed to be entirely due to wave formation. The speed of the hydroplane is to be 90 ft/sec; calculate its resistance at this speed if the resistance of a model at the corresponding speed was found to be 0·5 Lb. The linear dimensions of the model were $\frac{1}{20}$ of the hydroplane. What is the speed of the model?

As wave resistance depends on the Froude number,

$$\text{corresponding speed of model} = 90\sqrt{\frac{l_m}{l}}$$

$$= 90 \times \sqrt{\frac{1}{20}}$$

$$= 20{\cdot}01 \text{ ft/sec}$$

Then $$\frac{R}{R_m} = \left(\frac{l}{l_m}\right)^3$$

Hence $$R = 20^3 \times 0{\cdot}5$$

$$= 4{,}000 \text{ Lb}$$

12.4. Methods of Obtaining True Dynamical Similarity. It was shown in § 12.2 and § 12.3 that it is possible to predict the resistance of a large body by testing a small model of the same shape. This can be done only if there is dynamical similarity between the body and the model. It was shown that true dynamical similarity occurs when such factors as ν/lv or lg/v^2 are the same for both body and model.

1. MODEL TESTING FOR HIGH-SPEED BODIES. In the case of surface ships, true dynamical similarity for wave resistance can be attained between the ship and the model, because the required corresponding speed for the model is within practical possibilities. For this type of resistance it was shown that

$$v_m = v\sqrt{\frac{l_m}{l}} \quad (\S\ 12.3)$$

Hence, the required model speed is less than that of the large craft under consideration.

In a test for frictional or viscous resistance, the corresponding speed of the model is given by the equation

$$v_m = v \times \frac{l}{l_m} \times \frac{\nu_m}{\nu} \text{ (§ 12.1)}$$

Hence, if the model is tested in the same fluid as used for the large craft, the corresponding speed is very high and, consequently, unattainable. This difficulty can be overcome by testing the model in a fluid of lower viscosity or of a higher density, thus reducing the corresponding speed.

It is the practice to test models of aeroplanes and aeroplane wings in wind tunnels containing air compressed to 25 or 30 atmospheres. This causes alterations in the density and coefficient of viscosity of such magnitudes that the ratio ν_m/ν is sufficiently reduced to make the corresponding speed a practical amount. In this way tests for frictional resistance with true dynamical similarity are made possible. Compressed-air wind tunnels, of 6 ft diameter, have been constructed for this purpose.

When compressed-air wind tunnels are not available for model testing, or when speeds up to 100 m.p.h. only are possible, dynamical similarity cannot be attained, as the air speed is too low; hence, some other device must be used for predicting the resistance of the craft. The equation for the resistance of the full-sized craft and of the model will be of the form

$$D = k_D \times \rho A v^2$$

If the model is tested at any convenient speed and the value of its k_D found from the test, this value cannot be used to predict the resistance of the full-sized craft because the speed of the test was not the corresponding speed to give dynamical similarity. In order to find the value of k_D for the large craft it is necessary to investigate the variation of k_D with the Reynolds number R_e. In order to do this, the model is tested with various values of R_e and each corresponding value of k_D calculated. A curve with k_D and R_e as ordinates is then plotted. Then, by extrapolating, the value of k_D, for the same Reynolds number as the large craft, can be obtained.

This variation of k_D with Reynolds number is known as the *scale effect*.

In the testing of models at supersonic speed a supersonic wind tunnel is used. In order to obtain the air current at a supersonic speed, high-pressure air is expanded through a converging-diverging nozzle (Chapter 19). This produces a high supersonic speed in the diverging cone of the nozzle. The model is then placed in the diverging cone.

2. Model Testing for Oiled Bearings. It was shown in § 11.8 that the non-dimensional factor for an oiled bearing was $p/\eta N$.

Let suffix 1 and suffix 2 apply to two geometrically similar bearings under comparison. Then

$$\frac{R_1}{R_2}=\frac{\eta_1 N_1 D_1^3\phi\left(\dfrac{p_1}{\eta_1 N_1}\right)}{\eta_2 N_2 D_2^3\phi\left(\dfrac{p_2}{\eta_2 N_2}\right)} \quad . \quad . \quad . \quad (9)$$

For true dynamical similarity the non-dimensional constants $p_1/\eta_1 N_1$ and $p_2/\eta_2 N_2$ must cancel.

Then $$\frac{p_1}{\eta_1 N_1}=\frac{p_2}{\eta_2 N_2}$$

from which $$N_2=N_1\frac{p_2\eta_1}{p_1\eta_2}$$

This is the corresponding speed for true dynamical similarity; if this value of N_2 is used, the non-dimensional constants in eq. (9) will cancel.

Substituting this corresponding speed in eq. (9),

$$\frac{R_1}{R_2}=\frac{\eta_1 N_1 D_1^3}{\eta_2\left(\dfrac{N_1 p_2\eta_1}{p_1\eta_2}\right)D_2^3}$$

$$=\frac{p_1 D_1^3}{p_2 D_2^3}$$

Hence $$R\propto pD^3$$

That is, for similar bearings under the same pressure, the resistance moment is proportional to the cube of the linear dimensions.

3. Model Testing for Propellers. It was shown in § 11.10 that there were two non-dimensional factors governing the model test on a propeller; namely,

$$\frac{\eta}{\rho dv}\text{ and }\frac{dn}{v}$$

The former represents the Reynolds number.

If a geometrically similar model of a propeller is tested in a fluid in order to predict the performance of the propeller, it will be impossible to satisfy both of these conditions in the same test. The method adopted is to neglect the Reynolds number, which is found to be relatively small, and to use the second non-dimensional factor dn/v as the criterion for dynamical similarity. This means that the model is tested at such a speed that

$$\frac{dn}{v}\text{ for model}=\frac{dn}{v}\text{ for large propeller}$$

The efficiency of the propeller can be obtained by comparing the work done per second by the thrust T to the work done per second by the torque on shaft. Or,

$$\text{efficiency of propeller} = \frac{T \times v}{\text{torque on shaft} \times \omega}$$

where ω = angular velocity of propeller shaft in radians per second.

EXERCISES 12

1. What is meant by dynamical similarity?
What are the conditions necessary for dynamical similarity in the case of the flow of two different fluids through similar pipes of different diameter? (*I.Mech.E.*)

2. In model experiments to determine the resistance of a ship, what is meant by "corresponding speeds"?
Hence show that in similar speed boats in which the resistance is mainly due to wave formation, if S is the ratio of the linear dimensions the resistances at corresponding speeds vary as S^3. (*Lond. Univ.*)

3. In the rotation of similar discs in a fluid in which the motion of the fluid is turbulent, show by the method of dimensions that a rational formula for the frictional torque M of a disc of diameter D rotating at a speed of N in a fluid of viscosity η and density ρ is

$$M = D^5N^2\rho\, \phi. \left(\frac{\eta}{D^2N\rho}\right)$$

Hence show that in similar discs rotating in the same fluid the frictional torques at the corresponding speeds vary as the diameters of the discs. What is the ratio of the corresponding speeds? (*Lond. Univ.*)

Ans. $\dfrac{N_1}{N_2} = \left(\dfrac{D_2}{D_1}\right)^2$

4. Assuming that the thrust T of a screw propeller is dependent upon the diameter d, speed of advance v, fluid density ρ, revolutions per second n, and the coefficient of viscosity η, show, using the principle of dimensional homogeneity, that it can be represented by

$$T = \rho d^2\, v^2\, \phi \left\{\frac{\eta}{\rho d v}, \frac{dn}{v}\right\}$$

and hence explain the condition of dynamical similarity usually assumed for propellers.
The characteristics of a propeller of 12 ft diameter and rotational speed 100 r.p.m. are examined by means of a geometrically similar model of 18 in. diameter. When the model is rotated at 360 r.p.m. by a torque of 16 Lb-ft the thrust developed is 52 Lb and the speed of advance is 4·8 knots. Determine the torque, thrust, speed of advance and efficiency of the full-scale propeller. (1 knot = 6,080 ft/hr.) (*Lond. Univ.*)

Ans. Torque = 40,600 Lb-ft; T = 16,500 Lb; v = 10·68 knots; eff. = 70 per cent.

5. A model of an air duct is built to 1/30 scale and tested with water which is 50 times more viscous and 800 times more dense than air. When tested under dynamically similar conditions, the pressure drop is 33 Lb/in.^{2} in the model. Find the corresponding full-scale pressure drop, and express it in inches of water. (*I.Mech.E.*) *Ans.* 0·326 in. of water.

6. The resistance to the uniform flow of fluids through similar pipes is given by $\frac{p}{l} = \frac{\rho v^2}{d} \phi \left(\frac{\eta}{\rho d v}\right)$, in which $\frac{p}{l}$ is the pressure drop per unit length, and d is the diameter of the pipe, ρ is the density, η the viscosity, and v the velocity of the fluid.

Hence, find the pressure drop, expressed in inches of water, in a pipe 8 in. in diameter, 1,000 ft long, in which air is flowing at a velocity of 6·21 ft/sec, given that the loss of head is 6·1 ft when water flows through a similar pipe 1 in. in diameter and 100 ft long, at the corresponding speed. For water $\rho =$ 62·4 Lb/ft^3, and $\eta = 1{\cdot}01 \times 10^{-2}$ c.g.s. units; for air $\rho = 0{\cdot}0751$ Lb/ft^3, and $\eta = 1{\cdot}86 \times 10^{-4}$ c.g.s. units. (*Lond. Univ.*) *Ans.* 0·405 in.

7. What are the requirements of dynamical similarity and how may they be fulfilled when testing a model of a hydraulic structure? (*I.Mech.E.*)

8. The resistance of geometrically similar plates when towed edgewise through a fluid is given by $R = \rho l^2 v^2 \phi \left(\frac{\rho l v}{\eta}\right)$ in which ρ is the fluid density and η the coefficient of viscosity of the fluid, l the linear dimensions, and v the velocity. Determine the torque necessary to rotate a thin disc 24 in. in diameter at 3,000 r.p.m. in air for which $\rho = 1{\cdot}2 \times 10^{-3}$ and $\eta = 1{\cdot}86 \times 10^{-4}$ c.g.s. units, given that the torque necessary to rotate a similar disc 9 in. in diameter in water at the corresponding speed is 0·079 Lb-ft. For water $\rho = 1{\cdot}00$, and $\eta = 0{\cdot}0101$ c.g.s. units. (*Lond. Univ.*) *Ans.* 0·0596 Lb-ft.

9. What is meant by "corresponding speeds" in model experiments? Deduce the law of corresponding speeds (*a*) for viscous resistance, (*b*) for resistances due to the effects of gravity. In the case of a hydroplane the resistance is mainly due to wave formation. If the scale of a model hydroplane is 1/25 and if its resistance at a speed of 20 ft/sec is 0·4 Lb, what will be the resistance of the large hydroplane at the corresponding speed? (*Lond. Univ.*) *Ans.* 6,250 Lb.

10. What are meant by "corresponding speeds" in connexion with a model and its original? What would be the ratio of the corresponding speeds: (*a*) for a weir and its model, (*b*) for a closed pipe line and its model? (*I.Mech.E.*)

Ans. (*a*) $\frac{v_1}{v_2} = \frac{H_2 v_1}{H_1 v_2}$; (*b*) $\frac{v_1}{v_2} = \frac{d_2 v_1}{d_1 v_2}$

11. For the rotational speeds of similar wheels in a fluid, the power dissipated in windage is dependent upon the diameter D and the speed N rev/sec of the wheel, the density ρ and the viscosity η of the fluid. Hence, making use of the principle of dimensional homogeneity, show that the power P can be expressed

$$P = D^5 N^3 \rho \,.\, \phi(\eta/D^2 N \rho)$$

In similar impulse steam turbines it may be assumed that $\phi(\eta/D^2 N\rho) = A(\eta/D^2 N\rho)$, in which A is a dimensionless constant; hence, obtain a formula for P in terms of the diameter in feet, and the rotational speed in revolutions per second of the wheel, the density and viscosity of the steam in Lb-ft-sec (abs.) units, given that the resisting torque is 8·75 Lb-ft for a wheel 34 in. in diameter rotating at 50 rev/sec in steam of density 0·04 and viscosity 8×10^{-6} Lb-ft-sec (abs.) units. (*Lond. Univ.*) *Ans.* $P = 10{\cdot}96 D^3 N^2 \eta$.

12. By using the principle of dimensional similarity, obtain an expression for the compression wave resistance of a streamline body moving in a gaseous fluid and show that it is a function of the Mach number. (*Lond. Univ.*)

13. A vee notch is employed to measure the flow of a fluid which has a kinematic viscosity 8 times that of water. If the measured head is 10 in., calculate the head for water giving dynamical similarity. From experiments on water the empirical formula $Q = 2{\cdot}48\,H^{2{\cdot}47}$ has been deduced, Q being in cubic feet per second, and H in feet. Calculate the flow of fluid (*Lond. Univ.*)

Ans. 2·5 in.; 1·64 ft³/sec.

14. In order to estimate the surface wave resistance of the ship, 600 ft long, at a speed of 30 knots in sea water, a scale model of the ship, 6 ft long, is tested at the corresponding speed for surface wave resistance in a tank containing sea water. After allowing for the frictional resistance, the surface wave resistance of the model was found to be 1·85 Lb. Find (1) the corresponding speed of the test, (2) the surface wave resistance of the large ship, and (3) the propulsion horse-power of the ship required to overcome the wave resistance at the given speed. 1 knot = 1·69 ft/sec. (*Lond. Univ.*)

Ans. (1) 3 knots; (2) 1,850,000 Lb; (3) 170,500 h.p.

15. In order to estimate the frictional head lost in a pipe 1 ft in diameter and 800 ft long, through which air is flowing at 12 ft/sec, a test was made on a pipe ½ in. in diameter and 10 ft long with water flowing through at the corresponding speed for frictional resistance. The head lost was found by measurement to be 22·4 ft of water. Using the following data, calculate the corresponding speed of the test for dynamical similarity, and the pressure drop, in pounds per square inch, in the 800 ft pipe. Neglect any change of density of the air as small. Weight per cubic foot of air and water = 0·075 and 62·4 Lb respectively. Coefficient of viscosity of air and water = 0·0002 and 0·01 c.g.s. abs. units respectively. (*Lond. Univ.*)

Ans. 17·32 ft/sec; $p = 0{\cdot}01865$ Lb/in.²

16. By using the principle of dimensional similarity deduce an expression for the surface wave resistance of a ship.

In order to predict the performance of a large sea-going ship, 600 ft long, having a wetted surface area of 40,000 ft² and a speed of 30 knots, a scale model is made 12 ft long, and is tested at the corresponding speed for surface wave resistance. It is found from the model test that a total propulsive force of 10·7 Lb is required to tow the model at this speed in sea water. Estimate the wave resistance of the large ship and the total propulsive horse-power required due to the frictional resistance of its wetted surface plus surface wave resistance.

Froude's frictional coefficient $f' = 0{\cdot}01$ for ship and model. 1 knot = 1·69 ft/sec. (*Lond. Univ.*)

Ans. $v_m = 4{\cdot}1$ knots; $R_w = 375{,}000$ Lb; 123,000 h.p.

17. Using the principle of dimensional similarity, prove that the frictional resistance between a fluid and a surface is a function of the Reynolds number.

In order to estimate the frictional resistance of a 6 in. pipe, 200 ft long, through which air is flowing isothermally at 10 ft/sec, the resistance of a scale model of the pipe, ½ in. in diameter, is measured experimentally when water is flowing through at the corresponding speed for dynamical similarity. The pressure drop due to friction in the water pipe was found to be 13·95 Lb/in.² Estimate the pressure drop of the air in the 6 in. pipe in pounds per square inch.

η for air = 0·00000036 engineers' units,

w for air = 0·075 Lb/ft³,

η for water = 0·0000275 engineers' units. (*Lond. Univ.*)

Ans. $v_w = 10{\cdot}7$ ft/sec; $dp = 0{\cdot}0147$ Lb/in.²

18. In order to predict the frictional resistance of air flowing isothermally through a large pipe, a test was carried out on a small pipe through which water

was flowing. The pipe used for the air was 50 ft long and 6 in. in diameter, and the air had a velocity of 30 ft/sec. The mean pressure was 30 Lb/in.^{2} and the temperature was 60°F. The density of the air may be assumed constant.

The model pipe under test had water flowing through at the corresponding speed for dynamical similarity; it was 6 ft long and $\frac{1}{2}$ in. in diameter. The frictional head lost in this pipe was found to be 128·5 ft of water. Find the corresponding speed of the water, and calculate the pressure drop in the large air pipe, in pounds per square inch. Coefficients of viscosity of water and air are 25×10^{-6} and $0{\cdot}37 \times 10^{-6}$ engineers' units, respectively.

R for air $= 53{\cdot}3$ ft-Lb in Fahrenheit units. (*Lond. Univ.*)

Ans. $v_w = 60{\cdot}8$ ft/sec; $dp = 0{\cdot}0233$ Lb/in.^{2}

CHAPTER 13

COMPRESSIBLE FLUIDS

13.1. Properties of Gases. Problems dealing with the pressure and flow of gases are more complex than those applied to liquids. As a gas is easily compressible, in comparison with a liquid, the variation in its density is considerable. It is this variation in density which complicates the calculations on gas flow.

The behaviour of gases under changes of temperature, pressure and volume is governed by several well-known laws, all of which have been verified experimentally.

Let p = pressure of gas in pounds per square foot,
V = volume in cubic feet,
t = temperature as measured by a thermometer,
T = absolute temperature
= $(t + 273)$ °C abs. = degrees Kelvin (°K)
= $(t + 460)$ °F abs. = degrees Rankine (°R),
ρ = density of gas in absolute units,
w = weight per cubic foot of gas
= ρg.

Then, volume 1 Lb of gas $= \dfrac{1}{w}$ ft³

Let c_p and c_V = specific heats at constant pressure and constant volume in heat units,
γ = ratio of specific heats
$= c_p/c_V$,
J = Joule's equivalent of heat
= 1,400 ft-Lb per degree Centigrade
= 778 ft-Lb per degree Fahrenheit,
U = internal energy of 1 Lb of gas reckoned above a given temperature base.

Let suffix 1 represent the initial condition of the gas and suffix 2 the final condition.

BOYLE'S LAW. This law states that if a gas is expanded or compressed at constant temperature the product of its pressure and volume at any instant is a constant. Or,

$$p \times V = \text{a constant}$$

Then

$$p_1V_1 = p_2V_2$$

CHARLES' LAW. If a gas is expanded or compressed at constant pressure, then

$$\frac{V}{T} = \text{a constant}$$

If the change takes place at constant volume, then

$$\frac{p}{T} = \text{a constant}$$

CHARACTERISTIC EQUATION OF A GAS. By combining the laws of Boyle and Charles the characteristic equation of a gas is obtained. That is,

$$pV = RT$$

where V is the volume of 1 Lb of the gas at pressure p and absolute temperature T. R is known as the gas constant.

For atmospheric air, $R = 96$ ft-Lb Centigrade units
$= 53{\cdot}3$ ft-Lb Fahrenheit units

13.2. The Internal Energy of a Gas. The internal energy stored in a gas is the amount of energy in the form of heat; that is, the energy due to the vibration of its molecules. In gases such as oxygen, nitrogen and hydrogen, the internal energy is registered by the temperature. It can be proved from Joule's experiment* that the internal energy of 1 Lb of gas at an absolute temperature T, and reckoned above the freezing point of water, is given by the equation,

$$U = c_V\,(T - 273) \text{ C.H.U., when } T \text{ is in Centigrade units}$$
$$= c_V\,(T - 460) \text{ B.Th.U., when } T \text{ is in Fahrenheit units}$$

If the temperature of a gas changes from T_1 to T_2, then the increase of internal energy per pound of gas is

$$U_2 - U_1 = c_V(T_2 - T_1) \text{ heat units} \qquad . \quad . \quad . \; (1)$$

13.3. Application of Law of Conservation of Energy. If a gas is expanded in such a manner that it performs some form of external work, it follows that the heat absorbed, the work done, and the change of internal energy must balance in accordance with the principle of conservation of energy. This may be written in the form of an equation—

$$\left\{\begin{matrix}\text{heat absorbed}\\ \text{by 1 Lb of gas in}\\ \text{heat units}\end{matrix}\right\} = \left\{\begin{matrix}\text{work done by}\\ \text{1 Lb of gas in}\\ \text{heat units}\end{matrix}\right\} + \left\{\begin{matrix}\text{increase of in-}\\ \text{ternal energy per}\\ \text{pound of gas}\end{matrix}\right\}$$

or $$Q = W + (U_2 - U_1)$$

* See author's text-book *Thermodynamics Applied to Heat Engines* (Pitman).

It should be noted that, if the gas is rejecting heat, then Q is negative; if external work is being done on the gas in compressing it, then W is negative. If the term $(U_2 - U_1)$ is negative, it then represents a decrease in internal energy.

13.4. Isothermal Process. If a gas is expanded or compressed whilst its temperature is maintained constant, the process is called isothermal. External work of some description is performed by the

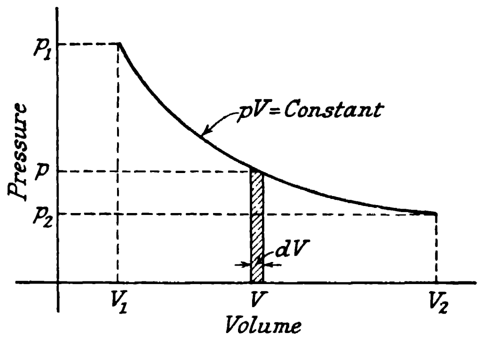


FIG. 152

gas. As the temperature is constant the expansion follows Boyle's law; then

$$pV = \text{constant}$$

and

$$p_1V_1 = p_2V_2$$

As there is no change of temperature there cannot be any change of internal energy; this will be seen from eq. (1).

Applying the law of conservation of energy to the isothermal process,

$$Q = W + (U_2 - U_1)$$
$$= W + 0 \text{ heat units}$$

Hence it follows that the heat absorbed by the gas during this process is equal to the work done by the gas.

The work done during an isothermal process can be calculated by finding the area under the isothermal curve of Fig. 152. Let this curve represent the expansion pV = constant, and consider a thin strip at pressure p and volume V. Let the thickness of this strip be $\mathrm{d}V$.

Work done $= W =$ area under curve

$$= \int_{V_1}^{V_2} p \, \mathrm{d}V \tag{2}$$

But $$pV = p_1V_1$$

Hence $$p = \frac{p_1V_1}{V}$$

Substituting this value in eq. (2),

$$W = p_1V_1\int_{V_1}^{V_2}\frac{dV}{V}$$

$$= p_1V_1\Big[\log_e V\Big]_{V_1}^{V_2}$$

$$= p_1V_1 \log_e \frac{V_2}{V_1}$$

Let r = ratio of expansion or pressure ratio

$$= \frac{V_2}{V_1}$$

Then $$W = p_1V_1 \log_e r \text{ (ft-Lb)} \quad . \quad . \quad . \quad (3)$$

From the characteristic equation of a gas,

$$p_1V_1 = RT$$

Then, for 1 Lb of gas, $$W = RT \log_e r \text{ (ft-Lb)} \quad . \quad . \quad . \quad (4)$$

This equation represents the heat energy absorbed during the expansion. If the process is a compression, r is less than unity and $\log r$ becomes negative; then the work done by the gas is negative. This means that work is done on the gas and heat is consequently rejected.

13.5. Adiabatic Process. If a gas is expanded or compressed in such a manner that no interchange of heat takes place during the process, the process is called adiabatic. External work of some description is performed by the gas. It can be proved* that the equation representing an adiabatic process is

$$pV^\gamma = \text{constant}$$

Then $$p_1V_1^\gamma = p_2V_2^\gamma$$

where γ is the ratio of the specific heats.

Applying the law of conservation of energy to this process, and putting the heat supplied as 0,

$$Q = W + (U_2 - U_1)$$

Then $$0 = W + (U_2 - U_1)$$

or $$W = (U_1 - U_2) \text{ heat units} \quad . \quad . \quad (5)$$

Thus, the work done by the gas is performed at the expense of its own internal energy.

* See author's text-book *Thermodynamics Applied to Heat Engines* (Pitman).

Some authorities apply the terms "adiabatic" and "isothermal" only to the reversible processes described in § 13.4 and § 13.5. Other authorities apply the term "adiabatic" to all insulated processes, and the term "isothermal" to all constant-temperature processes; the more general application of these terms may cause considerable confusion.

An equation for the work done during this process can be obtained from the adiabatic curve of Fig. 153. Consider the thin strip of pressure p, volume V and thickness dV.

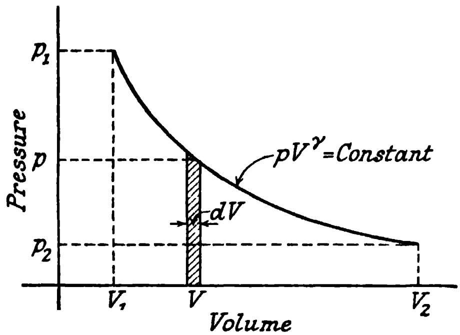


FIG. 153

$$\text{Work done} = W = \int_{V_1}^{V_2} p \, dV$$

But $$pV^\gamma = p_1 V_1^\gamma$$

from which $$p = p_1 V_1^\gamma V^{-\gamma}$$

Then substituting for p,

$$W = p_1 V_1^\gamma \int_{V_1}^{V_2} V^{-\gamma} \, dV$$

$$= \frac{p_1 V_1^\gamma}{1-\gamma} \left[V^{1-\gamma} \right]_{V_1}^{V_2}$$

$$= \frac{p_1 V_1^\gamma}{1-\gamma} (V_2^{1-\gamma} - V_1^{1-\gamma})$$

But $$p_1 V_1^\gamma = p_2 V_2^\gamma$$

Hence $$W = \frac{(p_2 V_2 - p_1 V_1)}{1-\gamma}$$

$$= \frac{(p_1 V_1 - p_2 V_2)}{\gamma - 1} \text{ ft-Lb} \quad . \tag{6}$$

As $p_1 V_1 = RT_1$ and $p_2 V_2 = RT_2$, this equation may be written

$$W = \frac{R(T_1 - T_2)}{\gamma - 1} \text{ ft-Lb} \tag{7}$$

It will be seen from eq. (5) that eqs. (6) and (7) represent the decrease of internal energy per pound of gas during an adiabatic expansion, and the increase of internal energy during an adiabatic compression. Or,

$$(U_1 - U_2) = \frac{p_1V_1 - p_2V_2}{(\gamma - 1)J} \text{ heat units} \quad . \quad . \quad (8)$$

As $p_1V_1^{\gamma} = p_2V_2^{\gamma}$, and as $p_1V_1/T_1 = p_2V_2/T_2$ it follows that

$$\frac{p_1}{p_2} = \left(\frac{V_2}{V_1}\right)^{\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Also

$$\frac{T_1}{T_2} = \left(\frac{V_2}{V_1}\right)^{\gamma-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

From eqs. (9), (10) and (11) the relation between pressure, volume and temperature during an adiabatic change can be found.

13.6. Total Heat, or Enthalpy, of Gases. The total heat, or enthalpy, of a gas at a given temperature is defined as the amount of heat required to raise the temperature of unit weight of the gas from 0°C to the given temperature, when heated at constant pressure.

Hence total heat $= H = c_p(T - 273)$ . . . (12)

where T is the absolute temperature in degrees Centigrade. Or

$$H = c_p(T - 492) \quad . \quad . \quad . \quad . \quad (13)$$

where T is the absolute temperature in degrees Fahrenheit.

Sometimes, for convenience, the total heat is reckoned above a lower basic temperature. If reckoned above absolute zero temperature,

$$H = c_pT + \text{latent heat}$$

The total heat of a gas is independent of the pressure except at extremely high pressures; it is usually expressed in heat units.

The values of the total heats given by the above equations assume the specific heat to be constant. Actually, the value of c_p varies with the temperature, the increase being considerable over a large temperature range. In modern calculations it is usual to take this variation into account, and tables have been prepared giving the exact values of H at all temperatures and for different gases. A table giving the properties of air, including the values of H, is given in Appendix 3.

The total heat can also be found from the sum of the internal energy of the gas and the external work done by the gas, due to its

increase in volume. If the total heat is reckoned above absolute zero temperature, and assuming the gas occupies no volume at that temperature,

$$H = U + \frac{pV}{J} \qquad . \quad . \quad . \quad . \quad . \quad (14)$$

where $U = c_V T$ (c_V being in heat units).

If reckoned above 0°C,

$$H = U + \frac{p(V - V_0)}{J}$$

where $U = c_V(T - T_0)$ and the suffix 0 refers to 0°C conditions.

In the above, the values of c_p and c_V are in heat units.

13.7. Entropy of Gases. The term *entropy* may be defined as a measure of the availability of heat energy for conversion into work.* An increase in temperature is accompanied by a reduction in the rate of availability of the heat energy for conversion into work. The entropy of unit weight of a substance, at a given temperature and pressure, is a physical property of the substance and can be stated in units of entropy measured from a datum of 0°C.

If a body is heated in any manner, the heat absorbed at any instant, divided by its absolute temperature at that instant, is equal to the change of entropy.

Let dQ = heat absorbed by 1 Lb of substance during small interval of time,
T = absolute temperature of substance at that instant,
S = entropy of substance,
dS = change of entropy during small interval of time.

Then
$$\mathrm{d}S = \frac{\mathrm{d}Q}{T} \qquad . \quad . \quad . \quad . \quad . \quad (15)$$

Integrating,
$$\int_{S_1}^{S_2} \mathrm{d}S = \int_{T_1}^{T_2} \frac{\mathrm{d}Q}{T}$$

Hence
$$S_2 - S_1 = \int_{T_1}^{T_2} \frac{\mathrm{d}Q}{T} \qquad . \quad . \quad . \quad . \quad . \quad (16)$$

If the heating takes place adiabatically, then d$Q = 0$, as there is no transfer of heat during an adiabatic operation; hence, d$S = 0$. Thus, an adiabatic operation takes place at constant entropy and is consequently termed *isentropic*.

If the heating takes place isothermally, T remains constant.

Then
$$S_2 - S_1 = \frac{Q}{T}$$

* For full explanation and use of entropy see text-books on Heat or Thermodynamics.

where Q = total heat absorbed during the process

$$= \frac{p_1 V_1 \log_e r}{} \quad (\S\ 13.4)$$

It will be noticed from eq. (15) that

$$dQ = T\,dS$$

Then integrating,

$$Q = \int T\,dS \quad . \quad . \quad . \quad . \quad (17)$$

If a curve be plotted to represent the heating of 1 Lb of a substance, having S as base and T as the vertical ordinate (Fig. 154), it follows from eq. (17) that the area under the curve represents the heat absorbed by the substance. Such a diagram is known as a temperature-entropy diagram, or a T-S diagram.

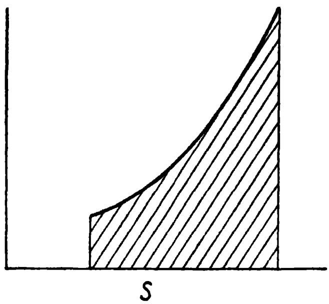


FIG. 154

13.8. Heat-entropy Chart for Gas. The properties of 1 Lb of a particular gas can be plotted on a chart having a base representing entropy, the vertical ordinate representing respectively T, H or U, to different scales. The effect of the variation of the specific heats with temperature can be allowed for in the values of H and U used in the plotting. Such a chart for 1 Lb of air is shown plotted on the folding inset, Fig. 155, facing this page. In this chart, black lines, representing lines of constant pressure, and red lines, representing lines of constant specific volume, have been drawn. It will be seen that 1 Lb of air, at a given temperature and pressure, is represented by a particular point on the chart; this point is known as a *state point*, and the value of its specific volume, entropy, total heat and internal energy can be read off from the corresponding scales.

If 1 Lb of air at a known condition is expanded adiabatically to a given lower pressure, this expansion is represented by a vertical line on the chart, because an adiabatic operation occurs at constant entropy and is termed *isentropic*. Referring to Fig. 156 let the point A represent the condition of the air at a given temperature T_1, pressure p_1 and specific volume V_{s_1}, and let it expand adiabatically to a pressure p_2. By drawing a vertical line from A to cut the constant-pressure line p_2 at B, the values of T_2, V_{s_2}, U_2 and H_2 at the point B can be read off from the chart. Thus, the vertical height AB, measured on the total heat scale, gives the loss in total heat during the expansion.

It was shown in § 13.5 that the work done during an adiabatic expansion is the change of internal energy; hence

$$\text{work done} = U_1 - U_2 \text{ heat units}$$

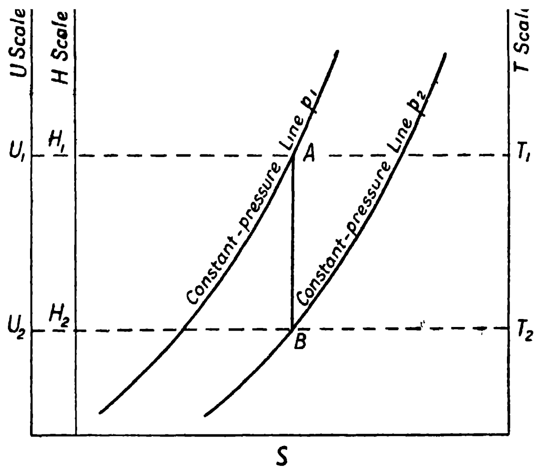


FIG. 156

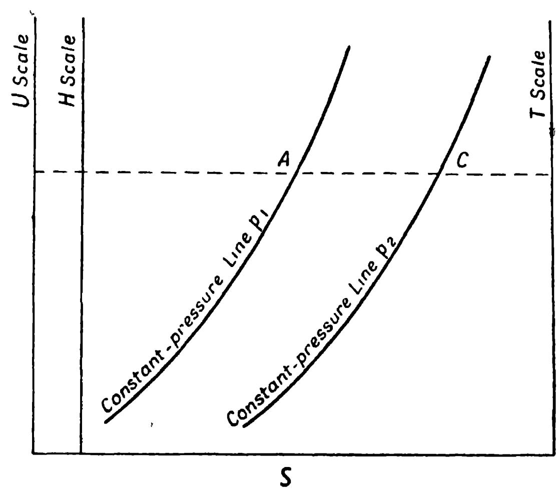


FIG. 157

The values of U_1 and U_2 can be read off the chart from the internal energy scale.

If the 1 Lb of air at A be expanded isothermally to p_2, the expansion is now represented by the horizontal line AC (Fig. 157) because it occurs at constant temperature. It will be noticed that there is

no change in total heat or internal energy during the process. Also, the area under the line AC represents the heat absorbed during the process (§ 13.7); this also equals the external work done.

It should be noted that, although the expansion AC takes place at constant total heat, there has been a reduction in the available energy per pound of gas for conversion into work. This is denoted by the drop in pressure from p_1 to p_2, as pressure is the medium by which the gas produces mechanical work, and the atmospheric pressure may be assumed to be the lowest pressure limit possible for the production of work.

EXAMPLE 1

One pound of air at a pressure of 30 Lb/in.^{2} and a temperature of 60°F is expanded abiabatically to a pressure of 14·7 Lb/in.^{2} Assuming no losses occur during the expansion and using the heat-entropy chart, read off the values of the total heat, internal energy, and specific volume of the air at the beginning and end of the expansion. What is the final temperature of the air?

The adiabatic expansion is represented by a vertical line from the initial point to the 14·7 Lb/in.^{2} pressure line.

At beginning of expansion,

$H = -6{\cdot}8$ B.Th.U., $U = 4{\cdot}8$ B.Th.U., $V_s = 6{\cdot}25$ ft^3/Lb.

At end of expansion,

$H = -17$ B.Th.U., $U = -12{\cdot}1$ B.Th.U., $V_s = 10{\cdot}6$ ft^3/Lb,

$t = -38$°F.

13.9. Bulk Modulus of a Fluid. The bulk elastic modulus of a fluid is the ratio between the increase of pressure and the volumetric strain caused by this pressure increase. This applies to liquids and gases.

Let a quantity of fluid at pressure p and volume V be subjected to an increase of pressure $\mathrm{d}p$. Let this increase of pressure cause a change of volume $-\mathrm{d}V$.

$$\text{Then,}\quad \text{volumetric strain} = \frac{-\mathrm{d}V}{V}$$

$$\text{Bulk modulus} = K = \frac{\text{increase of pressure}}{\text{volumetric strain}}\ \text{ft-Lb units}$$

$$= \frac{\mathrm{d}p}{-\mathrm{d}V/V}$$

$$= -V\frac{\mathrm{d}p}{\mathrm{d}V} \quad . \quad . \quad . \quad . \quad . \ (18)$$

The bulk modulus for a liquid is large on account of the small amount of compressibility. For a gas, the value of K is relatively small as $\mathrm{d}V$ is large.

13.10. Velocity of Pressure Wave in a Fluid. The velocity of a pressure wave transmitted through a fluid depends on the bulk modulus and density of the fluid.

Consider a tube of fluid of unit cross-sectional area (Fig. 158) through which a pressure wave is being transmitted from right to left, having a velocity of v. Now bring the wave to rest by imagining the fluid to have a velocity of v in the opposite direction. Consider a section A at which the intensity of pressure is p and the velocity v;

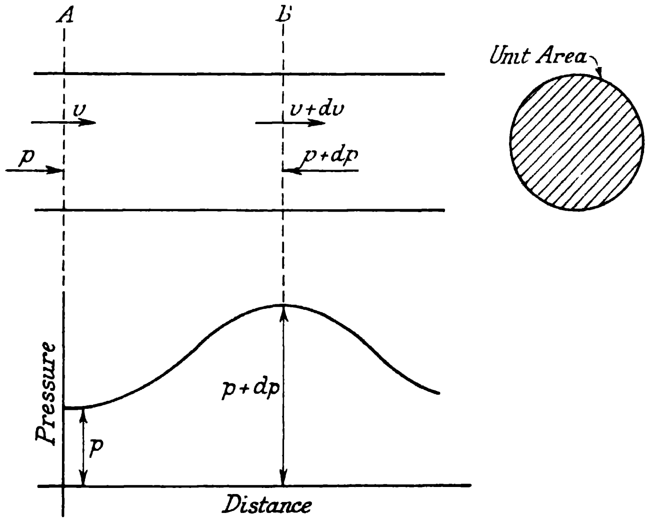


FIG. 158

let this section represent that portion of the fluid which is at the least pressure. Let the section B represent the adjacent portion of the fluid at which the pressure is a maximum. Let the pressure between the sections A and B increase by dp and let the velocity change by dv; this is due to a local circulation caused by the wave. The pressures are shown plotted on the pressure-distance diagram projected vertically below the tube (Fig. 158). It will be noticed that section A corresponds to the trough of the pressure wave and section B to the crest. Then,

$$\text{pressure of fluid at } B = p + \mathrm{d}p$$

and $$\text{velocity of fluid at } B = v + \mathrm{d}v$$

Let V = volume of fluid at A compressed per second by wave,

and ρ = density of fluid at A.

Let volume compressed per second increase by dV between sections A and B.

Then, at section A, $V = a \times v = v$ (as a is unity)

At section B, volume compressed per second

$$= V + \mathrm{d}V$$
$$= a(v + \mathrm{d}v)$$

But, $a = 1$ and $V = v$; hence,

$$\mathrm{d}V = \mathrm{d}v \qquad (19)$$

Now, force on fluid between sections A and B

= change of momentum per second

= mass per second $\times$ change of velocity

That is $[p - (p + \mathrm{d}p)]a = \rho a V \times \mathrm{d}v$

or $\mathrm{d}p = -\rho V\,\mathrm{d}v$

But, from eq. (19), $\mathrm{d}v = \mathrm{d}V$

Hence $\mathrm{d}p = -\rho V\,\mathrm{d}V$

or

$$\frac{\mathrm{d}p}{\mathrm{d}V} = -\rho V \qquad (20)$$

But it was proved in eq. (18) that

$$K = -V\frac{\mathrm{d}p}{\mathrm{d}V}$$

or

$$\frac{\mathrm{d}p}{\mathrm{d}V} = -\frac{K}{V} \qquad (21)$$

From eqs. (20) and (21),

$$\rho V = \frac{K}{V}$$

But as $V = v$, then, $\rho v = \dfrac{K}{v}$

or

$$v = \sqrt{\frac{K}{\rho}} \qquad (22)$$

Thus, the velocity of a pressure wave in a fluid depends on the values of its bulk modulus and density. As sound is propagated by means of a pressure wave transmitted in a fluid, eq. (22) gives the velocity of sound in a fluid.

Example 2

Calculate the velocity of sound in water of weight 62·4 Lb/ft^3 if the value of its bulk modulus is 300,000 Lb/in.^{2}

Using eq. (22),

$$v = \sqrt{\frac{K}{\rho}}$$

Now

$$\rho = \frac{w}{g} = \frac{62{\cdot}4}{32{\cdot}2}$$

and $$K = 300{,}000 \times 144 \text{ Lb/ft}^2$$

Hence $$v = \sqrt{\frac{300{,}000 \times 144 \times 32{\cdot}2}{62{\cdot}4}}$$
$$= 4{,}720 \text{ ft/sec}$$

13.11. Wave Velocity for Isothermal Process. The velocity of a pressure wave in a fluid depends on whether the transmission is an isothermal process or an adiabatic. If the process is assumed to be isothermal, then

$$pV = C$$

where C is a constant.

Differentiating, $p\,\mathrm{d}V + V\,\mathrm{d}p = 0$

Then $$p\,\mathrm{d}V = -V\,\mathrm{d}p$$

or $$\frac{\mathrm{d}p}{\mathrm{d}V} = -\frac{p}{V} \quad . \quad . \quad . \quad . \quad (23)$$

But from eq. (21),

$$\frac{\mathrm{d}p}{\mathrm{d}V} = -\frac{K}{V}$$

Equating these two values of $\mathrm{d}p/\mathrm{d}V$,

$$\frac{K}{V} = \frac{p}{V}$$

Hence $$K = p$$

Substituting this value of K in eq. (22),

$$\text{velocity of wave} = v = \sqrt{\frac{p}{\rho}} \quad . \quad . \quad . \quad . \quad (24)$$

Eq. (24) was deduced by Newton for the velocity of sound in a gas, but it is found from tests that this equation does not give good results. An adiabatic process should be assumed for the velocity of sound in a gas, as the temperature does not remain constant during the pressure variation of the wave.

Eq. (22) is found to give good results for a liquid; this is because the temperature change in a liquid due to a variation of pressure is extremely small and, consequently, the process approximates to an isothermal one.

13.12. Wave Velocity for Adiabatic Process. The transmission of a pressure wave in a gas is approximately adiabatic; this is because the pressure variation takes place very suddenly and, consequently, there is no time for any appreciable interchange of heat. Applying the law for an adiabatic process (§ 13.5),

$$pV^{\gamma} = C$$

where C is a constant.

Differentiating,

$$\gamma p V^{\gamma-1}\,\mathrm{d}V + V^{\gamma}\,\mathrm{d}p = 0$$

$$\gamma p V^{\gamma-1}\,\mathrm{d}V = -V^{\gamma}\,\mathrm{d}p$$

Hence
$$\frac{\mathrm{d}p}{\mathrm{d}V} = -\frac{\gamma p}{V} \quad . \quad . \quad . \quad . \quad (25)$$

But, from eq. (21),
$$\frac{\mathrm{d}p}{\mathrm{d}V} = -\frac{K}{V}$$

Equating these two values of $\mathrm{d}p/\mathrm{d}V$,

$$-\frac{K}{V} = -\frac{\gamma p}{V}$$

Hence
$$K = \gamma p$$

Substituting this value of K in eq. (22),

$$\text{velocity of wave} = v = \sqrt{\frac{\gamma p}{\rho}} \quad . \quad . \quad . \quad (26)$$

This equation was deduced by Laplace for the velocity of sound in a gas. Results of tests are found to be in agreement with this equation. Eq. (26) is also used for obtaining the value of γ for a gas; the velocity of sound at a known pressure being measured experimentally.

Example 3

Calculate the velocity of sound in air at a pressure of 14·7 Lb/in.² and at a temperature of 0°C. γ for air = 1·41.

Using the characteristic equation of a gas,

$$pV = RT$$

where
$$V = \frac{1}{w},\ R = 96 \text{ and } T = 273^\circ\text{C abs.}$$

Hence
$$p \times \frac{1}{w} = 96 \times 273$$

or
$$w = \frac{14{\cdot}7 \times 144}{96 \times 273}$$

$$= 0{\cdot}081 \text{ Lb/ft}^3$$

Then,
$$\rho = \frac{w}{g} = \frac{0{\cdot}081}{32{\cdot}2}$$

Using eq. (26),
$$v = \sqrt{\frac{\gamma p}{\rho}}$$

$$= \sqrt{\frac{1{\cdot}41 \times 14{\cdot}7 \times 144 \times 32{\cdot}2}{0{\cdot}081}}$$

$$= 1{,}090 \text{ ft/sec}$$

13.13. Variation of Atmospheric Pressure with Altitude. The pressure of the atmosphere at the earth's surface is due to the weight of the column of air above. This will not be directly proportional to the head, as in liquids, on account of the varying density of the air.

If the temperature of the column of air is assumed constant, the variation in pressure will follow an isothermal process and obeys Boyle's law. This is not the case in practice; it is found that the atmosphere gets colder as the altitude increases. On this account, it is more accurate to consider the pressure variation to follow the adiabatic law; that is, to assume there is no interchange of heat between the layers. The ratio between the temperature drop and the altitude is known as the *temperature gradient*.

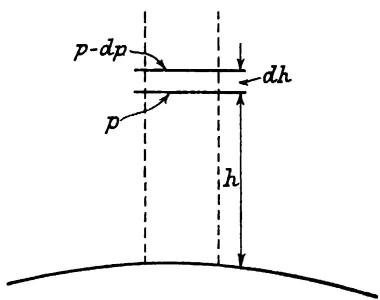


FIG. 159

In the solution of problems on the pressure of the atmosphere, the atmosphere is assumed to be *quiet* and to contain a constant amount of moisture; it is then said to be in *convective equilibrium*.

It is found from measurements that the true condition of the atmosphere lies between the isothermal and adiabatic processes, and it appears to follow a law

$$pV^n = \text{constant}$$

In aeronautical problems, a condition called the *International Standard Atmosphere* is assumed. This assumes the above law of variation, the value of n being 1·235. The solution of problems based on this law can be obtained from the equations for the adiabatic process (§13.5) by substituting the value of n for γ.

The variation of atmospheric pressure can also be obtained direct from the temperature gradient, if the latter is known.

Consider the column of atmospheric air of unit cross-sectional area (Fig. 159). Consider a section of the column at a height h above the earth's surface, and let the pressure at this height be p Lb/ft^2. Now consider another section at a height of dh above the first section and let the pressure at this section be p − dp.

The difference of pressure between the two sections is equal to the weight of air between them. Or

$$\cdot\, \mathrm{d}p = w(\mathrm{d}h \times 1)$$

that is, $$\mathrm{d}p = -w\,\mathrm{d}h \qquad (27)$$

Let V = volume of 1 Lb of air between the two sections. Then

$$w = \frac{1}{V}$$

Substituting this value of w in eq. (27),

$$dh = -V\,dp \quad . \quad . \quad . \quad . \quad . \quad (28)$$

But, from the characteristic equation of a gas,

$$pV = RT$$

Hence, substituting in eq. (28),

$$dh = -\frac{RT}{p}\,dp \quad . \quad . \quad . \quad . \quad (29)$$

Equations for the variation of atmospheric pressure with altitude will now be obtained for each of the four methods previously defined.

1. Assuming Isothermal Process. If an isothermal process is assumed for the pressure variation, the temperature of the column remains constant. Then eq. (29) becomes

$$dh = -RT\frac{dp}{p} \quad . \quad . \quad . \quad . \quad (30)$$

Let p_1 and p_2 be the pressures at two heights from the ground h_1 and h_2.

Integrating eq. (30),

$$\int_{h_1}^{h_2} dh = -RT\int_{p_1}^{p_2}\frac{dp}{p}$$

or

$$h_2 - h_1 = -RT\log_e\frac{p_2}{p_1} \quad . \quad . \quad . \quad (31)$$

Eq. (31) gives the pressure ratio between any two heights h_2 and h_1. If h_1 is zero it corresponds to the ground level; then p_1 is 14·7 Lb/in.^{2} Eq. (31) then becomes

$$h_2 = -RT\log_e\frac{p_2}{14{\cdot}7} \quad . \quad . \quad . \quad (32)$$

This equation gives the pressure p_2 in pounds per square inch at any altitude h_2 ft.

This assumption of an isothermal process does not give good results in practice for a large variation in altitude, on account of the alteration in temperature. For a large change in altitude the adiabatic law would be more accurate.

2. Assuming Adiabatic Process. The law for an adiabatic change is

$$pV^{\gamma} = p_1V_1^{\gamma}$$

where p and V apply to any height h above the earth's surface, and p_1 and V_1 apply to a particular height h_1.

For an adiabatic process (§ 13.5)

$$p = p_1 \left(\frac{T_1}{T}\right)^{\frac{\gamma}{1-\gamma}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

where T and T_1 are the absolute temperatures at altitudes h and h_1. Differentiating in terms of T,

$$dp = -\left(\frac{\gamma}{1-\gamma}\right) p_1 T_1^{\left(\frac{\gamma}{1-\gamma}\right)} T^{-\left(\frac{1}{1-\gamma}\right)} dT$$

Substituting the above values of p and dp in eq. (29),

$$dh = \frac{-R\left[-\left(\frac{\gamma}{1-\gamma}\right) p_1 T_1^{\left(\frac{\gamma}{1-\gamma}\right)} T^{-\left(\frac{1}{1-\gamma}\right)}\right] dT}{p_1 T_1^{\left(\frac{\gamma}{1-\gamma}\right)} T^{-\left(\frac{1}{1-\gamma}\right)}}$$

$$= \left(\frac{\gamma}{1-\gamma}\right) R\, dT \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Hence $$\frac{dT}{dh} = -\left(\frac{\gamma-1}{\gamma}\right)\frac{1}{R} \quad . \quad . \quad . \quad . \quad . \quad (35)$$

This is known as the temperature gradient.

Integrating eq. (34) between altitudes h_2 and h_1 and between the corresponding temperatures T_2 and T_1,

$$\int_{h_1}^{h_2} dh = \left(\frac{\gamma}{1-\gamma}\right) R \int_{T_1}^{T_2} dT$$

or $$(h_2 - h_1) = \left(\frac{\gamma}{1-\gamma}\right) R(T_2 - T_1)$$

$$= \left(\frac{\gamma}{1-\gamma}\right) T_1 R \left(\frac{T_2}{T_1} - 1\right) \quad . \quad . \quad . \quad (36)$$

But, from eq. (11),

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$$

Substituting this value of T_2/T_1 in eq. (36),

$$h_2 - h_1 = \left(\frac{\gamma}{1-\gamma}\right) RT_1 \left[\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right] \quad . \quad . \quad (37)$$

If eq. (37) is applied to find the pressure at any height h_2 from the ground, h_1, T_2, and p_1 will apply to the ground level; then $h_1 = 0$ and $p_1 = 14{\cdot}7$ Lb/in.^{2}

Then $$h_2 = \left(\frac{\gamma}{1-\gamma}\right) RT_1 \left[\left(\frac{p_2}{14{\cdot}7}\right)^{\frac{\gamma-1}{\gamma}} - 1\right] \quad (38)$$

From this equation the value of p_2 in pounds per square inch can be calculated.

3. Assuming the Temperature Gradient is Known. If the temperature gradient is known, the variation of atmospheric pressure with altitude can be calculated from eq. (29).

Let T_0 = absolute temperature at ground level,
p_0 = atmospheric pressure at ground level,
t = temperature drop per foot increase of altitude,
T = temperature at any altitude h ft,
p = pressure at altitude h ft,
p_1 = pressure at any given altitude h_1 ft.

Then $$T = T_0 - th$$

Using eq. (29),
$$\mathrm{d}h = -\frac{RT\mathrm{d}p}{p}$$
$$= -\frac{R(T_0 - th)\mathrm{d}p}{p}$$

Hence $$\frac{\mathrm{d}p}{p} = -\frac{\mathrm{d}h}{R(T_0 - th)}$$

Integrating between the pressure limits p_0 and p_1 and between the altitude limits of 0 and h_1,

$$\log_e \Big[p\Big]_{p_0}^{p_1} = \frac{1}{Rt} \log_e \Big[T_0 - th\Big]_0^{h_1}$$

that is, $$\log_e \left(\frac{p_1}{p_0}\right) = \frac{1}{Rt} \log_e \left(\frac{T_0 - th_1}{T_0 - 0}\right)$$
$$= \frac{1}{Rt} \log_e \left(1 - \frac{th_1}{T_0}\right) \quad . \quad . \quad . \quad . \quad (39)$$

This equation gives the pressure p_1 at any given altitude h_1.

4. Assuming Process Follows Law pV^n = Constant. The solution of this process is the same as for an adiabatic process except that n should be substituted for γ. Eq. (37) then becomes

$$h_2 - h_1 = \left(\frac{n}{1-n}\right) RT_1 \left[\left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}} - 1\right] \quad . \quad . \quad (40)$$

For the international standard atmosphere the value of n is taken as 1·235.

Example 4

If the atmosphere at sea level has a pressure of 14·7 Lb/in.² and a temperature of 60°F, find the pressure at a height of 10,000 ft, (1) assuming an isothermal change, (2) assuming an adiabatic change. $\gamma = 1{\cdot}41$ for air.

$$T_1 = 460 + 60 = 520°\text{F abs.}$$

(1) *Assuming Isothermal Change*

Using eq. (32),

$$h_2 = -RT \log_e \frac{p_2}{p_1}$$

that is,

$$10{,}000 = -53{\cdot}3 \times 520 \log_e \frac{p_2}{14{\cdot}7}$$

or

$$-0{\cdot}361 = 2{\cdot}303 \log_{10} \frac{p_2}{14{\cdot}7}$$

$$-0{\cdot}1565 = \log \frac{p_2}{14{\cdot}7}$$

Taking antilogs, $0{\cdot}6974 = \dfrac{p_2}{14{\cdot}7}$

from which $\quad p_2 = 10{\cdot}24$ Lb/in.²

(2) *Assuming Adiabatic Change*

Using eq. (38),

$$h_2 = \left(\frac{\gamma}{1-\gamma}\right) RT_1 \left[\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]$$

That is,

$$10{,}000 = -\left(\frac{1{\cdot}41}{0{\cdot}41}\right) 53{\cdot}3 \times 520 \left[\left(\frac{p_2}{14{\cdot}7}\right)^{\frac{0{\cdot}41}{1{\cdot}41}} - 1\right]$$

Then

$$0{\cdot}1048 = 1 - \left(\frac{p_2}{14{\cdot}7}\right)^{0{\cdot}2905}$$

or

$$0{\cdot}8952 = \left(\frac{p_2}{14{\cdot}7}\right)^{0{\cdot}2905}$$

Taking logs, $\quad -0{\cdot}0481 = 0{\cdot}2905 \log \left(\dfrac{p_2}{14{\cdot}7}\right)$

or

$$-0{\cdot}1654 = \log \left(\frac{p_2}{14{\cdot}7}\right)$$

Taking antilogs, $0{\cdot}6832 = \left(\dfrac{p_2}{14{\cdot}7}\right)$

from which $\quad p_2 = 10{\cdot}03$ Lb/in.²

EXAMPLE 5

Find the pressure and density of the atmosphere at a height of 12,000 ft when the pressure, temperature and density at ground level are 14·7 Lb/in.², 15°C, and 0·0765 Lb/ft³ respectively. Assume that the temperature of a quiescent atmosphere diminishes with the height at a uniform rate of 2°C per 1,000 ft. For air, $pV = 96T$. (*Lond. Univ.*)

Using eq. (39),

$$\log_e\left(\frac{p_1}{p_0}\right) = \frac{1}{Rt}\log_e\left(1 - \frac{th_1}{T_0}\right)$$

In this example, $t = 0{\cdot}002°\text{C}$

$$T_0 = 288°\text{C abs}$$

$$R = 96$$

and $p_0 = 14{\cdot}7 \text{ Lb/in.}^2$

Then $$\log_e\left(\frac{p_1}{14{\cdot}7}\right) = \frac{1}{96 \times 0{\cdot}002}\log_e\left(1 - \frac{0{\cdot}002 \times 12{,}000}{288}\right)$$

from which $$\frac{p_1}{14{\cdot}7} = 0{\cdot}638$$

Hence $$p_1 = 9{\cdot}38 \text{ Lb/in.}^2$$

From the characteristic equation of a gas,

$$\frac{p_1}{T_1} = \frac{96}{V_1}$$

Hence, $$\text{final density} = \frac{1}{V_1} = \frac{144 \times 9{\cdot}38}{96 \times 264}$$

$$= 0{\cdot}0534 \text{ Lb/ft}^3$$

13.14. Sensitive Manometers. 1. CHATTOCK TILTING GAUGE. A very sensitive instrument for measuring the difference of pressure between the two limbs of a Pitot tube is the Chattock tilting gauge (Fig. 160). The instrument consists of a lever *a*, carrying a scale *f* at one end, which is fixed horizontally by means of three levelling screws. The left-hand end carries a knife edge *c* on which is pivoted another lever *b*. The lever *b* can be tilted by means of the micrometer screw *d* which carries a large disc graduated around its edge. The vertical movement of the right-hand end of lever *b* can be measured by the pointer *e* and by the rotation of the micrometer disc.

Flasks *A* and *C* are fixed to the tilting lever *b*, as shown; a smaller flask *B* is fixed at the centre. The flasks *B* and *C* are connected at their bases by a glass tube, and contain water as shown. The upper portion of *B* is filled with oil. The flask *A* contains water; its base is connected to the oil in *B* by a glass tube which has its outlet in

the oil in *B*. The high-pressure limb of the Pitot tube is connected to the air in *A*, whilst the low-pressure limb is connected to the air in *C*.

The pressure in *A* causes a water bubble to form at the end of the tube in the oil of *B*. The level of the upper surface of this water bubble can be sighted on the cross-wire of a microscope supported at *g*. An increase of pressure in *A* increases the size of this water

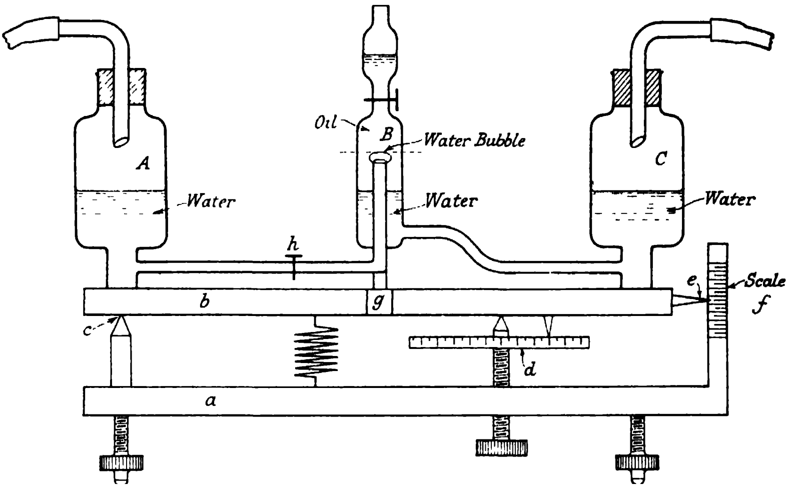


FIG. 160. THE CHATTOCK TILTING GAUGE

bubble in *B*; a decrease of pressure in *C* also causes the size of the bubble to increase.

The instrument is initially set by sighting on to the bubble's upper surface with the cock *h* closed; the readings of the pointer *e* and the micrometer disc being noted. The cock *h* is then opened; the pressure difference between *A* and *C* will now cause an increase in size of the bubble. The lever *b* is now tilted by revolving the micrometer disc until the upper surface of the bubble again coincides with the cross-wire of the microscope. This operation has brought the size of the water bubble back to the original state, showing that the total pressures of the water in *A* and *C* are again equal. The readings of *e* and the micrometer disc are now taken; by subtracting the initial readings from these, the difference of pressure head in the two limbs of the Pitot tube can be calculated.

The instrument is extremely sensitive; differences of pressure can be measured as small as 0·0001 in. of water. It is used for Pitot tube measurements of air velocities in wind channels. It may also be modified to be used for measuring pressure differences in other types of fluids.

2. INCLINED TUBE MANOMETER. If one arm of a U-tube is bent so that it has a small inclination to the horizontal, the movement of the liquid in this inclined tube will be considerable for a small change in pressure head. Hence, a U-tube can be made very sensitive by bending one of its arms until it is nearly horizontal; this is the principle of the inclined tube manometer.

A view of the Krell type of inclined tube manometer is shown in Fig. 161. The inclined tube D corresponds to one arm of the U-tube,

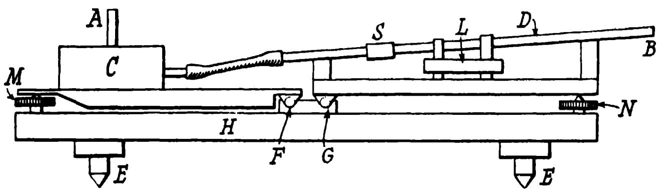


FIG. 161. KRELL TYPE OF INCLINED TUBE MANOMETER

the other arm being the large reservoir C. The tubes from the Pitot-static tube are connected to the arms C and D at the openings A and B, the higher pressure tube being connected at A. By making the liquid surface in C very many times larger than the cross-sectional area of the tube D, the variation of height of the liquid surface in C will be extremely small and can, therefore, be regarded as constant. The pressure difference in the two arms will consequently be given by the reading of the liquid surface in the inclined tube D.

The instrument is mounted on a base H, which can be set horizontally by three levelling screws E. The frame supporting the reservoir C is hinged at F and can be raised or lowered by turning the screw M; this adjusts the zero reading of the liquid in the inclined tube D. The frame supporting the inclined tube D is hinged at G; its inclination can be varied by means of the screw N. The liquid level in this tube can be read off a fixed scale by the sliding cross-wire S.

The instrument can be used for measuring two ranges of air speed by fixing the inclined tube at two different slopes. This is done by means of two spirit levels L, one on each side of the tube, which are permanently set at different inclinations. When it is desired to measure low air speeds, the inclined tube D is set at a small inclination by using the spirit level set with the least permanent slope. If it is desired to measure high air speeds, the inclined tube can be set at a greater slope as larger changes of head will now occur; this is done by setting the tube D to the other spirit level, which has been permanently fixed at a greater slope. It will be seen from this that there are two ranges for the instrument, depending on which spirit level is used for setting the inclination of the tube D.

Alcohol is found to be the best liquid for use in this instrument as it provides a suitable meniscus for the inclined tube. The instrument can be used as a sensitive U-tube for measuring any fluid pressure differences within its range; its use is not limited to the Pitot-static tube.

3. THE CURVED TUBE MANOMETER. This is the same in principle as the inclined tube manometer, but the tube is curved in such a form that a uniform scale of velocities is obtained. A view of the

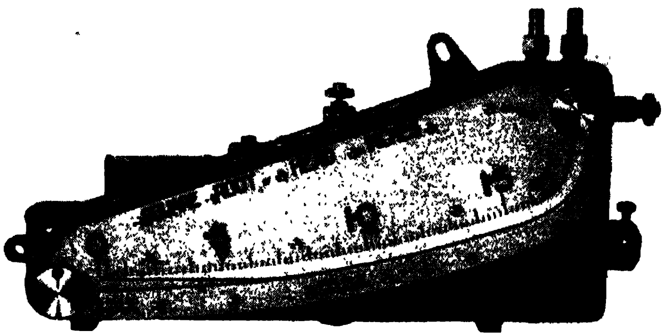


FIG. 162. THE KENT-HODGSON CURVED TUBE MANOMETER
(*Courtesy of George Kent, Ltd.*)

Kent-Hodgson Curved Tube Manometer is shown in Fig. 162. If this instrument is permanently installed in a fixed position for use in connexion with a particular apparatus, such as a Pitot-static tube, a velocity scale can be substituted for the pressure-difference scale, from which air velocities can be read off direct. If the installation is of a permanent nature, such as for the measurement of air flow through a duct or pipe of known dimensions, a quantity scale can be substituted from which the quantity of flow can be obtained directly from the scale reading.

A great advantage of the curved tube manometer is that a wide range of air speeds, or pressure differences, can be quickly read off with the one setting of the instrument. This manometer is for the measurement of low pressures only and will read pressure differences of 0·01 in. of water.

EXERCISES 13

1. Calculate the velocity of a pressure wave transmitted through a liquid having a specific gravity of 0·85 and a bulk modulus of 284,000 Lb/in.^{2}
Ans. 4,985 ft/sec.

2. Calculate the velocity of sound in air having a pressure of 9·38 Lb/in.^{2} abs. and a temperature of $-$ 9°C.
(*a*) Assuming an isothermal process.
(*b*) Assuming an adiabatic process.
R for air $= 96$ ft-Lb Centigrade units and $\gamma = 1{\cdot}4$.
Ans. (*a*) 900 ft/sec; (*b*) 1,065 ft/sec.

3. Find the pressure and density of the atmosphere at an altitude of 13,000 ft assuming an isothermal atmosphere. At sea level, $p = 14{\cdot}7$ Lb/in.^{2} and $T = 15°C$. R for air $= 96$ ft-Lb Centigrade units.

Ans. 9·2 Lb/in.^{2}; 0·048 Lb/ft^3.

4. If the pressure and temperature of the atmosphere at ground level are 14·7 Lb/in.^{2} and 15°C respectively, calculate the pressure and density at an altitude of 16,000 ft assuming an adiabatic atmosphere. Find also the mean temperature gradient up to this altitude. $R = 96$ ft-Lb units and $\gamma = 1{\cdot}4$.

Ans. 7·83 Lb/in.^{2}, 0·049 Lb/ft^3, 3°C per 1,000 ft.

5. Find the pressure and density of the atmosphere at a height of 3,000 ft when the pressure and temperature at ground level are 30 in. of mercury and 15°C. Assume that the temperature of the atmosphere diminishes with the height at a uniform rate of 1·5°C per 1,000 ft. For 1 Lb of air $pV = 96T$. (*Lond. Univ.*)

Ans. 13·18 Lb/in.^{2}; 0·0698 Lb/ft^3.

6. Show that a disturbance is propagated with velocity $\sqrt{(K/\rho)}$ in a fluid of elasticity K and density ρ. Determine this speed for water, $K = 300{,}000$ Lb/in.^{2} (*Lond. Univ.*)

Ans. 4,730 ft/sec.

7. Given that the barometer pressure is p_0 at ground level where the temperature is 15°C, prove that the pressure p at a height h ft is given by the expression

$$\log \frac{p}{p_0} = A \log (1 - Bh)$$

in which A and B are constants. If the temperature of a quiescent atmosphere diminishes with the height at a uniform rate of 2°C per 1,000 ft, and for air $pV = 96T$, find the values of A and B. (*Lond. Univ.*)

Ans. $A = 5{\cdot}21$; $B = 0{\cdot}00000695$.

8. Deduce the atmosphere pressure at 10,000 ft when it is 14·7 Lb/in.^{2} at sea level, weighs 0·075 Lb/ft^3 at sea level, and the temperature is constant. Find the pressure at 2 ft radius in a free vortex, at an altitude of 10,000 ft, when the motion is adiabatic and the speed 30 ft/sec at 40 ft radius. (*Lond. Univ.*)

Ans. 10·32 Lb/in.^{2}; 2·24 Lb/in.^{2}

CHAPTER 14

FLOW OF GASES

14.1. Types of Flow. The flow of a gas is a more complex problem than the flow of liquids on account of the great compressibility of the former. The gas flow may take place isothermally, adiabatically, or at constant total energy. The flow may be resisted by friction which partially or wholly reheats the gas; thus the loss of energy due to friction reappears in the gas in the form of internal energy. If large changes of temperature occur during the flow, the variation in the specific heats will be considerable and should be taken into account.

The type of flow is also affected by the velocity. It may be subsonic, sonic or supersonic, depending on its relation to the velocity of sound in the gas; the latter will vary according to its temperature or its pressure and density. At speeds above the sonic speed compression waves and shock waves are transmitted which affect the resistance to flow. Consequently, when dealing with gas flow of high velocity, that is, velocities in the vicinity of the velocity of sound, it is usual to refer all phenomena to the Mach number (§ 3.17).

The above remarks also apply to bodies moving with high velocities in a stationary gas; the resulting phenomena will depend on the relative velocity between the body and the gas.

14.2. Energy Equation for Flowing Gases. By applying the law of conservation of energy to a gas flowing through any type of duct an energy equation for flowing gases can be obtained. This corresponds to the Bernoulli equation for a liquid. With flowing gases, any frictional resistance does not cause a loss of energy to the gas, because the frictional work done reheats the gas thus increasing its temperature.

Consider the gas flowing through the tapering pipe of Fig. 163, and consider the portion of gas between sections (1) and (2). Assume 1 Lb of gas to flow through the pipe and cause a movement of dl_1 at section (1) and dl_2 at section (2).

Work done by pressure at section (1)

$$= (p_1 a_1) \times dl_1$$

$$= p_1 V_1 \text{ (as } V_1 = a_1 \times dl_1)$$

where V_1 is the volume of 1 Lb of gas at section (1).

Work done against pressure at section (2)

$$= (p_2 a_2) \times dl_2$$
$$= p_2 V_2 \text{ (as } V_2 = a_2 \times dl_2)$$

where V_2 is the volume of 1 Lb of gas at section (2).

Summing up the energies, above absolute zero temperature, at the two sections, and ignoring any change in altitude as small,

$$\begin{matrix}\text{work done}\\ \text{on gas}\end{matrix} + \begin{pmatrix}\text{heat}\\ \text{absorbed}\\ \text{by gas}\end{pmatrix} = \begin{pmatrix}\text{gain of}\\ \text{internal}\\ \text{energy}\end{pmatrix} + \begin{pmatrix}\text{gain of}\\ \text{kinetic}\\ \text{energy}\end{pmatrix}$$

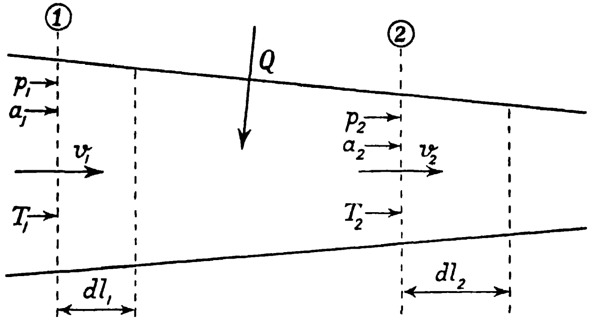


FIG. 163

Let Q = heat entering through pipe walls per pound of gas. Then,

$$\left(\frac{p_1 V_1 - p_2 V_2}{}\right) + Q = (U_2 - U_1) + \left(\frac{v_2^2 - v_1^2}{2gJ}\right)$$

from which

$$p_1 V_1 + U_1 + \frac{v_1^2}{2gJ} = p_2 V_2 + U_2 + \frac{v_2^2}{2gJ} - Q \qquad (1)$$

Eq. (1) is the equation of steady flow of a gas. If heat flows out of the pipe during this operation, Q becomes negative.

It should be noted that the term $p_1 V_1/J$ represents the work required to introduce 1 Lb of flowing gas between the sections considered; and the term $p_2 V_2/J$ represents the work expended in discharging 1 Lb of gas from section (2). The former is sometimes termed the *inhaling work* and the latter the *exhaling work*.

From eq. (14), § 13.6, it was shown that

$$H_1 = p_1 V_1 + U_1$$

and

$$H_2 = \frac{p_2 V_2}{J} + U_2$$

where H_1 and H_2 are the enthalpies, or total heats, of the gas.

Substituting these values in eq. (1),

$$H_1 + \frac{v_1^2}{2gJ} = H_2 + \frac{v_2^2}{2gJ} - Q \quad . \quad . \quad . \quad (2)$$

1. For an Isothermal Process. If the flow is isothermal, from § 13.4, $U_2 = U_1$ and $p_1V_1 = p_2V_2$. Hence,

$$Q = W = \text{work done by gas}$$

$$-\frac{p_1V_1}{J}\log_e\left(\frac{V_2}{V_1}\right) = \text{gain of kinetic energy},$$

Hence $$\frac{v_2^2 - v_1^2}{2g} = p_1V_1\log_e\left(\frac{V_2}{V_1}\right)$$

$$= p_1V_1\log_e\left(\frac{p_1}{p_2}\right) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

2. For an Adiabatic Process. If the flow is adiabatic, $Q = 0$. From eq. (8), § 13.5,

$$U_1 - U_2 = \frac{p_1V_1 - p_2V_2}{J(\gamma - 1)} \text{ heat units}$$

Substituting these values in eq. (1),

$$(p_1V_1 - p_2V_2) + \frac{p_1V_1 - p_2V_2}{(\gamma - 1)} = \frac{v_2^2 - v_1^2}{2g}$$

Hence, $$\frac{v_2^2 - v_1^2}{2g} = \frac{\gamma}{\gamma - 1}(p_1V_1 - p_2V_2)$$

$$= p_1V_1\frac{\gamma}{\gamma - 1}\left(1 - \frac{p_2V_2}{p_1V_1}\right)$$

Substituting for V_2/V_1 from eq. (9), § 13.5,

$$\frac{v_2^2 - v_1^2}{2g} = p_1V_1\frac{\gamma}{\gamma - 1}\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right] \quad . \quad . \quad (4)$$

As $p_1V_1 = RT_1$, eq. (4) may be written,

$$\frac{v_2^2 - v_1^2}{2g} = RT_1\frac{\gamma}{(\gamma - 1)}\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right] \quad . \quad . \quad (5)$$

Also, substituting for $(v_2^2 - v_1^2)/2g$ from eq. (2),

$$H_1 - H_2 = \frac{RT_1}{J}\frac{\gamma}{(\gamma - 1)}\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right] \quad . \quad . \quad (6)$$

The term $H_1 - H_2$ is known as the adiabatic heat drop, and is represented by a vertical line on the H–S chart.

14.3. Velocity of Gas from Heat Drop. When a gas is allowed to expand through an orifice or nozzle, the loss of total heat which occurs is converted into kinetic energy, thus producing the velocity. Applying the energy equation for a gas (eq. (1)) and working in heat units,

$$\frac{p_1V_1}{J} + \frac{v_1^2}{2gJ} + U_1 = \frac{p_2V_2}{J} + \frac{v_2^2}{2gJ} + U_2 - Q$$

But, from § 13.6,

$$\frac{p_1V_1}{J} + U_1 = H_1$$

and

$$\frac{p_2V_2}{J} + U_2 = H_2$$

Hence, substituting these values, the equation becomes

$$H_1 + \frac{v_1^2}{2gJ} = H_2 + \frac{v_2^2}{2gJ} - Q \quad . \quad . \quad . \quad (7)$$

from which

$$\frac{v_2^2 - v_1^2}{2gJ} - Q = (H_1 - H_2) \quad . \quad . \quad . \quad (8)$$

The term $H_1 - H_2$ is the reduction in total heat during the expansion and is known as the *heat drop.*

Let H_d = heat drop in heat units
$= H_1 - H_2$.

If the expansion is adiabatic, $Q = 0$; then H_d is known as the *adiabatic heat drop.*

If the gas is initially at rest and the expansion is adiabatic, $v_1 = 0$; then eq. (8) becomes—

$$\frac{v_2^2}{2g} = (H_1 - H_2)J \text{ ft-Lb units}$$
$$= JH_d$$

from which

$$v_2 = \sqrt{2gJ}\,\sqrt{H_d}$$
$$= 300\sqrt{H_d} \text{ (in Centigrade units)} \quad . \quad (9)$$
$$= 224\sqrt{H_d} \text{ (in Fahrenheit units)} \quad . \quad (10)$$

As the expansion is adiabatic, eq. (4) may be used. Then,

$$\frac{v_2^2 - v_1^2}{2g} \quad p_1V_1\left(\frac{\gamma}{\gamma - 1}\right)\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right]. \quad . \quad (11)$$

Hence, from eq. (6),

$$H_1 - H_2 = \frac{p_1V_1}{J}\left(\frac{\gamma}{\gamma - 1}\right)\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right]. \quad . \quad (12)$$

The value of the heat drop H_d for 1 Lb of air can also be read off the total-heat scale of the heat-entropy chart, Fig. 155, facing page 330.

EXAMPLE 1

Air at 200 Lb/in.^{2} pressure and 140°F is expanded adiabatically in a nozzle to a pressure of 40 Lb/in.^{2} Using the heat-entropy chart and neglecting all losses, find the final temperature and specific volume of the air, the total-heat drop during the expansion, and the final velocity of the air.

From the heat-entropy chart, Fig. 155,
final temperature = − 83°F; final specific volume = 3·43 ft^3/Lb

$$\text{total heat drop} = H_d = 26 - (-27{\cdot}5)$$
$$= 53{\cdot}5 \text{ B.Th.U./Lb}$$

Using eq. (10),

$$v = 224\sqrt{H_d}$$
$$= 224\sqrt{53{\cdot}5}$$
$$= 1{,}645 \text{ ft/sec}$$

14.4. Effect of Friction on Heat Drop. The frictional resistance of the walls of a nozzle or pipe tends to reduce the velocity of flow, which, in turn, reduces the effective heat drop, since the initial and

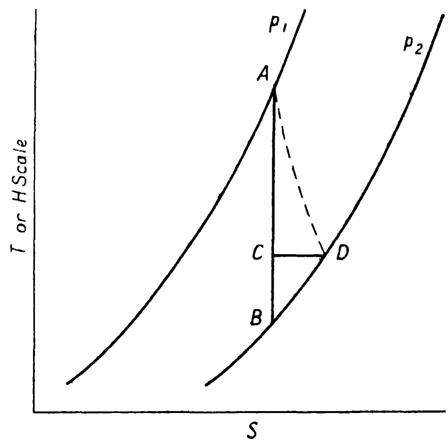


FIG. 164

back pressures are fixed. The loss of kinetic energy due to friction is converted into heat which reheats the gas. Hence, the effect of friction is to convert part of the kinetic energy of the gas into heat which increases its final total heat and internal energy; this operation is demonstrated in Fig. 164.

Let k = fraction of heat drop available after allowing for friction. Then,

$$\text{effective heat drop} = kH_d$$

Applying this to eqs. (9) and (10),

$$v_2 = 300\sqrt{kH_d} \text{ (in Centigrade units)} \quad . \quad . \quad (13)$$

$$v_2 = 224\sqrt{kH_d} \text{ (in Fahrenheit units)} \quad . \quad . \quad (14)$$

Referring to the H–S diagram of Fig. 164, let AB represent the frictionless adiabatic expansion of 1 Lb of the gas between pressures of p_1 and p_2. This is the adiabatic heat drop. If, owing to frictional resistance, the adiabatic heat drop is reduced by BC, then

$$\begin{aligned}\text{effective heat drop} &= AC \\ &= kAB\end{aligned}$$

Now the gas must finally have the pressure p_2, and hence the effect of friction is to reheat the gas at constant total heat from C to D; the point D is where the horizontal from C cuts the pressure line p_2. This reheating operation has actually taken place continuously during the whole expansion; hence the true condition of the gas throughout the expansion is represented by the dotted line AD. Then,

$$\begin{aligned}\text{resulting heat drop} &= H_A - H_D \\ &= H_A - H_C \\ &= k(H_A - H_B)\end{aligned}$$

It will be seen from Fig. 164 that the effect of friction is to reduce the heat drop, and thus reduce the final velocity, whilst the final temperature of the gas is increased from T_B to T_D.

The ratio between the actual effective heat drop and the frictionless adiabatic heat drop is known as the *adiabatic efficiency*. Or, from Fig. 164,

$$\text{adiabatic efficiency} = \frac{AC}{AB} = k \quad . \quad . \quad . \quad . \quad (15)$$

The operation shown in Fig. 164 can be drawn on the heat-entropy chart facing page 330, and the required values of H, H_d and V_s read off the respective scales.

EXAMPLE 2

Air at a pressure of 110 Lb/in.^{2} and a temperature of 73°F expands adiabatically through a nozzle to a pressure of 16 Lb/in.^{2}; 10 per cent of the total heat drop is lost in friction which is assumed to reheat the air.

Using the heat-entropy chart, find the final temperature and specific volume of the air and calculate its final velocity.

From the heat-entropy chart, and using the construction shown in Fig. 164,

$$\begin{aligned}\text{total heat drop} &= H_d = 54{\cdot}5 \text{ B.Th.U.} \\ \text{final temperature} &= -153^\circ\text{F} \\ \text{specific volume} &= 6{\cdot}9 \text{ ft}^3\text{/Lb}\end{aligned}$$

Using eq. (14),

$$\begin{aligned} v &= 224\sqrt{kH_d} \\ &= 224\sqrt{0{\cdot}9 \times 54{\cdot}5} \\ &= 1{,}563 \text{ ft/sec} \end{aligned}$$

EXAMPLE 3

Air at a pressure of 80 Lb/in.² and a temperature of 20°C flows through a pipe with a velocity of 300 ft/sec and is passed through an expansion nozzle in which it is expanded down to 14·7 Lb/in.² Using the heat-entropy chart, find the heat drop in the nozzle and the final temperature of the air. Assume 15 per cent of the heat drop is lost in friction.

From the heat-entropy chart,

$$\text{adiabatic heat drop} = 27{\cdot}7 \text{ C.H.U.}$$

Final temperature after allowing for reheating by friction (Fig. 164)

$$= -\,76°\text{C}$$

Using eq. (8),

$$\frac{v_2^2 - v_1^2}{2g} = kH_dJ$$

that is

$$\frac{v_2^2 - 300^2}{64{\cdot}4} = 0{\cdot}85 \times 27{\cdot}7 \times 1{,}400$$

then

$$v_2^2 - 90{,}000 = 2{,}120{,}000$$

Hence

$$\begin{aligned} v_2 &= \sqrt{2{,}210{,}000} \\ &= 1{,}489 \text{ ft/sec} \end{aligned}$$

14.5. Flow of Gas through Venturi Meter. The flow of a gas through a Venturi may be regarded as an isothermal process if the pressure drop is small, in which case the flow can be calculated by the method given in § 3.6. If the pressure drop is appreciable, the flow will be adiabatic, and there is a rapid fall of temperature at the throat. In this case eq. (4) can be used for calculating the quantity of flow.

Imagine a quantity of gas to be flowing through the Venturi meter of Fig. 34 (page 56). Let suffix 1 apply to the conditions of the gas at entrance, and suffix 2 to the conditions at the throat. Neglect all losses, and assume flow to be adiabatic.

Let a_1 and a_2 be the areas of cross-section at inlet and throat, in square feet, and v_1 and v_2 the corresponding velocities.

Weight of gas flowing per second

$$= W = \frac{a_1v_1}{V_1} = \frac{a_2v_2}{V_2}$$

where V_1 and V_2 represent the volume of 1 Lb gas at pressures p_1 and p_2 respectively.

Hence $$v_1 = \frac{a_2}{a_1}\left(\frac{V_1}{V_2}\right) v_2 \quad . \quad . \quad . \quad . \quad (16)$$

But, from eq. (9), § 13.5,

$$\frac{V_1}{V_2} = \left(\frac{p_2}{p_1}\right)^{1/\gamma}$$

Hence, from eq. (16), $$v_1 = \frac{a_2}{a_1}\left(\frac{p_2}{p_1}\right)^{1/\gamma} v_2 \quad . \quad . \quad . \quad . \quad (17)$$

Now, the flow through the meter is assumed to be adiabatic, and is consequently represented by eq. (4). Then, substituting eq. (17) in this equation,

$$\frac{v_2^2}{2g}\left[1 - \left(\frac{a_2}{a_1}\right)^2 \left(\frac{p_2}{p_1}\right)^{2/\gamma}\right] = p_1 V_1 \left(\frac{\gamma}{\gamma - 1}\right)\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right]$$

from which

$$v_2 = \sqrt{\frac{2g\left(\frac{\gamma}{\gamma-1}\right) p_1 V_1 \left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right]}{\left[1 - \left(\frac{a_2}{a_1}\right)^2 \left(\frac{p_2}{p_1}\right)^{2/\gamma}\right]}}$$

Then $$W = \frac{a_2 v_2}{V_2}$$

$$= \frac{a_2}{V_2}\sqrt{\frac{2g\left(\frac{\gamma}{\gamma-1}\right) p_1 V_1 \left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right]}{\left[1 - \left(\frac{a_2}{a_1}\right)^2 \left(\frac{p_2}{p_1}\right)^{2/\gamma}\right]}} \quad . \quad (18)$$

Eq. (18) gives the weight of gas flowing in pounds per second.

EXAMPLE 4

A horizontal air Venturi meter is installed in a 6 in. pipe line, its throat diameter being 2 in. The inlet pressure measured 140 Lb/in.^{2} abs. and the throat pressure 130 Lb/in.^{2} abs. The temperature at inlet was 15°C. Calculate the weight of air flowing per second. $R = 96$ ft-Lb units and $\gamma = 1{\cdot}4$.

As $$p_1 V_1 = RT_1$$

$$V_1 = \frac{96 \times 288}{144 \times 140}$$

$$= 1{\cdot}37 \text{ ft}^3/\text{Lb}$$

As expansion is assumed to be adiabatic,

$$V_2 = V_1 \left(\frac{p_1}{p_2}\right)^{1/\gamma}$$

$$= 1{\cdot}37 \left(\frac{140}{130}\right)^{\frac{1}{1\cdot4}}$$

$$= 1{\cdot}443 \text{ ft}^3/\text{Lb}$$

Also
$$\frac{p_2}{p_1} = \frac{130}{140}$$

$$= 0{\cdot}9285$$

Using eq. (18),

$$W = \frac{a_2}{V_2}\sqrt{\frac{2g\left(\frac{\gamma}{\gamma-1}\right)p_1V_1\left[1-\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right]}{\left[1-\left(\frac{a_2}{a_1}\right)^2\left(\frac{p_2}{p_1}\right)^{2/\gamma}\right]}}$$

$$= \frac{\frac{\pi}{4}\times\frac{1}{36}}{1{\cdot}443}\sqrt{\frac{64{\cdot}4\times\frac{1{\cdot}4}{0{\cdot}4}\times144\times140\times1{\cdot}37\left[1-(0{\cdot}9285)^{\frac{0\cdot4}{1\cdot4}}\right]}{\left[1-\frac{1}{81}(0{\cdot}9285)^{\frac{2}{1\cdot4}}\right]}}$$

$$= 0{\cdot}0151\sqrt{132{,}300}$$

$$= 5{\cdot}48 \text{ Lb/sec}$$

14.6. Flow of Gas through an Orifice or Nozzle. The flow of gas through an orifice or nozzle may be regarded as an adiabatic process if the pressure drop is large. For a small pressure drop the process may be assumed isothermal; the isothermal flow of a gas through orifices and pipes is dealt with in § 3.14 and § 7.13 respectively.

The adiabatic flow through an orifice (Fig. 165) is represented by eq. (4). In this case $v_1 = 0$, as the flow commences from rest. Let the final velocity of the gas be represented by v. Then, eq. (4) becomes

$$\frac{v^2}{2g} = p_1V_1\left(\frac{\gamma}{\gamma-1}\right)\left[1-\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right] \quad . \quad . \quad . \quad (19)$$

where p_1 and V_1 apply to the initial state of 1 Lb of the gas and p_2/p_1 is the ratio of the pressures on both sides of the orifice, or between the inlet and throat of the nozzle. Then, from eq. (19),

$$v = \sqrt{2gp_1V_1\left(\frac{\gamma}{\gamma-1}\right)\left[1-\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

Let a = effective area of orifice or throat area of nozzle in square feet
$= C_d \times$ area of orifice.

Weight of gas flowing per second $= W = \dfrac{av}{V_2}$

where V_2 is the volume of 1 Lb of gas at pressure p_2.

Then

$$W = \frac{a}{V_2}\sqrt{2gp_1V_1\left(\frac{\gamma}{\gamma-1}\right)\left[1-\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right]} \tag{20}$$

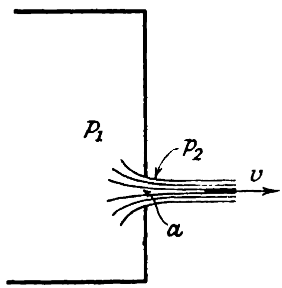


FIG. 165

But, it was proved in § 14.5 that

$$\frac{V_1}{V_2} = \left(\frac{p_2}{p_1}\right)^{1/\gamma}$$

Hence

$$V_2 = \frac{V_1}{\left(\frac{p_2}{p_1}\right)^{1/\gamma}}$$

Substituting this value of V_2 in eq. (20),

$$W = \frac{a}{V_1}\sqrt{2gp_1V_1\left(\frac{\gamma}{\gamma-1}\right)\left(\frac{p_2}{p_1}\right)^{2/\gamma}\left[1-\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right]}\,. \tag{21}$$

This equation gives the weight of discharge in pounds per second. It is more convenient to solve nozzle problems by using the heat-entropy chart for the given gas if such a chart is available (§ 13.8 and § 14.3).

14.7. Velocity at Throat for Maximum Discharge. From eq. (21) it will be seen that, for a gas in the given initial condition, the only variable is p_2/p_1.

Let $n = p_2/p_1$.

Hence, if this equation is differentiated in terms of n and equated to zero, the value of n for a maximum weight of discharge through the orifice, or nozzle, would be obtained.

From eq. (21),

$$W \propto n^{2/\gamma}\left[1 - n^{\frac{\gamma-1}{\gamma}}\right]$$

that is,

$$W \propto n^{2/\gamma} - n^{\frac{\gamma+1}{\gamma}}$$

Differentiating,

$$\frac{dW}{dn} = \frac{2}{\gamma} n^{\frac{2-\gamma}{\gamma}} - \frac{\gamma + 1}{\gamma} n^{1/\gamma} = 0$$

That is,

$$(\gamma + 1)n^{1/\gamma} = 2n^{\frac{2-\gamma}{\gamma}}$$

Hence

$$n^{\frac{\gamma-1}{\gamma}} = \frac{2}{\gamma + 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Eq. (22) gives the value of n, the pressure ratio, for a maximum weight of discharge for any orifice or nozzle and for any gas. Applying this equation to air, $\gamma = 1{\cdot}404$, eq. (22) then becomes

$$n^{\frac{1\cdot404-1}{1\cdot404}} = \frac{}{1{\cdot}404 + 1}$$

from which

$$n = 0{\cdot}528$$

Hence

$$n = \frac{p_2}{p_1} = 0{\cdot}528$$

or

$$p_2 = 0{\cdot}528p_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Thus, the maximum weight of discharge for air flowing through an orifice, or nozzle, is when the final pressure is 0·528 of the initial pressure.

It should be noticed that, as the velocity head is equal to the total adiabatic work done, then, from eq. (19),

$$\frac{v_2^2}{2g} = \frac{\gamma}{\gamma - 1} p_2 V_2 \left[\left(\frac{1}{n}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]. \quad . \quad (24)$$

where the suffix 2 applies to the throat of the nozzle, or outlet of orifice.

But

$$V_2 = \frac{1}{w_2} \quad \text{where } w_2 = \rho_2 g$$

Hence

$$V_2 = \frac{1}{\ }$$

Also, for maximum discharge (eq. (22)),

$$(n)^{\frac{\gamma-1}{\gamma}} = \frac{2}{\gamma+1}$$

Substituting these values in eq. (24),

$$\frac{v_2^2}{2g} = \left(\frac{\gamma}{\gamma-1}\right)\frac{p_2}{\rho_2 g}\left[\frac{\gamma+1}{2} - 1\right]$$

$$= \left(\frac{\gamma}{\gamma-1}\right)\frac{p_2}{\rho_2 g}\left(\frac{\gamma-1}{2}\right)$$

from which $$v_2 = \sqrt{\frac{\gamma p_2}{\rho_2}}$$

But it will be noticed from eq. (26), § 13.12, that the above equation is the equation for the velocity of sound in the gas, at the throat conditions. Hence, it follows that the critical velocity at the throat is equal to the velocity of sound. This result is very important as it provides the explanation of much of the phenomena which occur in nozzles after the sonic speed is reached.*

EXAMPLE 5

A nozzle having a throat diameter of 1 in. is fitted into the side of a tank containing air at a pressure of 120 Lb/in.² and at a temperature of 15°C. What should be the back pressure for a maximum discharge through the nozzle? Calculate the weight of this maximum discharge in pounds per second. Assume $C = 96$ and $\gamma = 1{\cdot}404$.

Using eq. (23) for maximum discharge,

$$\begin{aligned} p_2 &= 0{\cdot}528 p_1 \\ &= 0{\cdot}528 \times 120 \\ &= 63{\cdot}4 \text{ Lb/in.}^2 \end{aligned}$$

As $$p_1 V_1 = 96T$$

then $$\begin{aligned} V_1 &= \frac{96 \times 288}{144 \times 120} \\ &= 1{\cdot}6 \text{ ft}^3\text{/Lb} \end{aligned}$$

$$\begin{aligned} n &= \frac{p_2}{p_1} \\ &= 0{\cdot}528 \end{aligned}$$

Using eq. (21),

$$W = \frac{a}{V_1}\sqrt{2g\left(\frac{\gamma}{\gamma-1}\right)p_1 V_1 \left(\frac{p_2}{p_1}\right)^{2/\gamma}\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\ }{\gamma}}\right.}$$

* For a complete treatment of flow of gases through orifices and nozzles see § 17.8 and § 17.9 and the whole of Chapter 19.

$$\frac{\dfrac{\pi}{4 \times 144 \times 1{\cdot}6} \times}{\sqrt{64{\cdot}4 \times \dfrac{1{\cdot}404}{0{\cdot}404} \times 144 \times 120 \times 1{\cdot}6 \times 0{\cdot}528^{2/\gamma}\left[1 - 0{\cdot}528^{\frac{\gamma-1}{\gamma}}\right]}}$$

$$= 0{\cdot}0034\sqrt{420{,}000}$$

$$= 2{\cdot}2 \text{ Lb/sec}$$

EXAMPLE 6

Air is expanded in a converging-diverging nozzle which has a throat diameter of 1 in. The air enters the nozzle at a pressure of 60 Lb/in.² and a temperature of 15°C, and is expanded to 14·7 Lb/in.² at the mouth. The frictional losses all occur in the diverging cone and reduce the total heat drop in the nozzle by 15 per cent. Using the heat-entropy chart, find (1) the critical pressure at the throat; (2) the velocity at the throat; (3) the discharge in pounds per second; (4) the final velocity of the air leaving the nozzle; (5) the required diameter at the mouth.

(1) Critical pressure at throat $= p_2 = 0{\cdot}528 p_1$

$$= 0{\cdot}528 \times 60 = 31{\cdot}7 \text{ Lb/in.}^2$$

(2) From heat-entropy chart, and using suffix 2 for throat conditions,

total heat drop at throat $= H_{d_2} = 12{\cdot}0$ C.H.U.

and $V_{s_2} = 5{\cdot}0$ ft³/Lb

Then
$$v_2 = 300\sqrt{H_{d_2}}$$
$$= 300\sqrt{12}$$
$$= 1{,}039 \text{ ft/sec}$$

(3)
$$W = \frac{a_2 v_2}{V_{s_2}}$$
$$= \frac{\pi}{4} \times \left(\frac{1}{12}\right)^2 \times \frac{1{,}039}{5{\cdot}0} = 1{\cdot}131 \text{ Lb/sec}$$

(4) From heat-entropy chart, and using suffix 3 for mouth conditions,

total heat drop $= H_d = 23{\cdot}2$ C.H.U.

net heat drop $= 0{\cdot}85 H_d$

$$= 0{\cdot}85 \times 23{\cdot}2$$
$$t_3 = -84°\text{C}$$
$$V_{s_3} = 8{\cdot}7 \text{ ft}^3/\text{Lb}$$
$$v_3 = 300\sqrt{kH_d}$$
$$= 300\sqrt{0{\cdot}85 \times 23{\cdot}2} = 1{,}332 \text{ ft/sec}$$

(5) As $$W = \frac{a_3 v_3}{V_{s_3}}$$

$$a_3 = \frac{1{\cdot}131 \times 8{\cdot}7}{1{,}332} = 0{\cdot}0074 \text{ ft}^2$$

Hence $$d_3 = \sqrt{0{\cdot}0074 \times 144 \times \frac{4}{\pi}} = 1{\cdot}165 \text{ in.}$$

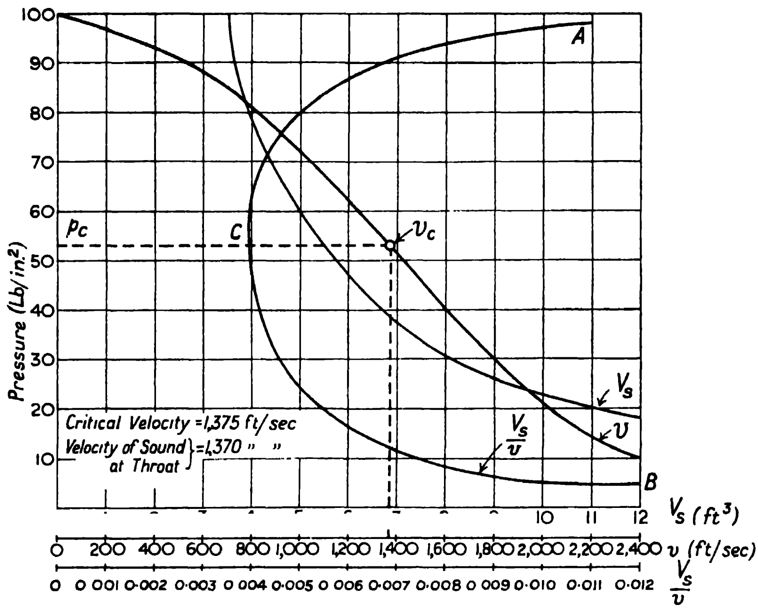


FIG. 166

14.8. Frictionless Adiabatic Flow of Jet. If a gas of initial pressure p_1 is expanded adiabatically through a nozzle to a back pressure p_b, its behaviour as it passes through the nozzle will depend on the initial pressure, the back pressure and the shape of the nozzle. As the gas expands its velocity and specific volume are continually increasing, but the weight of discharge per second, passing any section, must remain constant.

In Fig. 166 curves showing the variation of the velocity v, the specific volume V_s and the ratio V_s/v have been plotted for 1 Lb of air expanding adiabatically from an initial pressure of 100 Lb/in.^{2} to a back pressure of 5 Lb/in.^{2} absolute. The base represents respectively v, V_s and V_s/v, and the vertical ordinate represents the pressure at any instant. The values of V_s for the various pressures

were calculated from the gas laws for an adiabatic expansion, and the values of v from the equation

$$v = 300\sqrt{H_d}$$

where H_d is the drop in total heat in C.H.U.

Let a = cross-sectional area of jet in square feet at pressure considered. Now,

$$W = \frac{av}{V_s} \text{ Lb of air per second}$$

then $$\frac{V_s}{v} = a \text{ (for 1 Lb of air)} \quad . \quad . \quad . \quad . \quad (25)$$

From the curves it will be seen that, at a pressure of $0{\cdot}53p_1$, which is the pressure for the critical velocity, the velocity is the same as the sonic velocity at that pressure and density, which agrees with the results deduced in § 14.7. At this section the Mach number for the flow is unity.

On examining the curve of V_s/v, which is also the jet area per pound of flow, it will be noticed that its minimum value occurs at the critical velocity; this is equivalent to the throat of the jet. Also, for any given jet area above this minimum value, there are two values for the pressure, one given by the upper portion of the curve AC, and the other by the lower portion CB. From the velocity curve it will be seen that the curve AC represents a subsonic flow, whilst the curve CB corresponds to a supersonic flow. Hence, it follows that, for a given jet area, there are two pressure conditions for equilibrium, one on the curve AC producing subsonic flow and one on the curve CB producing supersonic flow. Once the section of critical velocity has been passed, the flow may follow either of these conditions, depending on the magnitude of the back pressure. For the ideal nozzle the minimum area, or throat, will be at the critical section C; beyond this section the expansion then follows the curve CA up to the back pressure at A. If the back pressure is less than that at A, the flow will at first follow the curve CB for a certain distance, depending on the back pressure, and may then suddenly increase in pressure towards the curve CA after which the flow remains subsonic. This sudden jump in pressure takes place at constant total energy except for a small loss due to friction.

It is interesting to notice the resemblance between the V_s/v curve of Fig. 166 and the specific energy curve for the flow of water in channels shown in Fig. 138. The former has its minimum value when the Mach number is unity, and the latter has its minimum value when the Froude number is unity. To the right of these conditions each curve shows two pressure values for equilibrium each of which produces two different types of flow. Hence, there is a close resemblance between the flow of gas through a nozzle and the flow of a liquid in a channel.

14.9. Variation of Pressure in a Nozzle. The flow of a gas through a converging-diverging nozzle is liable to become complicated once the flow is supersonic, on account of the sudden jump in pressure which may occur (§ 14.8). This phenomenon is explained by the curves of Fig. 167, which have been plotted for a given converging-diverging nozzle. The curves are plotted on a base representing the centre-line of the nozzle which is shown drawn to scale. They represent the same adiabatic flow of air dealt with in § 14.8, the required values of v and V_s being taken from Fig. 166.

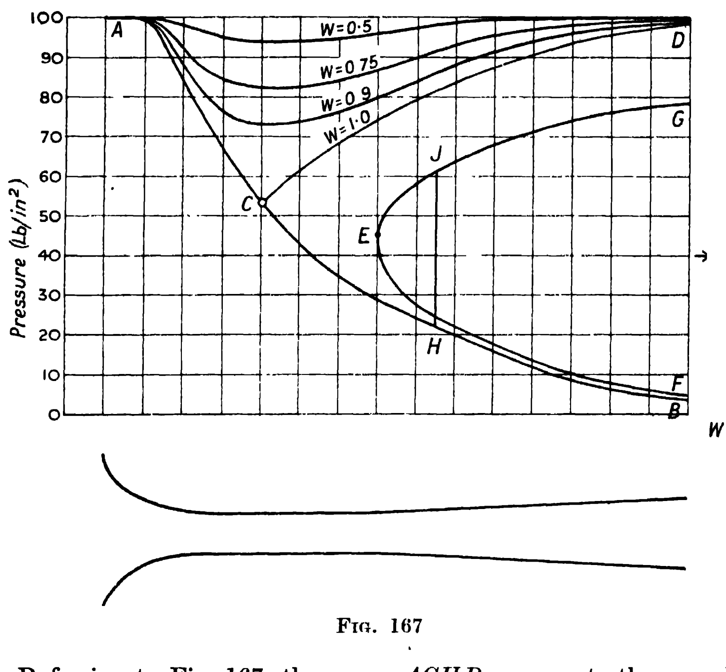


Fig. 167

Referring to Fig. 167, the curve *ACHB* represents the complete free adiabatic expansion through the given nozzle, from an initial pressure of 100 Lb/in.² to the back pressure p_B. This was plotted for unit discharge in pounds per second; then, from eq. (25),

$$\frac{V_s}{v} = a$$

The values of V_s/v for the different cross-sectional areas were thus obtained from the shape of the nozzle; from these values the pressures at the respective sections of the nozzle were then read off from the V_s/v curve of Fig. 166. It should be noticed that each given area occurs twice in the complete length of the nozzle, the same area occurring both in the converging portion and in the

diverging portion. The curve AC (Fig. 167) for the converging portion was obtained from the pressures given by the subsonic curve AC of Fig. 166, as the flow is below sonic to the left of the throat. The curve CB for the diverging length was obtained from the supersonic curve CB of Fig. 166. The curve CD (Fig. 167) was obtained in the same way from the subsonic curve AC of Fig. 166, there being two pressures in the diverging length for each nozzle area (§ 14.8). Thus, if the expansion follows the curve ACB, the flow is supersonic beyond the throat; if it follows the curve ACD, the flow is subsonic beyond the throat. The particular curve actually followed will depend on the magnitude of the back pressure.

The curve $GJEF$ shown in Fig. 167 was obtained in the same way as the curve $DCHB$. To obtain this curve, the set of curves shown in Fig. 166 were again plotted but this time the air was assumed to expand adiabatically from a lower initial pressure of 80 Lb/in.^{2} From the new V_s/v curve thus obtained the curve $GJEF$ of Fig. 167 was plotted in the same manner as curve $DCHB$, the curve GJE representing subsonic flow beyond the throat and curve EF representing supersonic flow beyond the throat. Now, curve EJG will also represent the adiabatic compression of the gas in the diverging cone of the nozzle due to a back pressure of 80 Lb/in.^{2} This is because the diverging cone of the nozzle acts as a diffuser, and the adiabatic compression thus caused is the reverse process, and in the opposite direction, of an adiabatic expansion in the same cone from an initial pressure of the same magnitude as the back pressure at G. Similar curves to $GJEF$ can also be plotted for other assumed back pressures.

The curves above the curve ACD (Fig. 167) were plotted for reduced discharges. Curve ACD represents a discharge W of 1 Lb/sec. The curves above ACD have been plotted for discharges of 0·9, 0·75 and 0·5 Lb/sec respectively, for the same nozzle and for the same initial pressure of 100 Lb/in.^{2} These curves were obtained in the same way as curve ACD. It should be noted that as

$$W = \frac{av}{V_s}$$

then

$$\frac{V_s}{v} = \frac{a}{W} \qquad . \quad . \quad . \quad . \quad (26)$$

Hence, the V_s/v curve of Fig. 166 may be again used if the effect of the reduced discharge is taken into account, as shown in eq. (26). It will be noticed from these curves that the pressure has a tendency to decrease rapidly at the throat and then to gradually increase in the diverging cone.

The ideal type of flow for the particular nozzle shown is when the back pressure has the value p_D; the flow then follows the curve ACD, reaching the sonic velocity at C, and the discharge has its maximum value.

If the back pressure is increased above p_D, the flow follows a curve of the type shown above the curve ACD and the discharge is reduced; in this case the velocity remains subsonic throughout.

If the back pressure is reduced below p_D, there is no further increase in the discharge, the velocity increases beyond the throat and the flow follows the curve CB; the jet may even separate from the nozzle walls if the velocity is very large. The flow is now supersonic and is in an unstable state. If the back pressure is higher than p_B, then at a certain section H, depending on the magnitude of the

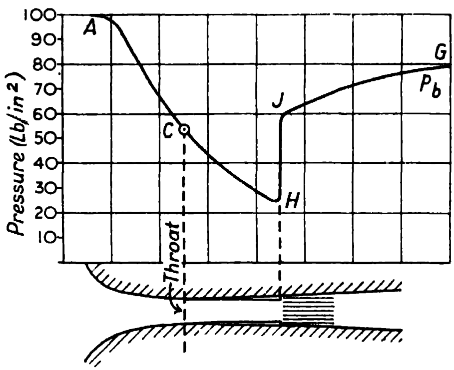


FIG. 168

back pressure, the pressure will suddenly increase to J; the flow then becomes subsonic and, if separation has occurred, the jet enlarges to the nozzle walls, as shown in Fig. 168. The flow now follows the curve JG, the point G representing the final back pressure. If there is no separation of the jet, the pressure jump from H to J takes place at constant cross-sectional area; this pressure jump resembles the hydraulic jump in a water channel (§ 10.11).

It will be seen from the above that, for a given nozzle, no increase in discharge can be obtained by lowering the back pressure below p_D. A graph showing the variation of discharge per second with the back pressure is shown to the right of Fig. 167.

In Fig. 168 is shown the pressure variation along the nozzle for a flow in which the pressure jump occurs. The behaviour of the jet is shown projected under the curve and a separation between the jet and the nozzle wall has been assumed, although this may not have occurred. It will be seen that the jet separation commences at C and continues up to the pressure jump at H. Photographs of this phenomenon in glass nozzles have been obtained by making use of the fact that a change of density affects the refraction of light; this enables the pressure jump to be registered on a photographic plate (§ 17.3).

Oblique shock waves, known as standing waves, also occur in the jet when the flow becomes supersonic; photographs of these are shown in Figs. 233 (*b*) and 233 (*c*), page 454.

14.10. Actual Pressure Variation in a Nozzle. Measurements of the distribution of pressure along the axis of a steam nozzle were

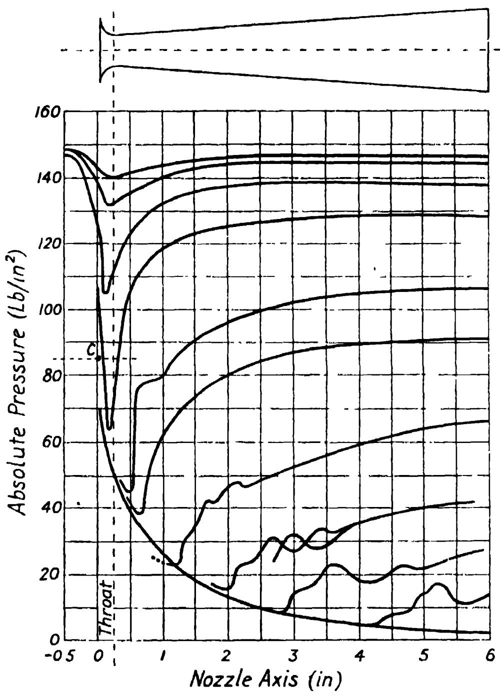


FIG. 169

first made by Stodola*; his results are shown plotted in Fig. 169. These curves, each representing a different back pressure, are plotted on a base representing the centre-line of the nozzle, a sectional view of which is shown at the top of the figure. The initial steam pressure was 150 Lb/in.^{2} in all cases. The critical pressure of the steam occurs at C, which should coincide with the nozzle throat.

It will be noticed that all the curves but the upper three attained supersonic velocities, having expanded below the critical pressure. In all of the lower curves a pressure jump has occurred in the diverging portion of the nozzle, thus demonstrating the explanation given in § 14.9. Other pressure waves, probably standing or shock

* See *Steam and Gas Turbines* by Stodola.

waves, are also noticeable superimposed on the main pressure curve. The shape of the main pressure curves should be compared with the curve of Fig. 168, which was plotted from calculated results obtained by the method described in § 14.8.

The three upper curves have not expanded to the critical pressure at C; hence their discharges will be below the maximum.

14.11. Isothermal Flow in Pipe. When gas flows isothermally along a pipe of uniform bore there is a loss of head due to the frictional resistance. This can be calculated from the equation

$$h_f = \frac{4flv^2}{2gd} \text{ ft of gas}$$

There is a continuous fall in pressure as the gas flows along the pipe, due to the overcoming of the frictional resistance. This fall of

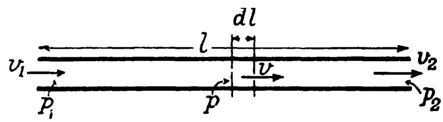


FIG. 170

pressure causes an expansion in volume and a corresponding reduction in density. The weight of gas per second flowing past any section must be constant; hence, an expansion in volume causes an increase in velocity. It will be seen from this that the frictional resistance causes the gas to expand; the temperature thus tends to fall. But as the flow is isothermal, the fall in temperature is prevented by heat being absorbed from the surroundings. The effect of friction also tends to maintain a constant temperature, because the energy lost in friction is converted into heat, most of which reheats the gas.

It follows from this that the pressure drop during an isothermal flow is less than that of an adiabatic flow for the same frictional resistance. Also, it can be shown that the change of velocity along the pipe, due to frictional causes, is small for moderate lengths of piping; hence, the mean velocity can be used in the frictional formula.

Referring to the pipe of Fig. 170, let suffix 1 apply to the inlet condition and suffix 2 to the outlet.

Let Q = amount of heat absorbed by gas in maintaining a constant temperature, in ft-Lb units.

Applying eq. (1) to the entrance and exit ends of the pipe, and including the frictional loss in heat units,

$$\frac{p_1V_1}{J} + \frac{v_1^2}{2gJ} + U_1 = \frac{p_2V_2}{J} + \frac{v_2^2}{2gJ} + U_2 + h_f - Q$$

where h_f = energy lost in friction per pound of gas (heat units).

As the flow is isothermal, $p_1V_1 = p_2V_2$ and $U_1 = U_2$. Hence, the equation becomes

$$\frac{v_1^2}{2g} = \frac{v_2^2}{2g} + h_f - Q \quad . \quad . \quad . \quad (27)$$

The work done against friction is converted to heat, which helps to reheat the gas to its former temperature. This reduces the amount Q which flows in through the pipe walls. If it is assumed that the whole of the frictional resistance is utilized in reheating the gas, the term h_f is not a loss of energy. Then eq. (27) becomes—

$$\frac{v_1^2}{2g} = \frac{v_2^2}{2g} - Q \quad . \quad . \quad . \quad . \quad (28)$$

In this case, Q is not the heat absorbed during a pure isothermal expansion as given by eq. (3) (§ 13.4), the expansion being affected by the frictional resistance. A flow at constant temperature which is frictionally resisted cannot be regarded as a true isothermal expansion, as defined in § 13.4.

For short pipes it is usually of sufficient accuracy to assume a constant density and velocity, and to treat the problem as was done with liquid flow (§ 7.13). An approximate solution* for pressure drop during isothermal flow is given in § 14.12.

14.12. Approximate Solution for Isothermal Flow in Pipes. The following approximate solution is satisfactory for most problems on isothermal flow through pipes. In this approximation the gain of kinetic energy of the gas is neglected as relatively small compared with the energy absorbed by the frictional resistance.

Consider a short length of the pipe dl (Fig. 170) at a point where the velocity is v, the pressure p, the density w, and the specific volume V. Then,

$$\text{weight of gas flowing} = wav = w_1av_1$$

Also, for isothermal flow, $\quad pV = p_1V_1$

or $\quad pv = p_1v_1$

as $v/v_1 = V/V_1$.

Then

$$v^2 = \frac{p_1^2v_1^2}{p^2} \quad . \quad . \quad . \quad . \quad (29)$$

As $p/w = p_1/w_1$, then

$$w = \frac{pw_1}{p_1} \quad . \quad . \quad . \quad . \quad (30)$$

* For exact solution for isothermal flow see § 18.11.

Combining eqs. (29) and (30),

$$wv^2 = \frac{pw_1}{p_1} \times \frac{p_1^2 v_1^2}{p^2}$$

$$= \frac{w_1 p_1 v_1^2}{p} \quad . \qquad (31)$$

From the characteristic equation for a gas,

$$p_1 V_1 = RT$$

Then

$$\frac{p_1}{w_1} = RT$$

or

$$w_1 = \frac{p_1}{RT} \quad . \quad . \quad . \quad . \quad . \qquad (32)$$

As $h_f = 4flv^2/2gd$, then for a short length of pipe dl,

$$\mathrm{d}(h_f) = \frac{\mathrm{d}p}{w} = -\frac{4fv^2\,\mathrm{d}l}{2gd}$$

where dp is the pressure change on length dl and is negative. Hence

$$\mathrm{d}p = -\frac{4fwv^2\,\mathrm{d}l}{2gd}$$

Substituting for wv^2 from eq. (31),

$$p\,\mathrm{d}p = -\frac{4fw_1p_1v_1^2\,\mathrm{d}l}{2gd}$$

Integrating between the two ends of the pipe,

$$\int_{p_1}^{p_2} p\,\mathrm{d}p = -\frac{4fw_1p_1v_1^2}{2gd}\int_0^l \mathrm{d}l$$

that is,

$$\frac{p_2^2 - p_1^2}{2} = -\frac{4fw_1p_1v_1^2 l}{2gd}$$

Substituting for w_1 from eq. (32),

$$\frac{p_2^2 - p_1^2}{2} = -\frac{4fp_1^2v_1^2 l}{2RTgd}$$

from which

$$p_2^2 = p_1^2\left(1 - \frac{8flv_1^2}{2gdRT}\right)$$

Then

$$p_2 = p_1\sqrt{1 - \frac{8flv_1^2}{2gdRT}} \quad . \quad . \quad . \qquad (33)$$

Eq. (33) is accurate for low velocities and small pressure drops only.

EXAMPLE 7

Compressed air is transmitted through 300 ft of 2 in. pipe. The supply pressure is 100 Lb/in.², and the flow is 80 ft³/min at the supply end. Calculate the delivery pressure assuming the temperature remains at 15°C throughout, and that $pV = 96T$ for 1 Lb of air. Prove any formula used. Take $f = 0{\cdot}005$. (*Lond. Univ.*)

$$T = 15 + 273 = 288°\text{C abs.}$$

$$a = \frac{\pi}{4}\left(\frac{1}{6}\right)^2$$

$$= 0{\cdot}0218 \text{ ft}^2$$

$$v_1 = \frac{Q}{a}$$

$$= \frac{80}{60 \times 0{\cdot}0218} = 61{\cdot}15 \text{ ft/sec}$$

Applying eq. (33),

$$p_2 = p_1\sqrt{1 - \frac{8flv_1^2}{2gdRT}}$$

$$= 100\sqrt{1 - \frac{8 \times 0{\cdot}005 \times 300 \times (61{\cdot}15)^2}{64{\cdot}4 \times \frac{1}{6} \times 96 \times 288}} \text{ Lb/in.}^2$$

$$= 100\sqrt{1 - 0{\cdot}151}$$

$$= 92{\cdot}1 \text{ Lb/in.}^2$$

14.13. Approximate Solution for Adiabatic Flow in Pipe. This type of flow occurs when the pipe is efficiently lagged so that no heat flows into the moving gas. As the gas flows along the pipe it is subjected to a frictional resistance causing a pressure drop. This pressure drop causes the gas to expand and thus increases its kinetic energy.

In practice, the energy lost in friction will partly reheat the gas, causing an increase in internal energy at the outlet end of the pipe which is theoretically equal to the loss due to friction.*

1. ADIABATIC FLOW, NEGLECTING REHEATING. Let v = mean velocity in pipe. Then

$$h_f = \frac{4flv^2}{2gd} \text{ (approximately)}$$

Applying eq. (4) to a horizontal pipe and allowing for the frictional loss h_f,

$$\left(\frac{\gamma}{\gamma - 1}\right) p_1V_1\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right] = \frac{v_2^2 - v_1^2}{2g} + h_f$$

* For exact solution for adiabatic flow see § 18.1.

That is

$$\left(\frac{\gamma}{\gamma - 1}\right) p_1 V_1 \left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right] = \frac{v_2^2 - v_1^2}{2g} + \frac{4flv^2}{2gd} \quad . \quad . \quad (34)$$

But from eq. (9), § 13.5,

$$\frac{v_2}{v_1} = \frac{V_2}{V_1} = \left(\frac{p_1}{p_2}\right)^{1/\gamma}$$

Hence

$$v_2 = v_1 \left(\frac{p_1}{p_2}\right)^{1/\gamma} \quad . \quad . \quad . \quad (35)$$

Substituting this value of v_2 in eq. (34),

$$\left(\frac{\gamma}{\gamma - 1}\right) p_1 V_1 \left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right] = \frac{v_1^2}{2g}\left[\left(\frac{p_1}{p_2}\right)^{2/\gamma} - 1\right] + \frac{4flv^2}{2gd} \quad . \quad (36)$$

The value of p_2 can be obtained from this equation by successive approximations. First assume the mean velocity v to be equal to v_1 and solve for p_2 by trial or by plotting. Repeat the process using the mean of v_1 and v_2 for v. No further adjustment will be necessary.

2. Adiabatic Flow, with Reheating. In this case it is assumed that the energy lost due to friction is absorbed by the gas in the form of heat. Hence, the total energy at the outlet end of the pipe is equal to the total energy at the inlet end. Owing to the effect of reheating the expansion is not a reversible adiabatic; Eq. (4) can be applied if the index n is substituted for γ, the law of expansion being of the form pV^n = constant. The index n is known from previous experimental results.

Then, for horizontal pipe,

$$\left(\frac{n}{n - 1}\right) p_1 V_1 \left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}}\right] = \frac{v_2^2 - v_1^2}{2g} \quad . \quad . \quad . \quad . \quad (37)$$

Substituting the value of v_2 from eq. (35),

$$\left(\frac{n}{n - 1}\right) p_1 V_1 \left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}}\right] = \frac{v_1^2}{2g}\left[\left(\frac{p_1}{p_2}\right)^{2/n} - 1\right] \quad . \quad . \quad (38)$$

If n, p_1, V_1, and p_2 are known, the value of v_1 can be calculated from this equation. Then, as $p_1 V_1^n = p_2 V_2^n$, V_2 can be obtained. Also, as $v_2/v_1 = V_2/V_1$, the velocity v_2 can be calculated.

14.14. Measurement of Air Speed. The following instruments are used for measuring the velocity of an air stream such as wind velocity, the speed of air flow in ducts, or the air flow in an experimental wind tunnel.* They may also be used for measuring the

* For a more detailed account of measurement of air speed, see *The Measurement of Air Flow* by E. Ower.

velocity of a body moving in air, such as an aeroplane. In this case, the Pitot-static tube is used and the velocity obtained is the relative velocity of the plane to the air. In order to obtain the absolute velocity of the plane, allowances must be made for the velocity of the wind.

1. Pitot-static Tube. A simple Pitot tube cannot be used for the measurement of air velocity if the air is under static pressure, as the latter would affect the reading of the instrument. In order to overcome this, the Pitot-static tube has been designed so that the measured pressure difference gives the required velocity head of the air.

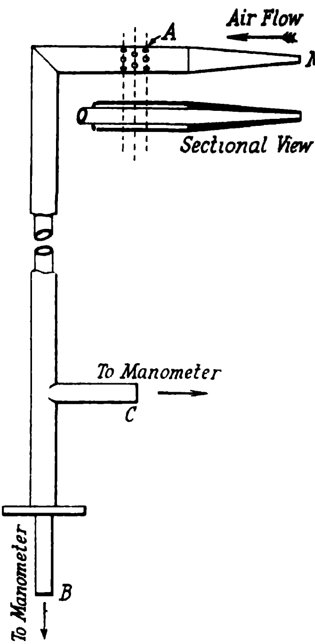


Fig. 171

An outside view and a sectional view of a Pitot-static tube are shown in Fig. 171. It consists of an internal L-shaped tube which forms the mouth of the instrument at M, the other end B being connected to one of the limbs of a suitable manometer. This inner tube is surrounded by an L-shaped outer tube so that an air space is provided between the two tubes; a ring of holes in the outer tube at A admits the air to this air space. The static pressure of the air can thus be transmitted along the air space to its outlet at C, which is connected to the other limb of the manometer. The instrument is placed with its mouth facing the air stream; the head measured is then the velocity head only of the air stream, as its static pressure is transmitted to both limbs of the manometer and is, therefore, eliminated. The instrument must first be calibrated for the range over which it is used in order to obtain the values of its coefficient k. Then,

$$v = k\sqrt{2gh}$$

It is found that good results are obtained with the Pitot-static tube if the centre of the ring of holes at A is about 2 in. away from the mouth. In the N.P.L. Standard Pitot-static Tube the inner tube has internal and external diameter of 0·16 in. and 0·0204 in. respectively; the thickness of the air space between the tubes is 0·032 in.

2. Vane Anemometer. This instrument, which is the same in principle as the current meter described in § 10.8, is used for

measuring wind velocities for meteorological purposes and for measuring air velocities in large ducts such as ventilating shafts. For the latter purpose it usually consists of a rotor containing eight vanes, and is the same in principle as an axial-flow turbine. The vanes may be flat plates set at a suitable angle to the direction of the air stream, or they may be curved in the same manner as a turbine blade. The rotor is connected by gearing to a revolution

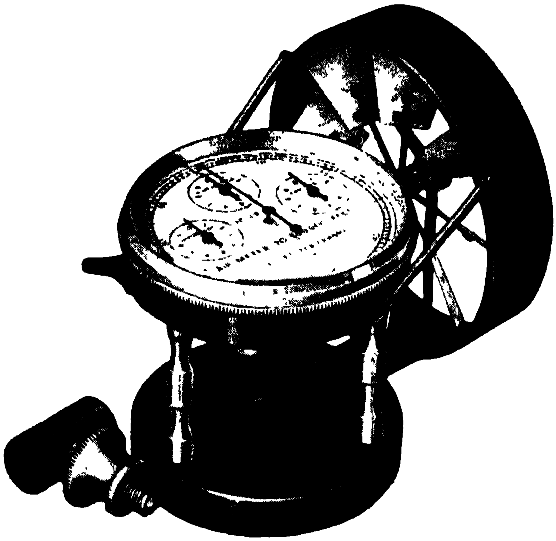


FIG. 172. VANE ANEMOMETER
(*Courtesy of Negretti and Zambra, Ltd.*)

counter, which indicates on the dials incorporated in the instrument the number of revolutions made. As the speed of revolution of the rotor is proportional to the air-stream velocity, the latter can be obtained by noting the number of revolutions made by the rotor over a known interval of time.

To use the instrument, it is placed in the air stream and the initial readings of the dials noted. Then, by using a stop watch, the time is taken for a given number of revolutions of the rotor. From these results the air velocity can be calculated from the calibration curve for the instrument used.

The Negretti and Zambra vane anemometer, shown in Fig. 172, is a portable instrument for indicating the number of linear feet of air. A number of light vanes are mounted on a spindle running on jewelled bearings; by means of a suitable gearing, the rotation of the spindle is communicated to the pointers.

The instrument is held in the air stream, preferably on a rod, and the number of feet of air passing the instrument is timed with the

aid of a stop watch. A correction is usually required, which is obtained from a calibration factor supplied. A disconnector is provided for throwing the indicating mechanism out of mesh, and a setting device for bringing the hands back to zero.

The vane anemometer gives accurate results over a limited range only; for any given range of air speeds an instrument should be employed which suits that particular range.

3. Hot-wire Anemometers. Another method of measuring the velocity of an air stream is by measuring the rate of heat loss from an electrically heated body immersed in the air stream; the rate of heat loss is proportional to the velocity of the air impinging on the hot body. The hot body usually consists of a short length of platinum or nickel wire which is arranged to form one of the arms of a Wheatstone bridge; a manganin resistance forms the opposite arm.

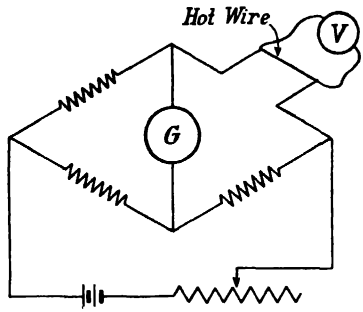


Fig. 173

There are two methods of measuring air velocity with the hot-wire anemometer: one by maintaining a constant temperature in the wire, the other by keeping the electric current constant. In the former method the resistance to the passage of the electric current through the wire remains constant, as the resistance is proportional to the temperature; consequently, the current required to maintain a constant temperature is proportional to the velocity of the air stream. In the latter method the electric current is kept constant, consequently the resistance varies with the temperature of the wire; the resistance is thus proportional to the air speed.

The constant-current method gives the best results, but the constant-resistance method is usually used as it is easier to maintain a constant resistance than a constant current.

(*a*) *Constant-resistance Method.* The arrangement of the Wheatstone bridge is shown in Fig. 173. The hot wire is exposed to the air stream, which tends to cool it; an increase in the electric current is then necessary to restore the temperature of the wire to its former value. By this means the temperature and resistance are maintained constant and the current is proportional to the air speed.

The bridge is kept in balance by varying the current which may be measured by an ammeter in series with the hot wire, or by a high-resistance voltmeter connected across the wire as shown in Fig. 173. Once the instrument is calibrated the air speed can be obtained from the reading of the ammeter or voltmeter.

(*b*) *Constant-current Method.* In this form of the instrument the electric current passing through the hot wire is kept constant. Referring to Fig. 173, the galvanometer G of the constant-resistance method is replaced by a milliammeter; this will register any out-of-balance current passing through the bridge. An increase in air speed tends to cool the hot wire, which, in turn, lowers its resistance to the flow of the electric current passing through it. This puts the bridge out-of-balance, the out-of-balance current being registered by the milliammeter. The out-of-balance current is thus proportional to the air speed, the value of which can be obtained from the calibration curve of the instrument. The constant-current method is used for low air speeds.

14.15. Electric Heater Air-flow Meter. A method of measuring the flow of air through a passage is by supplying the air with a known quantity of heat and measuring the rise in temperature.

A view of an electric heater air-flow meter is shown in Fig. 174; this instrument was designed to measure the air supply to a petrol engine, and had a diameter of 3 in. at the heater.* The air to be measured is passed through an electric grid heater JK which is consuming a known amount of electric current; a view of the heater is shown in the top right-hand portion of the figure. Equally spaced on each side of the heater are placed nickel wire resistance thermometers DB and EC which are made in the form of a grid. A view of a resistance thermometer is shown in the top left-hand portion of Fig. 174. The temperature of the air flowing through a resistance thermometer is measured by passing a known electric current through the nickel wire grid and measuring its resistance by the Wheatstone bridge method. As the resistance is a function of the temperature of the wire, the instrument can be calibrated to give the temperature of the air flowing through it.

The air is passed through the meter from right to left. It first flows through the resistance thermometer EC and its temperature is measured. It next passes through the electric heater JK and has its temperature increased. This increased temperature is measured as it flows through the resistance thermometer DB.

Let W = weight of air flowing per hour,

c_p = specific heat of air at constant pressure,

t = rise in temperature due to heater, degrees F,

C = no. of kilowatts consumed by heater per hour.

$$1 \text{ kW} = 3{,}412 \text{ B.Th.U.}$$

Then, heat gained by air = heat supplied to heater

* For details, see "Air consumption in I.-C. engines," by Dr. H. Moss, *Proc. Instn. Mech. Engrs.*, **1** (1924) p. 345.

that is, $$W \times c_p \times t = 3{,}412 \times C$$

from which $$W = \frac{3{,}412 \times C}{c_p \times t} \text{ Lb per hour}$$

The heater method was first used by Callendar for measuring the specific heat of gases. It was later developed by Professor Thomas

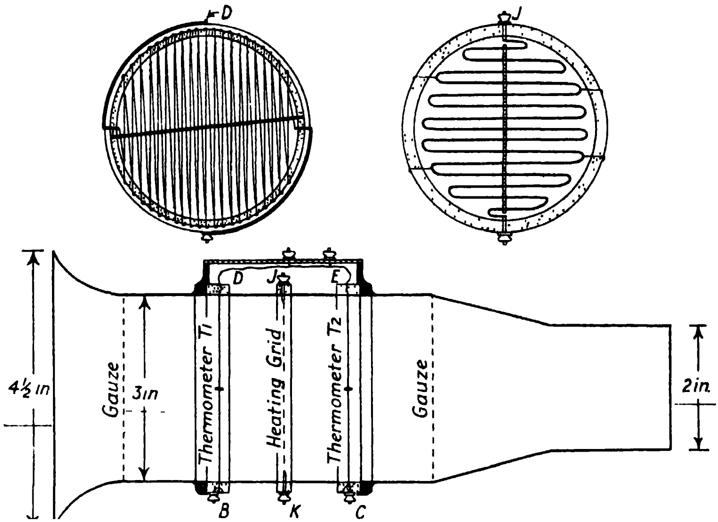


FIG. 174

as an air-flow meter for measuring the air supply to the furnaces of steam boilers; this meter had a diameter of 5 ft.

The electric heater air-flow meter does not cause any backwards and forwards surging of the air, which often occurs in the orifice method of measurement when applied to I.-C. engines.

EXERCISES 14

1. Air at 30 Lb/in.^{2} and 60°F is expanded adiabatically through a nozzle to a pressure of 14·7 Lb/in.^{2}; 10 per cent of the heat drop is lost in friction, the energy lost being utilized in reheating the air. Using the heat-entropy chart, find the net heat drop, the final temperature and velocity.

Ans. H_d = 18·36 B.Th.U.; − 86°F; 960 ft/sec.

2. Air at 14·7 Lb/in.^{2} and 20°C is compressed adiabatically in a compressor, the efficiency of the compression being 75 per cent. If all the losses are assumed to reheat the air and 10,860 ft-Lb of work per pound of air are required to perform the compression, find the final pressure and temperature of the air. Solve by means of the heat-entropy chart and sketch the operation on the chart.

Ans. p_2 = 19·2 Lb/in.^{2}

3. Calculate the weight of air flowing through a horizontal Venturi meter having an inlet diameter of 4 in. and a throat diameter of 2 in. The absolute pressures at inlet and throat were found to be 60 Lb/in.^{2} and 50 Lb/in.^{2} respectively; the temperature at inlet was 20°C. Assume $R = 96$ ft-Lb Centigrade units and $\gamma = 1{\cdot}4$. *Ans.* 3·41 Lb/sec.

4. Air from a large vessel discharges into the atmosphere from a small orifice placed in its side. The pressure and temperature of the air in the vessel are 30 Lb/in.^{2} abs. and 15°C respectively. The diameter of the orifice is 1 in. Assuming R and γ for air to be 96 ft-Lb Centigrade units and 1·4 respectively, calculate the weight of air discharging per second. The atmospheric pressure is 15 Lb/in.^{2} and C_d for the orifice = 0·64. *Ans.* 0·351 Lb/sec.

5. Describe the Pitot tube method of velocity measurement. Comment on its accuracy and working range. Describe a manometer suitable for use with a Pitot tube measuring air speeds of the order of 40 ft/sec.

A Pitot tube gives a pressure difference of 8 in. of water when placed in an air stream at 750 mm barometer and 18°C temperature. What is the speed?

$$\rho = 0{\cdot}0807 \text{ Lb/ft}^3 \text{ at N.T.P.}$$

(*Lond. Univ.*) *Ans.* 189·5 ft/sec.

6. A Venturi meter having an inlet diameter of 3 in. and a throat diameter of 1 in. is used for measuring the rate of flow of air through a pipe. Mercury U-gauges register gauge pressures at the inlet and throat equivalent to 250 mm and 150 mm of mercury respectively.

Determine the volume of air flowing through the pipe in cubic feet per second. Assume that flow takes place between the inlet and throat under adiabatic conditions ($\gamma = 1{\cdot}4$) and that the density of the air at inlet is 0·10 Lb/ft^3. The barometric pressure is 760 mm. (*Lond. Univ.*)

Ans. 2·2 ft^3/sec.

7. A Venturi meter, whose inlet and throat diameters are 12 in. and 4 in. respectively, is employed for measuring the flow of air.

Calculate the flow in cubic feet per minute at N.T.P. given the following data: the difference of pressure between the entrance to the meter and the throat is 0·6 in. of water; the pressure in the pipe at the entrance to the meter, as registered by a water manometer, is 5 in., the temperature is 20°C, and the barometric height is 29·83 in. of mercury. The coefficient of discharge for the meter is 0·96. Neglect compressibility and take $pV = 96T$ for air. (*Lond. Univ.*) *Ans.* 245 ft^3/min.

8. Find the diameter of a sharp-edged orifice suitable for measuring the discharge from an air compressor which deals with 50 ft^3/min of "free" air at 14·7 Lb/in.^{2} and 15°C.

The orifice is to be fitted to the top of a large cylindrical vessel into which air from the compressor passes and is then discharged into the atmosphere through the orifice. The pressure of air inside the vessel is 1 in. of water and the temperature is 20°C. Assume that the density of the air is constant through the orifice, and that $pV = 96T$ for air, and take $C_d = 0{\cdot}6$ for the orifice.

Sketch a manometer suitable for the measurement of the pressure inside the vessel. (*Lond. Univ.*) *Ans.* 2 in.

9. Sketch an arrangement of a Pitot-static tube combined with a suitable manometer for the measurement of current velocity of an air stream. Give the theory of the Pitot tube and comment on the accuracy to be expected from an instrument of this kind.

If the manometer registers 0·2 in. of water when the Pitot-static tube is placed in a current of air at a temperature of 20°C and pressure of 750 mm Hg, find the velocity of the air. Take $pV = 96T$. (*Lond. Univ.*)

Ans. 30·0 ft/sec.

10. State the energy equation for the frictionless adiabatic flow of a gas, and apply it to calculate the theoretical flow in pounds per hour of hydrogen gas through a horizontal Venturi meter given the following information: diameter of meter at inlet, 3 in., and at throat, 1 in.; the pressure is 800 mm of mercury and the temperature is 15°C at inlet, and the pressure is 765 mm of mercury at the throat. For hydrogen $pV^{1\cdot 4}$ = constant for adiabatic expansion, and $R = 1{,}385$ (ft-Lb)/(lb-°C). (*Lond. Univ.*) *Ans.* 110 Lb/hr.

11. Prove that the maximum continuous discharge of air through a convergent nozzle, fitted into the side of a large vessel, takes place when the pressure in the throat of the nozzle is 0·528 of the constant pressure of the air in the vessel.

Find the diameter of a nozzle suitable for measuring the discharge from an air compressor which deals with 250 ft³/min of atmospheric air at 14·7 Lb/in.² and 15°C. The nozzle is fitted into the side of a large vessel into which air is discharged from the compressor, and the pressure and temperature in the vessel are 33 Lb/in.² and 27°C. Assume a coefficient of discharge for the convergent nozzle of 0·99, that for 1 Lb of air $pV = 96T$, and that $\gamma = 1\cdot 4$. (*Lond. Univ.*) *Ans.* $d = 0\cdot 731$ in.

12. Prove that $\dfrac{v^2}{2g} + \displaystyle\int \frac{dp}{\rho}$ = constant for the steady flow of a compressible fluid. Hence, show that the energy equation for the flow of a gas can be written

$$\frac{v^2}{2g} + \frac{\gamma}{\gamma - 1} \cdot \frac{p}{\rho} = \text{constant}$$

in which the pressure and density are related by the adiabatic law (p/ρ^γ) = constant. Hence derive an expression for the theoretical flow of a gas in pounds per second through a Venturi meter in terms of the pressure and density of the gas, and the sectional area of the meter, at the entrance and at the throat sections. (*Lond. Univ.*)

13. The following results were obtained from a test on a Venturi meter—

k	0·90	0·92	0·94	0·96	0·98	0·99
R_e	1,500	2,800	6,000	19,000	160,000	3,000,000

where k is the coefficient of the meter and R_e the Reynolds number at the throat. Obtain the calibration curve of the meter by plotting k on a base representing log R_e.

In order to measure the quantity of compressed air flowing through a 2-in. diameter pipe, a Venturi meter, having a throat diameter of 1½ in., was fitted into the pipe. The difference of pressure head between the inlet and the throat was found to be 0·9 in. of water. The air in the pipe had a pressure of 60 Lb/in.² and a temperature of 65°F. Calculate the weight of air flowing through the pipe per minute, neglecting any change in density as small. $pV = 53\cdot 3T$ in Fahrenheit units for air. Obtain the correct value of the coefficient k from the calibration curve by successive approximations; the coefficient of viscosity, $\eta = 0\cdot 375 \times 10^{-6}$ engineers' units for air at the given temperature. (*Lond. Univ.*) *Ans.* $W = 15\cdot 37$ Lb/min.

14. Air flows adiabatically through a lagged Venturi meter. The pressure and temperature at inlet were found to be 30 Lb/in.^{2} and 500°F abs.; the pressure at the throat was 20 Lb/in.^{2} The diameters of the meter at inlet and throat were 2 in. and 1 in. respectively. Calculate the weight of flow through the meter in pounds per second. Neglect all losses.

$R = 53{\cdot}3$ ft-Lb in Fahrenheit units. $\gamma = 1{\cdot}4$ for air. (*Lond. Univ.*)

Ans. $W = 0{\cdot}68$ Lb/sec.

15. Compressed air flows isothermally through a horizontal pipe from a chamber, which is maintained at a constant pressure of 36 Lb/in.^{2}, into another chamber in which the pressure is maintained constant at 35·2 Lb/in.^{2} The temperature of the air is 60°F. The pipe is 2 in. in diameter and 40 ft long, and for turbulent flow Darcy's $f = \dfrac{0{\cdot}064}{R_e^{\,0{\cdot}23}}$

The coefficient of viscosity of air at 60°F $= 0{\cdot}37 \times 10^{-6}$ (engineers' units) and $pV = 53{\cdot}3T$.

Calculate the weight of air flowing per second taking all losses into account and using the mean density. Solve by a method of successive approximations, assuming $v = 60$ ft/sec in the first attempt. The results given by the second attempt are of sufficient accuracy. (*Lond. Univ.*)

Ans. $W = 0{\cdot}286$ Lb/sec.

16. A forced-draught fan takes hot flue gas of density 0·047 Lb/ft^3 from a boiler flue and delivers it to a second flue 15 ft above the first. A draught gauge 10 ft above the first flue and connected to it by a length of cold pipe reads 1·8 in. of water suction. Another gauge directly mounted upon the higher flue gives the suction there as 0·5 in. of water. Find the work done by the fan per pound of flue gas. Neglect any change of speed. (*I.Mech.E.*)

Ans. 148·8 ft-Lb.

CHAPTER 15

THE AEROFOIL AND ITS APPLICATION

15.1. Introduction. The developments in aeronautics have provided some new principles for use in mechanical engineering, the most important being the mechanics of the aerofoil. The aerofoil is now used in a meter for measurement of quantity of flow of a liquid (§ 15.11). It has also provided a new conception in the design of blading for turbines, centrifugal pumps and air compressors.

The conception of aerofoil blading, as distinct from momentum blading, has provided a new method of solution for turbine blading. In the types of blading in which the passage between the blading is wide compared with the length of blade, the aerofoil solution gives the more accurate results. In fact, if the blades are relatively very far apart, no solution can be obtained by the momentum theory. The aerofoil theory also explains why some types of turbine and compressor blading fail to act or become inefficient at high speeds.

It is known that, towards the end of the 19th century, a large multi-staged axial-flow air compressor, with blading designed on the momentum theory, failed to act when completed. Many years later, after the aerofoil theory had been developed, the aerofoil principles were applied to the blading of this particular compressor. It was found that the blades had been so designed that they were acting as stalled aerofoils.

15.2. Fluid Flow past an Inclined Plate. In Fig. 175 is shown a flat plate immersed in a fluid stream of velocity V relative to the plate, which is inclined at an angle α to the direction of flow; the fluid may be a liquid or a gas. Some of the fluid streams strike the underside of the plate and are deviated in a direction parallel to the surface by the pressure of the fluid stream beneath them, thus causing a tangential frictional drag on the plate. The impact of the fluid causes a normal pressure P on the plate which is proportional to its surface area and to the kinetic energy of the stream.

Hence $$P \propto \frac{\rho V^2}{2} \times \text{area of surface} \quad . \quad . \quad . \quad (1)$$

Now consider the upper surface of the plate. Local fluid streams will be deflected away from the upper surface of the plate by the action of its front, or leading edge. These are then forced downwards by the pressure of the fluid streams above, thus tending to form a vortex, as shown in the figure. This causes a negative, or vacuum, pressure on the upper surface. The resultant pressure P

will, therefore, be partly due to the positive pressure on the lower surface of the plate, and partly due to the effect of the negative pressure on the upper surface.

It will be noticed that the force P will have a component perpendicular to the stream of $P \cos \alpha$; this is sometimes known as the *lift*. There will also be a component in the direction of the stream

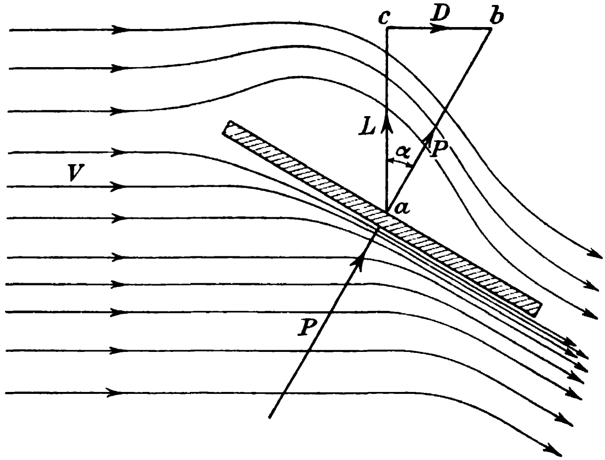


FIG. 175. FORCES ON FLAT PLATE IN A FLUID STREAM

of $P \sin \alpha$, causing a resistance to motion. The total resistance of the plate in the direction of the stream is known as the *drag*.

From § 11.2, eq. (1) may be written

$$L = k_L A \rho V^2 \quad . \quad . \quad . \quad . \quad (2)$$

where L = force on plate normal to direction of stream,
A = area of under surface of plate,
ρ = density of fluid (absolute units)
$= w/g$,
k_L = a coefficient to be determined experimentally for a flat plate inclined at an angle α.

The coefficient k_L is known as the *lift coefficient*; it varies with the type of plate and with the inclination α.

The resistance, or drag, of the plate in a direction parallel to the stream may be expressed in a similar type of equation—

$$D = k_D A \rho V^2 \quad . \quad . \quad . \quad . \quad (3)$$

where D = the drag or resistance of plate in the direction of motion.

The coefficient k_D is known as the *drag coefficient*; it varies with the type of plate and the angle of inclination and is determined experimentally.

The point on the surface through which the resultant force P acts is known as the *centre of pressure.*

Eqs. (2) and (3) are also used in the form

$$L = C_L \frac{A\rho V^2}{2}$$

and

$$D = C_D \frac{A\rho V^2}{2}$$

Hence, the values of C_L and C_D are twice those of k_L and k_D. If the drag is due to friction only, C_D is sometimes written C_f.

This problem of the fluid pressure on an inclined surface or plate frequently occurs in practice. The flat plate is the same in principle as an aeroplane wing, in which case the force L is the lift of the wing and D is its drag or resistance; a large part of the horse-power of the aeroplane is absorbed in overcoming this drag of the wings.

The propelling force on the main sail of a yacht, when tacking, is another example of this problem, the sail corresponding to the flat plate. Other examples of this problem are found in the flying of a kite, the turning force on the rudders of ships and aeroplanes, the dynamic lift on an airship or hydroplane, and the driving force of propeller and fan blades.

EXAMPLE 1

A flat plate 4 ft² in area is immersed in a fluid stream and inclined to the direction of motion. Find the force on the plate in a direction normal to that of the stream and also the resistance of the plate in a direction parallel to the stream—

(1) if the fluid is air;
(2) if the fluid is water.

The velocity of the fluid stream is 20 ft/sec, $k_L = 0{\cdot}2$, $k_D = 0{\cdot}05$, weight of 1 ft³ of air $= 0{\cdot}081$ Lb, weight of 1 ft³ of water $= 62{\cdot}4$ Lb.

(1) For air—

Using eq. (2),
$$\begin{aligned} L &= k_L A\rho V^2 \\ &= 0{\cdot}2 \times 4 \times \frac{0{\cdot}081}{32{\cdot}2} \times 20^2 \\ &= 0{\cdot}805 \text{ Lb} \end{aligned}$$

Using eq. (3),
$$\begin{aligned} D &= k_D A\rho V^2 \\ &= 0{\cdot}05 \times 4 \times \frac{0{\cdot}081}{32{\cdot}2} \times 20^2 \\ &= 0{\cdot}201 \text{ Lb} \end{aligned}$$

(2) For water—

$$\begin{aligned} L &= k_L A\rho V^2 \\ &= 0{\cdot}2 \times 4 \times \frac{62{\cdot}4}{32{\cdot}2} \times 20^2 \\ &= 620 \text{ Lb} \end{aligned}$$

$$D = k_D A \rho V^2$$
$$= 0{\cdot}05 \times 4 \times \frac{62{\cdot}4}{32{\cdot}2} \times 20^2$$
$$= 155 \text{ Lb}$$

15.3. The Pressure Distribution around Aerofoil. It was shown in § 15.2 that a fluid stream acting on an inclined flat plate causes a force L on the plate in a direction normal to the fluid stream, and

FIG. 176. CROSS-SECTION OF AN AEROFOIL

a drag D on the plate in a direction parallel to the stream. The force L can be increased by substituting for the flat plate a plane having a cross-section of the type shown in Fig. 176; such a section is known as an aerofoil.

Aerofoil sections vary in shape according to the work required of them.* Although the function of an aerofoil is the same as that of a flat plate, it is more efficient in its action, because it produces a larger force L and a smaller drag D when acting under similar conditions. Most of the force L is due to the negative pressure on the upper face.

The edge A of the aerofoil (Fig. 176) is known as the *leading edge*; the edge B as the *trailing edge*. The dotted line AB is known as the *chord*; this is the projected length of the aerofoil. The inclination α of the chord AB to the direction of motion of the aerofoil, or of the fluid stream, is known as the *angle of incidence* or *angle of attack*; by varying this angle the values of L and D are altered.

The distribution of the intensity of pressure around the surface of a particular aerofoil is shown in Fig. 177; these pressures were obtained† by measurements during a test on an aeroplane wing of this section whilst in flight. It will be noticed that there is a large negative pressure on the upper surface, which accounts for most of the lift.

Let c = length of chord AB in feet,

l = longitudinal length, or span, of aerofoil in feet (measured in horizontal plane perpendicular to AB),

* For more advanced work on the aerofoil see *Aerofoil and Airscrew Theory*, by H. Glauert, M.A. (The University Press, Cambridge). Also, see Bairstow's *Applied Aerodynamics* and Piercy's *Applied Aerodynamics*. For elementary work on this subject see *Elementary Applied Aerodynamics*, by Whitlock.

† These results are due to Stüper.

V = relative velocity of fluid stream to aerofoil in feet per second,

A = projected area of aerofoil in square feet
$= c \times l.$

The position of the centre of pressure of the aerofoil is also of importance in design. This is the point of application of the resultant force P (Fig. 175) and it varies with the angle of incidence.

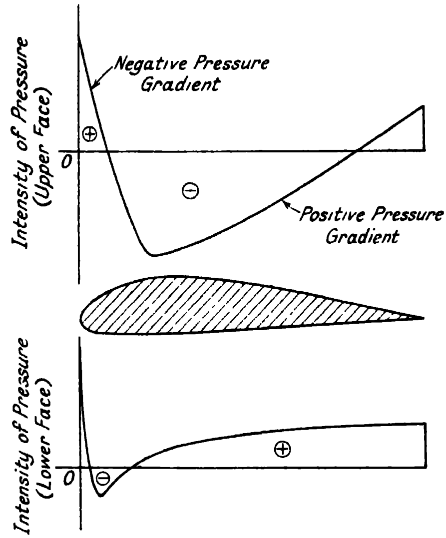


FIG. 177

The equations of § 15.2, which were derived for a flat plate, can be applied in the same way to an aerofoil. Then, force on aerofoil normal to direction of fluid stream

$$= L = k_L A \rho V^2$$

and resistance, or drag, of aerofoil

$$= D = k_D A \rho V^2$$

Horse-power expended in overcoming resistance

$$= \frac{D \times V}{550} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The total resistance and lift of the wings of an aeroplane can be calculated from these equations; in which case l is the length of both wings. If D_1 is the drag of the fuselage and the remaining parts of the structure other than the wings, then, total horse-power required to overcome drag

$$= \frac{(D + D_1)V}{550}$$

15.4. Characteristics of the Aerofoil. The chief characteristics, or properties, of a particular aerofoil are the values of its constants k_L, k_D, and the position of its centre of pressure C_p. These vary with the angle of incidence, and are determined from wind-tunnel tests carried out on an aerofoil of the shape considered, the tests being repeated for varying values of α.

The values of these characteristics for a well-known aerofoil section (R.A.F. 31) are shown plotted in Fig. 178 on a base representing the angle of incidence α. The ratio of lift to drag, or L/D ratio, is also shown plotted in this figure. The position of the centre of pressure is given as a fraction of the chord from the leading edge.

It will be noticed from these curves that the drag coefficient of this aerofoil is a minimum when α has a value of $-6°$. The maximum value of the lift-drag ratio occurs when $\alpha = -1°$; this is the most efficient angle for the aerofoil. It will be noticed from the centre of pressure curve that the position of C_p moves towards the leading edge as the angle of incidence increases.

Similar sets of curves are determined for the hundreds of different aerofoil sections used in practice; these are published in the various aeronautical handbooks,* and in the official aeronautical publications of most countries.

Although the curves of Fig. 178 were produced by test in an air stream for the purpose of providing data for the design of aeroplane wings, similar results would be obtained for an aerofoil immersed in a stream of other fluids, including water.

The aerofoil is also used to form a cross-section of propeller blades; the thrust of the blade corresponding to the lift of the aerofoil.

EXAMPLE 2

An aeroplane wing consists of an aerofoil section of the type given in Fig. 178. It has a length of 20 ft, a chord of 4 ft and is driven at a speed of 150 m.p.h. Calculate the lift, drag and horse-power required for this wing, when the angle of incidence is 4°. Find also the position of the centre of pressure at this angle. Weight of 1 ft³ of air = 0·08 Lb.

Using the curves of Fig. 178 and reading the values when $\alpha = 4°$,

$$k_L = 0{\cdot}375$$
$$k_D = 0{\cdot}023$$
$$C_p = 0{\cdot}3 \text{ of chord}$$

Now
$$A = c \times l$$
$$= 4 \times 20 = 80 \text{ ft}^2$$
$$V = 150 \times \frac{88}{60} = 220 \text{ ft/sec}$$

* See *Handbook of Aeronautics* (Pitman), *R. & M. Reports* (Air Ministry), *N.A.C.A. Reports* (U.S.A.).

Using the equations of § 15.3,

$$L = k_L A \rho V^2$$

$$= 0{\cdot}375 \times 80 \times \frac{0{\cdot}08}{32{\cdot}2} \times 220^2$$

$$= 3{,}600 \text{ Lb}$$

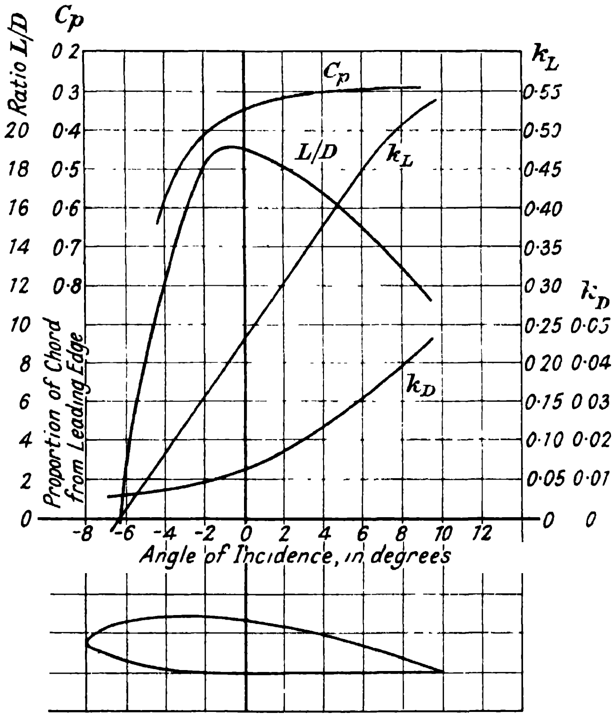


FIG. 178

$$D = k_D A \rho V^2$$

$$= 0{\cdot}023 \times 80 \times \frac{0{\cdot}08}{32{\cdot}2} \times 220^2$$

$$= 221 \text{ Lb}$$

$$\text{h.p.} = \frac{221 \times 220}{550} = 88{\cdot}5$$

$$\text{Position of } C_p = 0{\cdot}3 \times 4$$

$$= 1{\cdot}2 \text{ ft from leading edge.}$$

15.5. Aerofoil Blading for Turbines. In Chapter 6 the problem of the force on turbine blading was solved from the consideration of the change of momentum of the water stream when passing over the blades. Another method of solution is obtained by considering the turbine blades acting as aerofoils; the force on the blades and the blade resistance can then be calculated from the equations of § 15.3. In order to apply this method, the characteristic curves for the blades would first have to be obtained from tests.

When based on this theory turbine blades are made of a suitable aerofoil section, similar to those used for aeroplane wings, instead of the concave circular sections at present in use. Such blading is known as aerofoil blading.

In certain types of turbine and compressor aerofoil blading is more efficient than momentum blading because of the wider passage between the blades, the broader passage producing less frictional resistance.

The aerofoil will give its maximum lift only if it is clear of any near objects which may interfere with the passing of the fluid stream in its vicinity. Hence, the aerofoil blading of a turbine must be so spaced that there will be no interference of the fluid stream between any two adjacent blades. On the other hand, if the blades are spaced too far apart, some of the fluid stream will flow freely between them without doing any work. Energy will thus be wasted. The exact spacing of the blades to satisfy both of these conditions can be obtained only from tests.

Another factor to consider in the design of aerofoil blading is the effect of the vacuum pressure on the upper face of the aerofoil as shown in Fig. 177. If this negative pressure becomes too large, cavitation will occur in water turbines, thus interfering with the flow of the fluid stream. This will reduce the force on the blade and, consequently, reduce its efficiency.

15.6. Minimum Spacing of Aerofoil Blading. In order to investigate the minimum spacing of aerofoil blading in turbines, the author tested two model aerofoil sections in a fluid stream by means of the Hele-Shaw apparatus. This apparatus consists of a film of water flowing between two glass plates. At the inlet end of the plates thin streams of coloured liquid are injected into the water stream across the whole width of the film. If there is no obstruction the coloured streams of liquid will flow in straight parallel bands. By placing an object in the fluid stream between the glass plates, the deviation of the streamlines may be observed from the contours of the colour bands.

The two model aerofoil sections were placed between the glass plates in a parallel position, both having the same angle of incidence. Then, by observing the deviation of the coloured streambands, it was possible to see if interference was taking place.

A photograph of the streambands is shown in Fig. 179. In the position shown, the spacing of the blades has a pitch/chord ratio of 0·67. It will be noticed from the photograph that a slight deviation of the streambands occurs at this ratio; this will cause an interference with the lift and drag when compared with an isolated

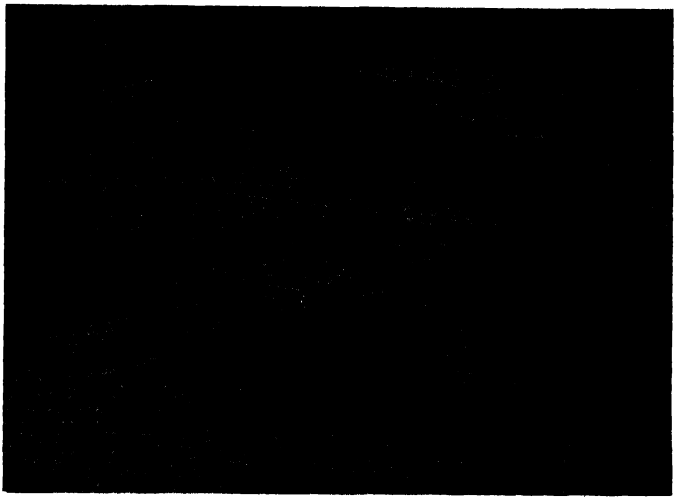


FIG. 179. AEROFOIL BLADING FOR TURBINES, SHOWING INTERFERENCE

aerofoil, and must be taken into account in the values of the lift and drag coefficients used.

The criterion for the type of blading of a turbine or compressor is specified by the pitch/chord ratio of the blades. If the blades are relatively close together, the passage of the fluid is in the form of a curved jet and the force on the blade is best found from consideration of the rate of change of momentum of the jet. Such blading is known as "momentum blading" and should be designed by the momentum method of § 6.5.

If the pitch/chord ratio of the blades is large, the passage of the fluid between them cannot be regarded, with accuracy, as a curved jet. The blades now tend to act as aerofoils in a fluid stream, and the work done on them can be obtained from the application of the aerofoil theory of § 15.3. This type of blading is known as *aerofoil blading.*

15.7. Work Done on Aerofoil Blading. Consider the aerofoil blade of an axial-flow reaction water turbine shown in Fig. 180. The velocity diagram at the blade inlet is shown in the figure. The water enters from the guide blade with an absolute velocity V and

at an angle μ. The aerofoil blade is moving with a velocity v, as shown; the difference between these vectors gives the relative velocity V_r and its direction θ. As this is the direction of the fluid stream relative to the blade, it follows that the aerofoil chord should be sloped at an angle of $\theta - \alpha$ to the direction of the blade's motion, where α is the best angle of incidence for the aerofoil section used.

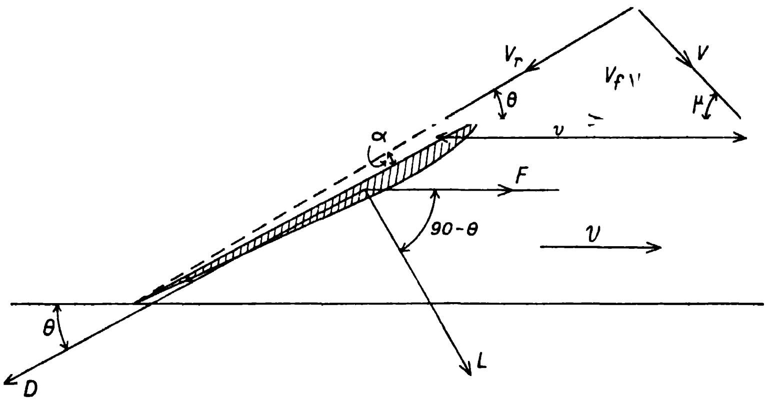


FIG. 180

Let c = length of chord of aerofoil blade, in feet,
S = length of span of aerofoil blade, in feet
= length of blade.

Then $$A = c \times S \text{ ft}^2$$

Let r = mean radius of blade circle, in feet,
N = number of blades on runner,
H = total head of water supplied,
Q = quantity of water supplied, in cubic feet per second,
v = mean blade velocity
$= \omega r$,
C_L and C_D = lift and drag coefficients at angle α.

Then $$L = \frac{C_L \rho A V_r^2}{2}$$

and $$D = \frac{C_D \rho A V_r^2}{2}$$

Let F = tangential force on one blade.

$$F = L \sin\theta - D \cos\theta$$

$$= \frac{\rho A V_r^2}{2}(C_L \sin\theta - C_D \cos\theta) \quad . \quad (5)$$

Work done on wheel per second

$$= F \times v \times N$$

$$\text{Horse-power developed} = \frac{FvN}{550} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Quantity of water used per second

$$= Q = \text{net axial area of flow} \times V_f$$
$$= k2\pi rSV_f \quad . \quad . \quad . \quad . \quad (7)$$

where k = blade area coefficient
and V_f = velocity of flow (§ 6.4).

Theoretical efficiency of wheel

$$= \frac{\text{work done per second}}{\text{energy supplied per second}}$$

$$= \frac{F \times v \times N}{wQH} \quad . \quad . \quad . \quad (8)$$

In designing the blading the values of Q and H are known. The wheel radii, speed and angle θ are chosen from previous experience. Then, from eq. (7),

$$V_f = \frac{Q}{k2\pi rS}$$

The velocity diagram of Fig. 180 can now be drawn. From this is obtained the value of V_r and the guide blade angle μ. The best values of α, C_L and C_D are obtained from wind-tunnel tests on the type of aerofoil used and should be corrected for the Mach number of the flow (§ 17.6–§ 17.8).

As the water passes over the blade its pressure is falling; this tends to increase the value of V_r, but this increase is mainly offset by the frictional resistance to the flow. The angle θ at discharge is less than that at inlet because the fluid is deviated slightly in its passage over the blades. The amount of this deviation can be obtained only from tests. It is usually arranged that the final discharge of the fluid is as near axial as possible.*

If the length of the blade is considerable when compared with the wheel radius, v cannot be regarded as constant over the whole length; hence, the shape of the velocity diagram of Fig. 180 will vary for different radii. In this case the solution is obtained by dividing the blade length into a number of short sections and applying the above solution to each section in turn. The total horse-power developed by the wheel will then be the sum of the horse-powers obtained for each section.

* For an application of this method of solution to an axial-flow air compressor, see § 15.9.

It will be noticed that as the wheel radius decreases the blade velocity at the section considered gets less, as $v = \omega r$. The effect of this on the velocity diagram is to increase the angle θ. This means that the blade is given a twist to a steeper angle as it approaches the wheel centre, the angle θ being greatest nearest to the centre.

EXAMPLE 3

A reaction water turbine of axial-flow type is fitted with four aerofoil blades and has a speed of 120 r.p.m. The mean radius of the blade circle is 5 ft and the blade length, in a radial direction, is 2 ft. The chord of the aerofoil blade is inclined at 25° to the direction of motion, and the chord length is 8·2 ft. The values of C_L and C_D for the angle of incidence used are 0·7 and 0·04 respectively. The turbine is supplied with water under a head of 25 ft. Neglecting the area occupied by the blade thickness and assuming a velocity of flow of 15 ft/sec, calculate the horse-power developed and the theoretical efficiency of the turbine.

$$v = \omega r$$
$$= 2\pi \times \frac{120}{60} \times 5 = 62{\cdot}8 \text{ ft/sec}$$

The velocity diagram is similar to that of Fig. 180; this diagram can now be drawn to scale.

From velocity diagram,

$$V_r = 36 \text{ ft/sec}$$
$$\text{Chord area} = A = 8{\cdot}2 \times 2 = 16{\cdot}4 \text{ ft}^2$$

Applying eq. (5),
$$F = \frac{\rho A V_r^2}{2}[(0{\cdot}7 \times \sin 25°) - (0{\cdot}04 \times \cos 25°)]$$
$$= \frac{62{\cdot}4 \times 16{\cdot}4 \times 36^2}{2 \times 32{\cdot}2}(0{\cdot}2958 - 0{\cdot}0362)$$
$$= 5{,}340 \text{ Lb}$$
$$\text{Horse-power} = \frac{F \times v \times N}{550}$$
$$= \frac{5{,}340 \times 62{\cdot}8 \times 4}{550}$$
$$= 2{,}446$$
$$\text{Weight of water used} = W = 62{\cdot}4 \times 2\pi r \times S \times V_f$$
$$= 62{\cdot}4 \times 2\pi 5 \times 2 \times 15$$
$$= 58{,}800 \text{ Lb/sec}$$
$$\text{Efficiency} = \frac{F \times v \times N}{WH}$$
$$= \frac{5{,}340 \times 62{\cdot}8 \times 4 \times 100}{58{,}800 \times 25}$$
$$= 91{\cdot}6 \text{ per cent}$$

EXAMPLE 4

Calculate the horse-power and efficiency of an axial-flow reaction turbine, fitted with aerofoil blading, from the following data—

Head = 40 ft of water	C_D = 0·025
Outer blade dia. = 20 ft	Chord angle = θ = 30° to direction of motion
Inner blade dia. = 10 ft	Quantity of water used = 6,000 ft³/sec
Speed = 75 r.p.m.	No. of blades = 4
Chord = 13·5 ft	Blade area coefficient = 0·95

Assume the blade angle θ is varied with the radius so that the tangential force on the blade is contant over its length.

First draw the velocity diagram for the outer periphery of the blade; this is similar to that shown in Fig. 180.

$$v = 2\pi r \times \text{r.p.m.}$$
$$= 2\pi \times 10 \times \frac{75}{60} = 78{\cdot}6 \text{ ft/sec}$$
$$V_f = \frac{Q}{\pi(r_1^2 - r_2^2)0{\cdot}95}$$
$$= \frac{6{,}000}{\pi(10^2 - 5^2)0{\cdot}95} = 26{\cdot}8 \text{ ft/sec}$$

From velocity diagram,

$$V_r = \frac{26{\cdot}8}{\sin 30^\circ} = 53{\cdot}6 \text{ ft/sec}$$

Consider a 1 ft length of blade at outer periphery.

$$A = c \times 1 = 13{\cdot}5 \text{ ft}^2$$

Using eq. (5),

$$F = \frac{wAV_r^2}{2g}(C_L \sin 30^\circ - C_D \cos 30^\circ)$$
$$= \frac{62{\cdot}4 \times 13{\cdot}5 \times 53{\cdot}6^2}{2 \times 32{\cdot}2}[(0{\cdot}6 \times 0{\cdot}5) - (0{\cdot}025 \times 0{\cdot}866)]$$
$$= 10{,}430 \text{ Lb per foot length of blade}$$

This is assumed to be constant over the whole length of blade. Consider a short length of blade dx at radius x, having a velocity v_x. Then,

$$\text{force on short length} = F_x = 10{,}430\, dx$$
$$v_x = \omega x$$
$$= \frac{2\pi 75}{60}x$$
$$= 7{\cdot}85x$$

Work done on short length per second

$$= F_x \times v_x$$
$$= 10{,}430 \, dx \times 7{\cdot}85x$$
$$= 82{,}000x \, dx$$

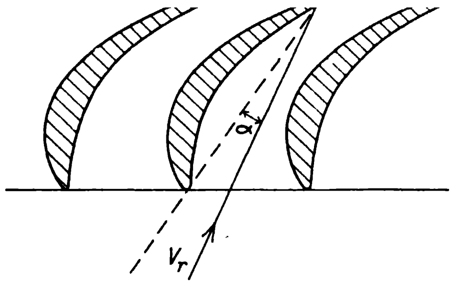


FIG. 181

Total work done per blade per second

$$= 82{,}000 \int_5^{10} x \, dx$$
$$= 82{,}000 \left[\frac{x^2}{2}\right]_5^{10}$$
$$= \frac{82{,}000}{2}(10^2 - 5^2)$$
$$= 3{,}075{,}000 \text{ ft-Lb}$$

Total horse-power developed by 4 blades

$$= \frac{3{,}075{,}000 \times 4}{550}$$
$$= 22{,}320$$

$$\text{Efficiency of turbine} = \frac{\text{work done per second}}{WH} \times 100$$
$$= \frac{3{,}075{,}000 \times 4 \times 100}{62{\cdot}4 \times 6{,}000 \times 40}$$
$$= 82{\cdot}1 \text{ per cent}$$

15.8. Effect of Blade Pitch on Lift and Drag. The lift-coefficient curve of Fig. 178 was obtained from a test on an isolated aerofoil and would not hold for a series of similar aerofoils placed close together in parallel, as on the rim of a turbine runner. This is obvious from the position of the streambands shown in Fig. 179.

The effect of the interference of adjacent aerofoils on the lift and drag has been investigated experimentally by Youssef.* Five model Parsons reaction steam-turbine blades, of the shape shown in Fig. 181 were placed in parallel at a known pitch and were tested in a wind tunnel; the lift and drag of one blade was measured experimentally at various angles of incidence. This was repeated for other

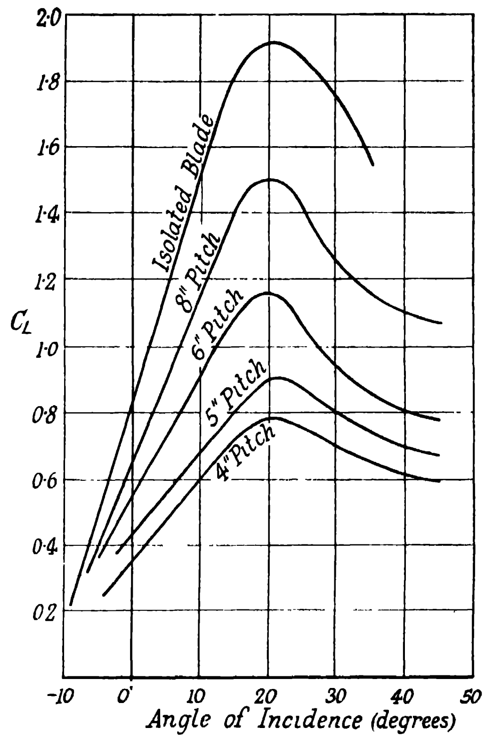


FIG. 182

pitches. The lift- and drag-coefficient curves were plotted for each pitch on a base representing the angle of incidence. These curves are shown in Figs. 182 and 183. The lift-coefficient curve for a single isolated blade was also obtained experimentally and is shown plotted in Fig. 182.

It will be noticed from the curves of Fig. 182 that the lift coefficient decreases considerably as the pitch of the blades is reduced, the maximum value of C_L at 4 in. pitch being less than one-half of the corresponding value for an isolated blade. From the drag-coefficient

* See "Wind tunnel experiments on model reaction turbine blades," by Dr. M. R. Youssef, *Engineering*, **153** (1942), p. 138.

curves of Fig. 183 it will be seen that there is a large decrease in drag as the pitch is reduced.

The model aerofoil used in these experiments had a maximum thickness of 1·4 in. and a chord of 9·2 in., and the pitch of the aerofoils was varied between 4 in. and 8 in. during the tests. If, in order to obtain a large lift, the 8 in. pitch were used, there would

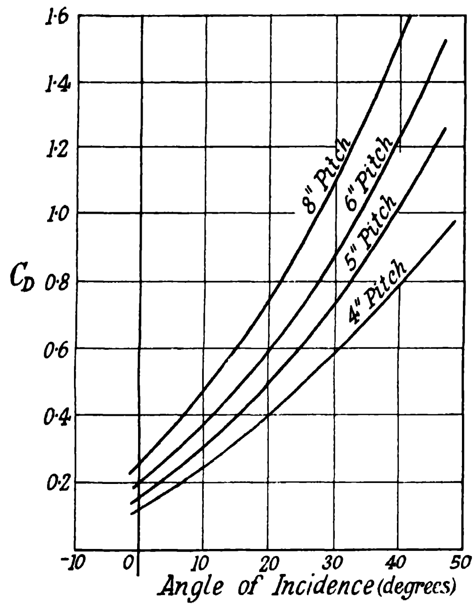


FIG. 183

be a relatively large space between the aerofoils, in which case a considerable quantity of fluid would pass between the aerofoils without giving up much of its total energy. Hence, in steam-turbine practice, it is necessary to have several rings of blades arranged in series and keyed on a common shaft in order to absorb the energy of the fluid.

By this method the fluid passes through the first blade ring causing it to rotate; it is then exhausted on to a fixed ring of guide blades which re-direct the fluid at the correct angle on to a second ring of moving blades. This is repeated on each blade ring of the series until almost the whole of the energy of the fluid is absorbed. It will be noticed that in passing over the aerofoil blades the fluid loses velocity and pressure in overcoming the drag.

As the values of C_L and C_D of an aerofoil blade depend on the pitch/chord ratio of the blading, it is possible to obtain an approximate value of these coefficients from the results of a test on an isolated aerofoil.

Let C_{L_1} and C_{D_1} = coefficients for the isolated aerofoil for a given value of α,

and C_L and C_D = the corresponding values for a grid of blading of the same aerofoil section, of a known pitch/chord ratio.

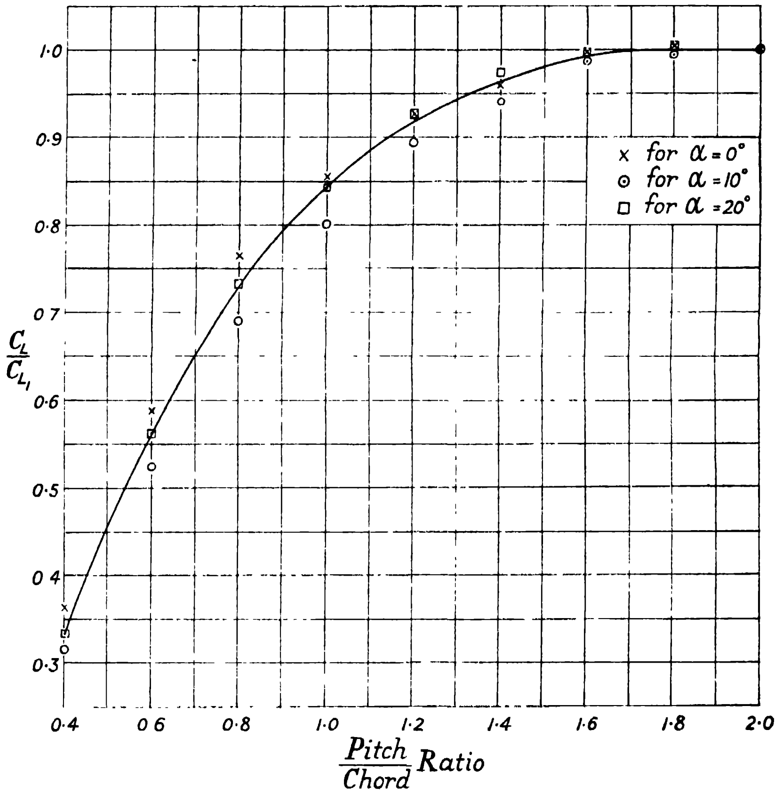


FIG. 184

The values of the ratio C_L/C_{L_1} and C_D/C_{D_1} have been calculated from the curves of Figs. 182 and 183 for values of α of 0°, 10° and 20°. These results are shown plotted on a pitch/chord base in Figs. 184 and 185. It will be noticed that the mean curve drawn is approximately independent of α, the slight deviations being due to experimental measurement errors. The values of both ordinates are non-dimensional which eliminates the effect of α. Some values not given in Fig. 183 were obtained by extrapolation.

From the mean curves of Figs. 184 and 185 the required values of C_L and C_D for any given pitch/chord ratio can be obtained by using

the experimental values from an isolated aerofoil test, such as those shown in Fig. 178. These can be obtained from aeronautical handbooks (see footnote on page 384).

The values of C_L and C_D used must also be corrected for the Mach number of the flow of the fluid relative to the blade (§ 17.6

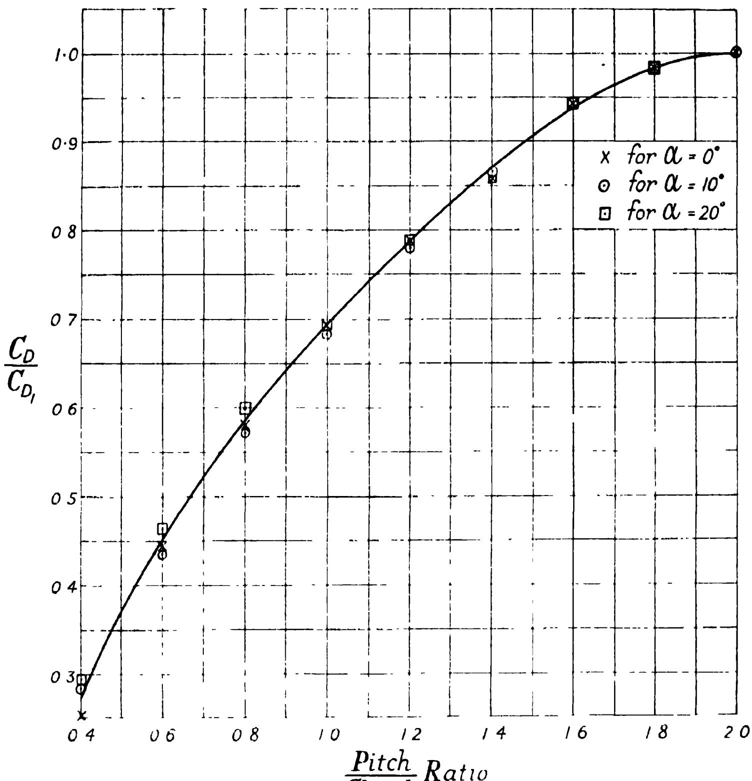


FIG. 185

and § 17.7); this correction is very important if the relative velocity is high and has a Mach number exceeding 0·6.

Aerofoil blading has been used successfully in propeller-type water turbines and in axial-flow air compressors for gas turbines.

15.9. Application of Aerofoil Theory to Air Compressor Blades. In order to apply the aerofoil method of solution of § 15.7 to the blading of an axial-flow air compressor, the values of C_L and C_D of the blade section used must be known for the conditions of flow

which occur. These values are affected by the pitch/chord ratio (§ 15.8) and by the relative Mach number of the flow over the blade (§ 17.6 and § 17.7); they can be found only from wind-tunnel tests.*

The velocity diagram at inlet (§ 15.7) must be drawn, as shown in Fig. 186, and the blade must be sloped so that its chord makes an

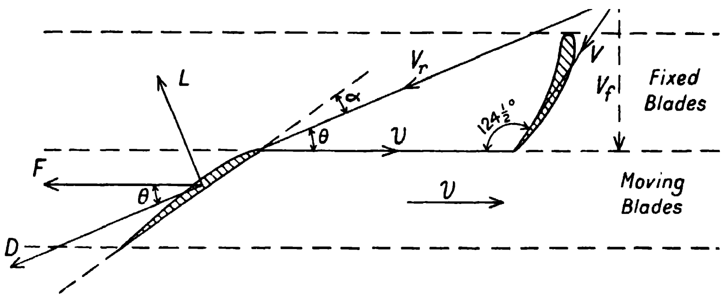


FIG. 186

angle α to the relative velocity direction, as shown, where α is the best angle of incidence.

Referring to Fig. 186, the air enters from the fixed guide blade with a velocity V. By combining this vectorially with the moving blade velocity v, the relative velocity V_r is obtained, in magnitude and direction. The blade chord is then placed at an angle α to the direction of V_r.

Now $$L = \frac{C_L \rho A V_r^2}{2}$$

and $$D = C_D \rho A V_r^2$$

From eq. (5), $$F = L \sin \theta + D \cos \theta \quad . \quad . \quad (9)$$

From eqs. (12) and (15), Chapter 14,

$$\text{work done per second} = F \times v \times N \times \text{adiabatic eff.}$$

$$= p_1 V_1 \left(\frac{\gamma}{\gamma - 1}\right)\left[\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right] . \quad (10)$$

where V_1 = volume of free air compressed per second.

Horse-power required for stage considered

$$= \frac{F \times v \times N}{550}$$

The final pressure of the air p_2 at the outlet of the stage can be calculated from eq. (10).

* For theory of centrifugal compressors, see author's text-book *Thermodynamics Applied to Heat Engines* (Pitman).

Alternatively, eq. (10) can be solved by using the heat-entropy chart (Fig. 155, facing page 330), the losses of energy over the blade being assumed to reheat the air (§ 14.3). The diagram representing this process on the heat-entropy chart is shown in Fig. 187.

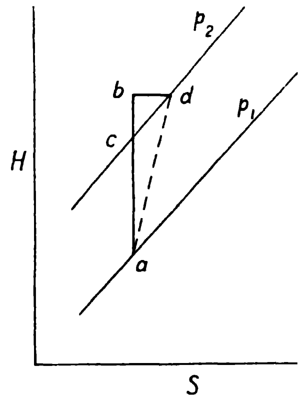


FIG. 187

Let W = wt. of air compressed per second. Then

$$\text{work done per pound of air} = \frac{F \times v \times N}{JW} \text{ heat units}$$

$$= ab \text{ (Fig. 187)} = H_b - H_a$$

adiabatic efficiency of compression

$$= \frac{ac}{ab}$$

If the adiabatic efficiency be assumed, the point c can be calculated, and the final pressure p_2 can be obtained from the chart. The final condition of the air is represented by d.

The method is illustrated by the following worked-out example.

EXAMPLE 5

The following particulars refer to the blades of the first stage of an axial-flow air compressor fitted with aerofoil blading—

Speed = 8,700 r.p.m.
Mean dia. of blade ring = 17·66 in.
Number of blades = N = 27
Height of blades = S = 3·8 in.
Chord length = c = 1·93 in.
Chord angle = $34\frac{1}{2}°$ to direction of motion
Adiabatic eff. = 75 per cent
Blade area coefficient = 0·93
Guide blade angle = $124\frac{1}{2}°$ to direction of motion
Weight of air compressed = 43 Lb/sec.
Initial pressure of air = p_1 = 14·7 Lb/in.²
Initial temperature = T_1 = 293°K
C_L = 0·426

Calculate—

(1) the relative Mach number of the flow over the blades;

(2) the theoretical horse-power required for this stage;

(3) the final temperature and pressure of the air leaving the stage. Use the heat-entropy chart facing page 330. (Allow for the reheating of the air by friction but neglect any heat losses, and neglect change of density of air.)

$$\text{Area of flow} = \frac{0{\cdot}93\pi dS}{144}$$

$$= \frac{0{\cdot}93\pi \times 17{\cdot}66 \times 3{\cdot}8}{144} = 1{\cdot}36 \text{ ft}^2$$

$$v = \frac{\pi \times 17{\cdot}66}{12} \times \frac{8{,}700}{60} = 669 \text{ ft/sec}$$

$$\text{Chord area} = A = \frac{3{\cdot}8 \times 1{\cdot}93}{144} = 0{\cdot}051 \text{ ft}^2$$

As $p_1 V_1 = 96wT_1$,

$$w = \frac{144 \times 14{\cdot}7}{96 \times 293} = 0{\cdot}0752 \text{ Lb/ft}^3$$

$$\text{Volume of flow} = Q = \frac{W}{w} = \frac{43}{0{\cdot}0752} = 572 \text{ ft}^3\text{/sec}$$

$$\text{Velocity of flow} = V_f = \frac{Q}{\text{area of flow}}$$

$$= \frac{572}{1{\cdot}36} = 420 \text{ ft/sec}$$

Initial sonic velocity

$$= v_s = \sqrt{\gamma RTg} \ (\S\ 17.2)$$

$$= \sqrt{1{\cdot}4 \times 96 \times 293 \times 32{\cdot}2}$$

$$= 1{,}125 \text{ ft/sec}$$

The velocity vector diagram can now be drawn (Fig. 186). From velocity diagram,

$$\theta = 23\tfrac{1}{2}^\circ$$

$$V_r = 1{,}050 \text{ ft/sec}$$

Hence, angle of incidence

$$= \alpha = 34\tfrac{1}{2}^\circ - 23\tfrac{1}{2}^\circ$$

$$= 11^\circ$$

(1) Relative Mach No.

$$= \frac{V_r}{v} = \frac{1{,}050}{1{,}125} = 0{\cdot}932$$

(2) $$L = \frac{C_L w A V_r^2}{2g} \left(\text{as } \rho = \frac{w}{g}\right)$$

$$= \frac{0{\cdot}426 \times 0{\cdot}0752 \times 0{\cdot}051 \times (1{,}050)^2}{2 \times 32{\cdot}2} = 27{\cdot}9 \text{ Lb}$$

$$D = \frac{C_D w A V_r^2}{2g}$$

$$= \frac{0{\cdot}23 \times 0{\cdot}0752 \times 0{\cdot}051 \times (1{,}050)^2}{2 \times 32{\cdot}2} = 15{\cdot}0 \text{ Lb}$$

Applying eq. (9),

$$F = L \sin\theta + D\cos\theta$$

$$= (27{\cdot}9 \sin 23\tfrac{1}{2}^\circ) + (15 \times \cos 23\tfrac{1}{2}^\circ) = 24{\cdot}8 \text{ Lb}$$

Work done per second

$$= F \times v \times N$$

$$= 24{\cdot}8 \times 669 \times 27 = 448{,}000 \text{ ft-Lb}$$

Theoretical horse-power required for stage

$$= \frac{448{,}000}{550} = 814$$

(3) Rise in total heat during compression

$$= \frac{448{,}000}{J \times W}$$

$$= \frac{448{,}000}{1{,}400 \times 43} = 7{\cdot}45 \text{ C.H.U.}$$

Referring to the heat-entropy chart facing page 330, the compression process on this chart is as shown in Fig. 187.

a = state point for air at p_1 and T_1

ab = adiabatic pressure rise neglecting reheating
= 7·45 C.H.U.

ac = ab × adiabatic efficiency
= 7·45 × 0·75 = 5·58 C.H.U.

d = state point for final condition of air, after reheating

From chart, $p_2 = 18{\cdot}8$ Lb/in.^{2}
$T_2 = 323°$K

15.10. Rotating Cylinder in Moving Fluid. If a cylinder in a transversely moving fluid be rotated about its longitudinal axis, a transverse force is found to act on the cylinder. The cylinder thus develops a lift coefficient and acts in a similar manner to an aerofoil. This phenomenon is known as the *Magnus effect.*

In Fig. 188 a cylinder is shown rotating clockwise in a fluid which is moving from left to right. The effect of the rotating motion is to deviate the streambands, as shown in the figure. The linear velocity of the perimeter of the cylinder is assumed to be greater than that of the fluid.

At the edge a the velocity of the fluid stream is increased by the movement of the cylinder which exerts a viscous drag on the fluid

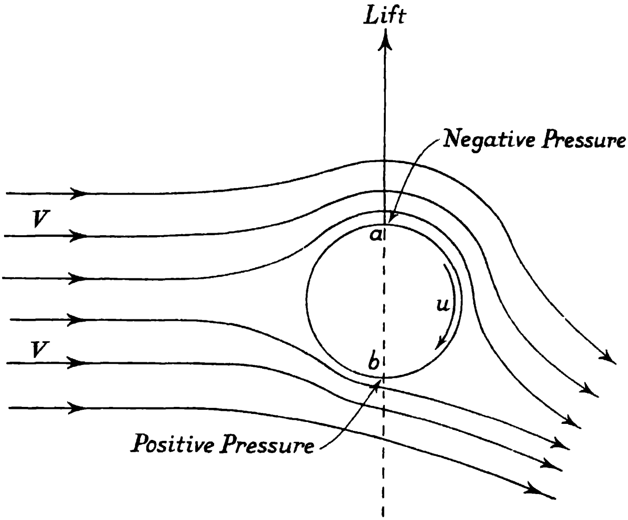


FIG. 188

and thus increases its velocity; hence, by considering the application of Bernoulli's equation to these streambands, the pressure at a is reduced. At the edge b the velocity of the adjacent streambands is reduced by the drag of the cylinder which now acts in a direction opposite to that of the fluid; hence, from the application of Bernoulli's equation, the pressure at b is increased. The effect of these pressure changes produces a lateral force on the cylinder causing it to act as an aerofoil. It is found from tests that the lift coefficient C_L for a rotating cylinder may be as high as 9.

Let V = velocity of fluid stream,

u = peripheral velocity of cylinder.

The variation of C_L with the ratio u/V is shown plotted in Fig. 189; these results were obtained experimentally by Betz with a rotating cylinder which was fitted with end discs in order to prevent axial end flow.

This aerofoil effect of a rotating cylinder is noticeable in the firing of shells from guns across a transverse wind. The shell leaves

the gun barrel with a high linear velocity and also with a high rotational velocity due to the rifling of the inside of the barrel. It thus becomes a rotating cylinder in a transversely moving fluid. This causes a vertical force on the shell which will affect its range, depending on the strength and direction of the wind. The same effect acts on the bullet from a rifle when fired across the wind.

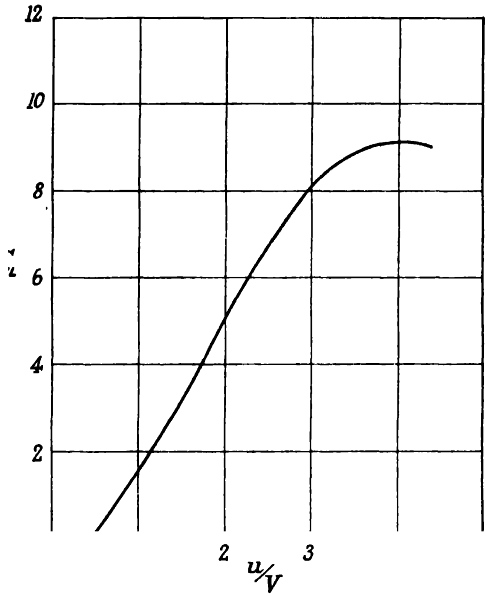


FIG. 189

Another example of a practical use of this phenomenon is in the Flettner rotor ship. A schooner named *Buckau* was fitted with two rotating cylindrical towers as a method of propulsion.* These towers were built above the deck and were revolved by an electric motor, the current being produced by a 45 h.p. Diesel engine. They were 9·1 ft in diameter and 60 ft high and could be rotated at various speeds up to 125 r.p.m.; the direction of rotation was reversible. The projected area of the towers was only one-tenth of that occupied by the former rigging of the *Buckau* as a sailing schooner. The weight of the towers and driving plant was 7 tons, against a total weight of 35 tons of the former rigging. The ship was fitted in this manner by Flettner, who considered that the propulsion of a ship by rotating cylinders exposed to the wind would be more efficient than the ordinary sails.

* See *Engineering* (23rd January, 1925).

The *Buckau*, by making use of the wind on her rotating towers. made a double crossing of the North Sea from Germany to Scotland, but the journey, although of scientific interest, was not regarded as a success.

In 1926, the *Buckau*, renamed *Baden-Baden*, crossed the Atlantic to New York by this method of propulsion. Another ship, the

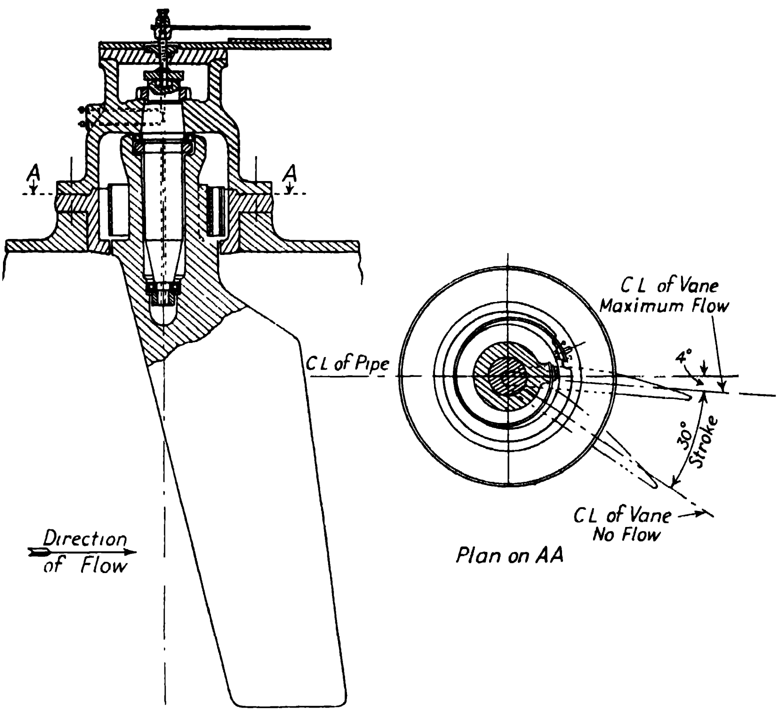


FIG. 190
(*Courtesy of The Institution of Mechanical Engineers*)

Barbara, of 3,000 tons, was fitted with three rotating towers of 13·2 ft diameter and 56 ft in height. Each rotor required 35 h.p. for rotation at 150 r.p.m. The weight of the rotors and gear was 40 tons; this was equivalent to 180 tons of sails and rigging. This ship had a speed of 10½ knots and was used as a freight carrier.

Flettner also used this principle in the design of windmills, the sails consisting of four rotating cylinders; each cylinder was rotated about its longitudinal axis by the power of a small engine.

15.11. The Aerofoil Flow Recorder. The aerofoil flow recorder is a very sensitive instrument for measuring the quantity of water flowing through a pipe or channel. It is based on the fact that a

freely suspended aerofoil blade automatically adjusts its angle of incidence to correspond to the velocity of flow of the fluid in which it is immersed. The angle of rotation of the aerofoil is thus proportional to the quantity of flow in a given channel. If the blade is geared to an automatic recorder, such as a drum rotated by clockwork, the angle of incidence may be plotted on a time base. This can be calibrated to give a continuous record of the flow, plotted on a time base.

A sectional view of an aerofoil recorder is shown in Fig. 190. The aerofoil blade rotates about the vertical axis, the angle of rotation being proportional to the velocity of the fluid. The rotation is spring controlled, the action of the spring tending to bring the blade back to its zero position. The aerofoil blade is mounted on ball bearings immersed in grease.*

The angular movement of the blade is shown in the plan view (Fig. 190); it will be seen that the maximum angle of rotation is 30 degrees.

The turning moment on the blade is caused by the lift of the fluid acting at the centre of pressure of the aerofoil; both of these quantities will vary with the fluid velocity (§ 15.3). The aerofoil blade causes a small amount of resistance due to its drag.

EXERCISES 15

The following values of ρ are to be used in these exercises—

$$\text{for air} \quad \rho = \frac{0{\cdot}075}{g} \text{ engineers' units}$$

$$\text{for water} \quad \rho = \frac{62{\cdot}4}{g} \text{ engineers' units}$$

1. Using the curves of Fig. 178, calculate the lift and resistance of this aerofoil when inclined at an angle of incidence of 2°. The chord is 6 ft, the length 32 ft, and the air speed 150 ft/sec. *Ans.* 1·33 tons; 166·0 Lb.

2. Using the curves of Fig. 178, calculate the horse-power required by a monoplane, having wings of this section, when travelling at 250 m.p.h. The area of each wing is 50 ft² and the angle of incidence 6°. Assume the propeller efficiency to be 70 per cent, and the air resistance of all parts other than the wings to be 30 per cent of the total wing resistance. What is the lift at this speed? *Ans.* 1,155 h.p.; 6·21 tons.

3. An axial-flow impulse turbine has a mean blade ring diameter of 4 ft and a speed of 150 r.p.m. The runner blades are of aerofoil section of the shape given in Fig. 178 and are set with an angle of incidence of 8° to the axis of the runner. Assuming the relative velocity of the water impinging on the blade is 80 ft/sec and that its direction is axial, find the horse-power developed. There are 30 blades each having a chord of 3·5 in. and a length of 4 in. *Ans.* 1,050 h.p.

* See "Recent developments in hydro-electrical engineering" by Dr. P. W. Seewer, *Proc. Instn. Mech. Engrs.*, **134**, p. 286.

4. Calculate for the turbine of Question 3, the guide blade angle and the head of water supplied. If the runner blades have a maximum thickness of ½ in., calculate the weight of water supplied per second and the hydraulic efficiency of the turbine.

Ans. 68·5°; 114·7 ft; 9,400 Lb/sec; 53·5 per cent.

5. An aeroplane is travelling at 180 m.p.h. and its rudder consists of a flat surface of area 2·5 ft². Find the force on the rudder normal to direction of flight when turned through an angle of 30°. Assume k_L for a flat surface at an angle of incidence of 30° to be 0·35. *Ans.* 142·2 Lb.

6. The following particulars refer to a propeller turbine having aerofoil blading—

No. of blades = 4; speed = 75 r.p.m.; mean radius of blade circle = 5·75 ft; length of blade in a radial direction = 2 ft; blade angle = 49° to direction of motion; head of water = 37·75 ft; velocity of flow = 24 ft throughout; blade area coefficient = 0·95; C_L and C_D at angle of incidence used = 0·6 and 0·025 respectively; length of chord = 18 ft.

Calculate the horse-power developed and the theoretical efficiency of the turbine. *Ans.* 5,100 h.p.; eff. = 76 per cent.

7. Calculate the lift obtained and the h.p. required to drive an aeroplane at a speed of 350 m.p.h. The chord area of each of the two wings is 80 ft² and the air resistance of all parts other than the two wings is equal to 30 per cent of the total wing resistance. The propulsion efficiency of the propeller is 80 per cent.

The lift and drag coefficients, C_L and C_D, at the angle of incidence of the flight are 0·36 and 0·02 respectively. The atmospheric pressure and temperature at the altitude of the flight are 10 Lb/in.² and 240°K respectively.

Ans. L = 6·56 tons; 1,240 h.p.

8. Describe the pressure and velocity distributions near a cylinder placed in a stream of moving fluid.

How is the result modified if the cylinder is rotated about its axis? (*Lond. Univ.*)

9. The following particulars apply to the second stage of an eight-stage axial-flow air compressor having aerofoil blading—

Speed = 8,700 r.p.m.	Blade area coefficient = 0·93
Mean dia. of blading = 18·17 in.	γ for air = 1·4
Height of blades = 3·49 in.	Temperature of entering air = 322°K
Number of blades = 27	Pressure of entering air = 18·6 Lb/in.²
Length of blade chord = 1·91 in.	
Chord angle = 34½° to direction of motion	for angle of incidence used
Guide blade angle = 115½° to direction of motion	Wt. of air compressed = 43 Lb/sec

Neglect change of density of air when passing over the blades as small, and assume an adiabatic efficiency of 75 per cent. Calculate the final pressure of the air leaving this stage, the relative Mach number of the air flow over the blades, and the angle of incidence of the blade. (The values of C_L and C_D given above have been corrected for pitch/chord ratio and for the Mach number of the air flow over the blade.) *Ans.* p_2 = 22·9 Lb/in.²; M_a = 0·812; α = 10½°.

CHAPTER 16

THE BOUNDARY LAYER

16.1. The Boundary-layer Theory. When a fluid is flowing past a body, or a surface, it can be noticed that there exists a layer of fluid adjacent to the surface, through which the variation of velocity between the fluid and the surface is transmitted. This layer is known as the boundary layer, and the whole of the viscous or frictional resistance between the fluid and the surface occurs in this layer.

The layer may be imagined to consist of a number of thin parallel streambands each having a slightly larger velocity than its inner neighbour. The band immediately adjacent to the surface of the body is found to adhere to the surface and has no velocity. Working outwards from the surface, the next band has an extremely small velocity; each successive band beyond will have a slightly higher velocity than its inner neighbour, until, finally, a band is reached which has approximately the full velocity of the fluid. This last band is the outside limit of the boundary layer; no further fluid resistance is transmitted to the surface beyond this outer limit.

The same reasoning holds if the body is moving in a stationary fluid. The boundary layer occurs between any surface and fluid which are in contact, and between which there is a relative velocity.

The existence of the boundary layer was first observed by Hele-Shaw, but the use of the conception of a laminated boundary layer transmitting the fluid resistance to the surface is due to Prandtl.

The flow within the boundary layer may be laminar or turbulent, according to the particular problem or to its distance from the leading edge of the surface. Sometimes, and under certain conditions, the boundary layer will leave the surface and curl up into a vortex or whirlpool; this phenomenon is known as *break-away* or *separation*.

The thickness of the boundary layer increases with its distance from the leading edge in proportion to the square root of the distance; it will depend also on the value of the Reynolds number of the body (§ 3.17).

The flow of a liquid past a circular-sectioned body is shown in Figs. 191, 192 and 193. These are due to Prandtl and were obtained by sprinkling small particles of aluminium on the surface of the liquid. The metal particles reflect the light and thus enable the streamlines to be photographed. In Fig. 191 the boundary layer is adhering to the surface throughout. In Fig. 192 the velocity of the stream has been increased. In this photograph the boundary layer

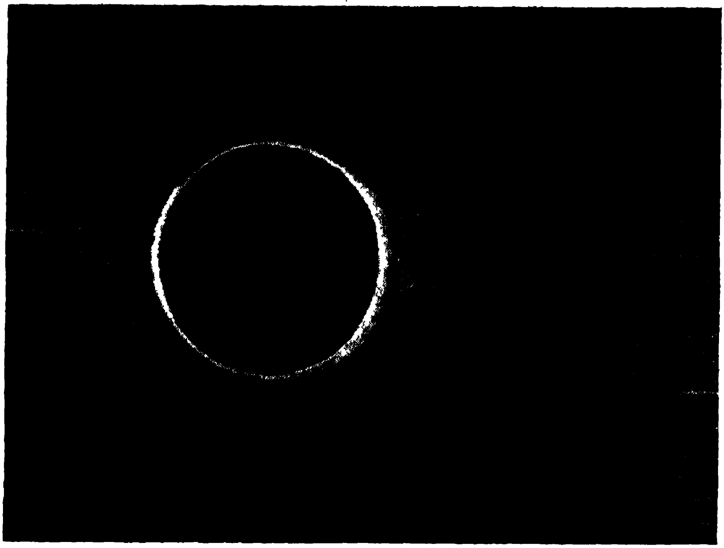


FIG. 191. FLOW PAST A CYLINDER—NO SEPARATION
Courtesy of The Royal Aeronautical Society)

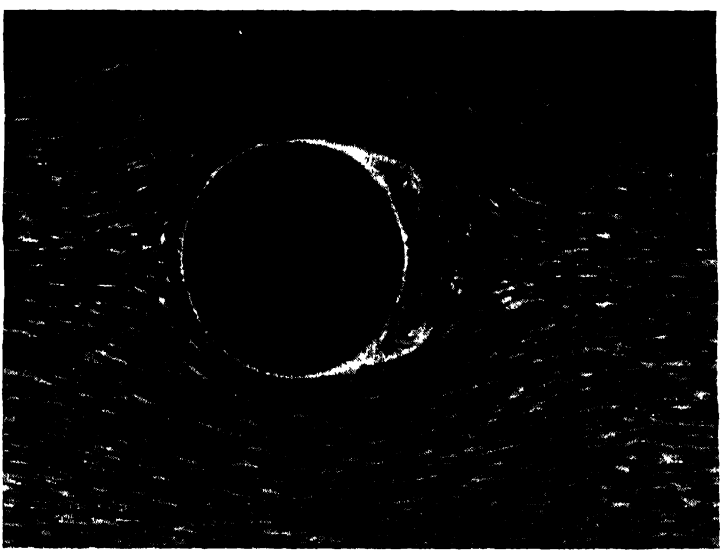


FIG. 192. FLOW PAST A CYLINDER—SEPARATION COMMENCING
(Courtesy of The Royal Aeronautical Society)

can be seen to have left the surface towards the wake, and vortices are commencing to form; this photograph clearly shows a break-away of the boundary layer. In Fig. 193 the speed of the stream has been further increased; an earlier break-away of the boundary layer is noticeable, causing the formation of more pronounced vortices in the wake.

The trail of vortices occurring in the wake of a body, after boundary-layer separation has taken place, is known as the *Kármán street*; these are shown in the photograph, Fig. 194.

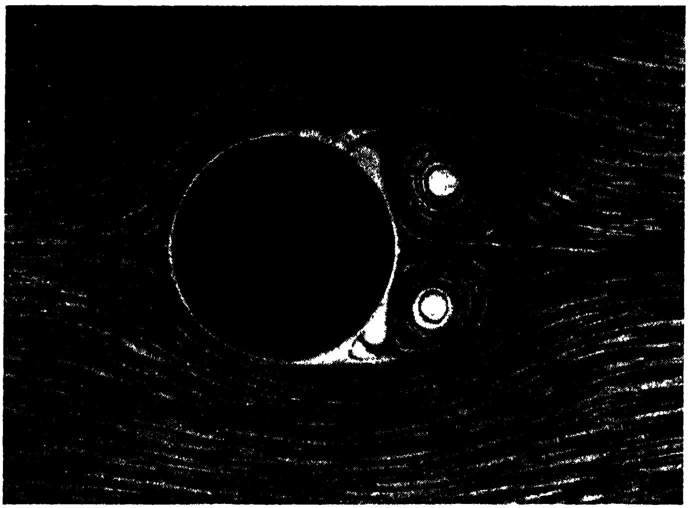


FIG. 193. FLOW PAST A CYLINDER—VORTICES FULLY FORMED
(Courtesy of The Royal Aeronautical Society)

The formation of a boundary layer occurs at the surface of all bodies immersed in a relatively moving fluid. It also occurs on the inner surfaces of short pipes* through which a fluid is being transmitted. It is the deciding factor of the magnitude of fluid resistance in such problems as pipe flow, flow in channels, resistance of ships in water, and resistance of aeroplanes and airships.

16.2. Variation of Velocity within Boundary Layer. Let Fig. 195 represent a surface, shown shaded, past which a fluid is flowing with a velocity V, and let the band adjacent to the surface represent the boundary layer. Consider a vertical section ab through the layer situated at a distance x from the leading edge of the body.

* In long pipes the boundary layers will intersect at the centre of the pipe and thus interfere with the flow.

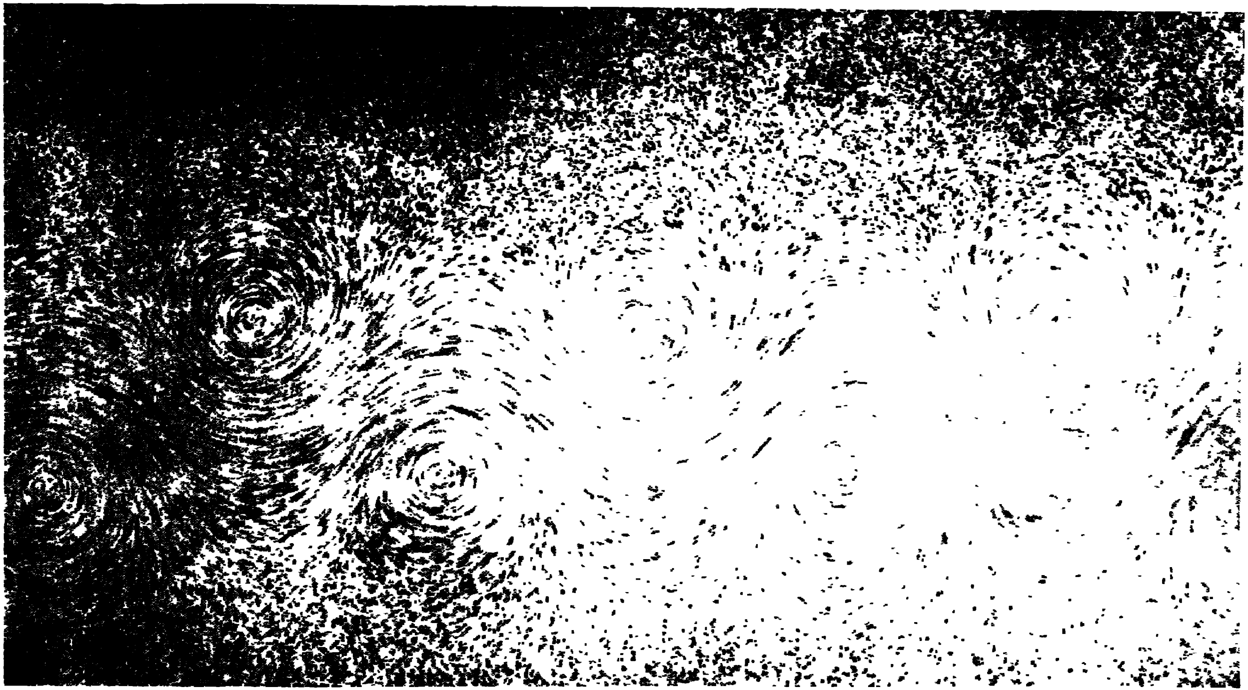


FIG. 194. PHOTOGRAPH OF A KÁRMÁN STREET

Let δ = thickness of boundary layer at section considered,
u = velocity of the fluid within the boundary layer at any distance y from the surface,
R_e = Reynolds number for the body immersed in the fluid $= Vl/\nu$.

where l represents the linear dimension of the body or surface.

If the Reynolds number is low, say, less than 500,000, the flow within the layer is wholly laminar. If the Reynolds number is high

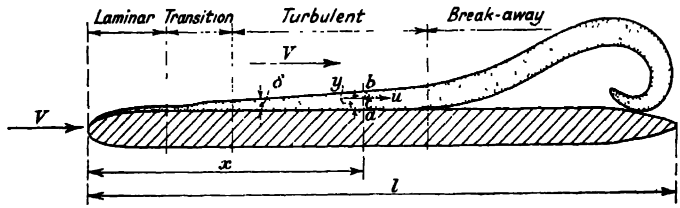


FIG. 195

the flow is mainly turbulent. Usually the flow is laminar for a short distance from the nose. This is followed by a short length of layer in which the flow changes from laminar to turbulent; in this portion of the layer the flow is unstable. The length of this transition portion of the layer is found to be about the same length as the laminar flow portion. If the Reynolds number is greater than 1,000,000, fully developed turbulence is obtained in the remaining length of the layer. For a short body the flow within the boundary layer may be laminar for its whole length. In a long body the flow may pass through all three stages: laminar, transition and turbulent, after which break-away may occur as shown in the figure.

It is found that the laminar layer, which commences at the leading edge, continues adjacent to the surface throughout the transition and turbulent layers. It is extremely thin and thus forms an inner lining to the turbulent layer. The portion of the laminar layer within the turbulent layer is called *the laminar sub-layer* (§ 16.8).

A curve showing the variation of velocity within the layer is shown in Fig. 196. If the velocities on any section ab (Fig. 195) are measured with a pitot tube at perpendicular distances y measured along ab, they are found to vary as shown by the curve. In this curve the vertical ordinate represents the distance y, and the horizontal ordinate the velocity. The particular value of y at which the velocity approximately reaches the value of the stream velocity V gives the thickness of the boundary layer at this section. Or, the thickness of the layer is the value of y at which the graph becomes vertical. It is found that the thickness δ varies with $\sqrt{x}$ and increases towards the trailing edge.

The boundary-layer theory has been applied to a flat plate surrounded by a fluid flowing longitudinally. For the portion of the boundary layer in which the flow is laminar, the variation of velocity at any section through the layer follows the Prandtl-Blasius law and is closely represented by the equation—

$$= 2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4 \quad . \quad . \quad . \quad (1)$$

For the portion of the boundary layer in which the flow is turbulent, the velocity distribution on any section is given approximately by the equation—

$$\frac{u}{V} \quad \frac{y}{\delta}\Big)^{1/7} \quad . \quad . \quad (2)$$

This may be written

$$\frac{u}{V} = \left(\frac{y}{\delta}\right) \qquad (3)$$

where n is a constant to be determined experimentally. The value of n is found from tests to vary between $\frac{1}{5}$ and $\frac{1}{7}$, the actual value depending on the Reynolds number.

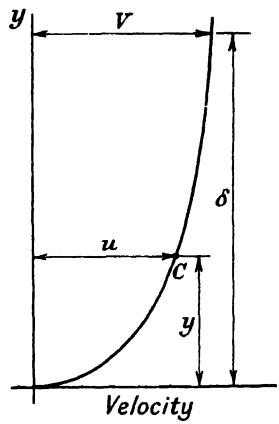


FIG. 196

VELOCITY VARIATION ACROSS BOUNDARY LAYER

It should be noted that in the testing of small models the Reynolds number is low and the boundary-layer flow is usually laminar throughout. The boundary layer surrounding full-scale sea-going ships, submarines, aeroplane wings, and airships is mainly turbulent.

16.3. Thickness of Boundary Layer. As stated in § 16.2, the thickness of the boundary layer increases from the leading edge to the trailing edge in proportion to $\sqrt{x}$. Pohlhausen deduced the following approximate value for a laminar flow past a flat plate—

$$\delta = \frac{5{\cdot}83\sqrt{lx}}{\sqrt{R_e}} \quad . \quad (4)$$

Another value for the thickness of the layer was found to be—

$$\delta = 4{\cdot}5\sqrt{\frac{\nu x}{V}} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It will be noticed by substituting for R_e in eq. (4), that these two equations are of the same type except for the difference of the constants.

The thickness of the boundary layer for turbulent flow past a flat plate has been found experimentally to vary between

$$\delta = 0{\cdot}303 \left(\frac{1}{R_e}\right)^{1/5} \sqrt{lx} \quad . \quad . \quad . \quad (6)$$

and

$$\delta = 0{\cdot}18 \left(\frac{1}{R_e}\right) \sqrt{lx} \quad (7)$$

The boundary-layer theory has been applied to a large American rigid airship* and to a model of the airship of about $\frac{1}{300}$ of the size.

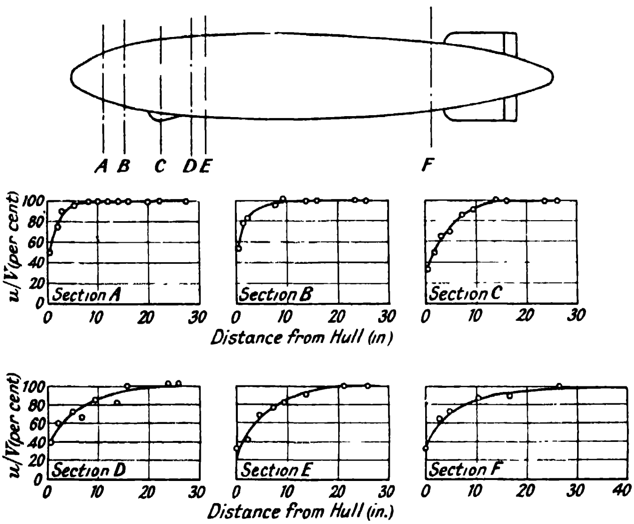


FIG. 197. EXPERIMENTAL VELOCITY CURVES FOR A RIGID AIRSHIP

The velocities within the boundary layer of the airship were measured at several sections by means of pitot tubes, whilst the airship was in flight. By plotting these velocities on each section line, similar velocity curves to Fig. 196 were obtained; it was possible to estimate the thickness of the layer from these velocity curves. The velocity curves obtained are shown in Fig. 197. These correspond to the sections *A*, *B*, *C*, *D*, *E*, and *F* marked on the airship profile. The Reynolds number for the airship at the speed of this test was 635,000, and its length was 785 ft.

The boundary-layer thicknesses obtained from the curves are shown plotted in Fig. 198 on a base representing the longitudinal position of the section, expressed as a percentage of the length from the nose. It will be noticed that the boundary layer had a thickness of about 40 in. at 75 per cent of the length from the nose.

* *Aerodynamic Theory*, Vol. VI, p. 64.

A model of this airship was similarly tested in a wind tunnel, and the boundary-layer thicknesses were obtained in the same manner. These are shown plotted in Fig. 199, the thicknesses being expressed

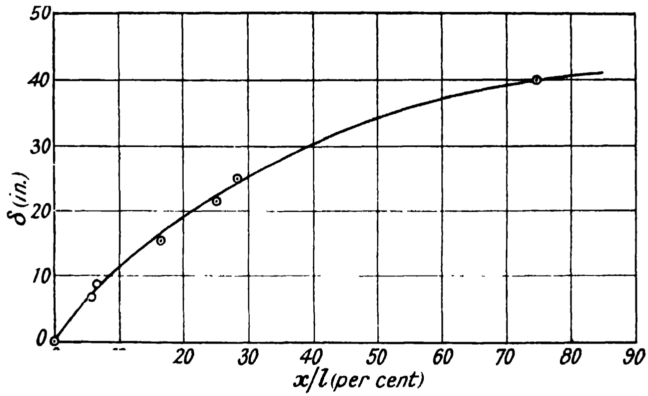


FIG. 198. BOUNDARY-LAYER THICKNESS FOR A RIGID AIRSHIP

as a percentage of the length. The thicknesses obtained from an experiment on a flat plate of the same Reynolds number are also shown plotted in this figure. It will be noticed that the variation is

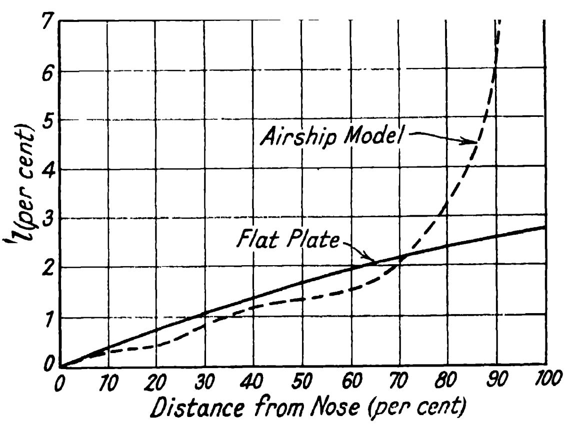


FIG. 199

slight between the results for the airship model and those of the flat plate, except at the extreme tail where there is considerable discrepancy.

The boundary-layer thicknesses obtained from the model do not correspond with those obtained from the actual airship, on account of the difference in the Reynolds number of the two tests.

If the curve of Fig. 198 is assumed to follow the law of eq. (6), the constant of this equation for an actual airship can be obtained by plotting the values of δ and $\sqrt{(x/l)}$ of this curve. This has been done in Fig. 200 and a straight line, passing through the origin, has been drawn representing the mean of the points obtained.

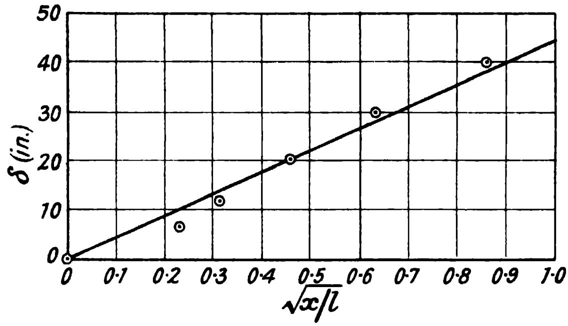


FIG. 200

From eq. (6)

$$\delta = \frac{kl}{(R_e)^{1/5}}\sqrt{\frac{x}{l}} \quad \text{(where } k \text{ is a constant)}$$

from which
$$k = \frac{R_e^{1/5}}{l} \times \frac{\delta}{\sqrt{\frac{x}{l}}}$$

Substituting the values of R_e and l for the airship, and the values of δ and $\sqrt{(x/l)}$ from Fig. 200,

$$k = \frac{(635{,}000)^{1/5}}{785} \times \frac{43}{12}$$
$$= 0{\cdot}0662$$

Hence, an approximation for the boundary-layer thickness for this airship is

$$\delta = 0{\cdot}0662\left(\frac{1}{R_e}\right)^{1/5}\sqrt{lx}$$

For the value of R_e of this test the boundary-layer flow was probably between laminar and turbulent over most of the length, as the results do not agree with either type of flow.

The variation in thickness of the boundary layer of a model of a British airship is shown by the curve of Fig. 201. This was obtained from the tests mentioned in § 16.5. The graph at first follows the square-root curve A: this is the laminar portion of the layer. At

about 20 per cent of the length there is a rapid thickening of the boundary layer as the flow changes from laminar to turbulent, after which the graph follows the turbulent curve B. At about 70 per cent of the length the graph becomes steeper on account of a further rapid thickening of the layer, due to irregularities of the flow at the tail.

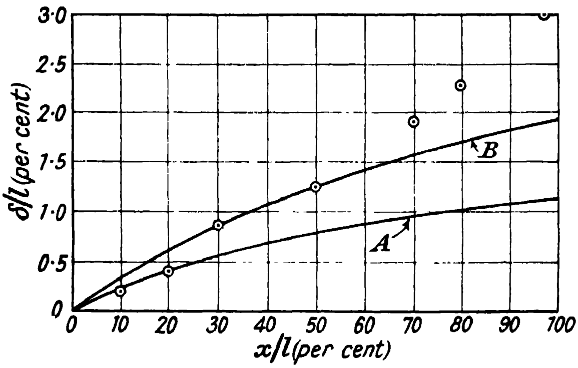


FIG. 201

Millikan found that the thickness of the boundary layer on a curved surface differs from that of a flat plate. A curved surface, such as a streamline body, produces a thinner layer at the nose and a thicker layer at the tail. This is demonstrated by the deviation of the points from the straight line in Fig. 200.

In the above, the airship was chosen as an example on the application of the boundary-layer theory because of the considerable amount of existing data which is available from airships and airship models.

16.4. Drag Coefficients from Boundary-layer Theory. The resistance of a body based on the boundary-layer theory can be obtained from eq. (3), Chapter 15, if the drag coefficient is known. Attempts have been made to obtain expressions for the drag coefficients for a flat plate, and to apply them as in an aerofoil problem. This drag coefficient will give the total resistance of the body, and includes the effect of the surfaces of both sides.

It is found that the drag coefficient for laminar flow varies with $\sqrt{(1/R_e)}$ (§ 16.6).

For laminar flow Blasius found the drag coefficient to be given by the following equation—

$$k_D = 1{\cdot}327 \sqrt{\frac{1}{R_e}}\ . \tag{8}$$

Then $$D = k_D A \rho V^2$$

For turbulent flow, the drag coefficient for a flat plate was found by Kármán to follow the equation*

$$k_D = 0{\cdot}072 \left(\frac{1}{R_e}\right)^{1/5} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The following equations for k_D for a flat plate have been deduced from experimental results; it is found to vary between

$$k_D = 0{\cdot}072 \left(\frac{1}{R_e}\right)^{1/5}$$

and

$$k_D = 0{\cdot}0375 \left(\frac{1}{R_e}\right)^{1/7} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The following table† gives a summary of the values of k_D for a flat plate compared with its Reynolds number. It should be noted that the change from laminar flow to turbulent flow causes an increase of drag—

R_e	3×10^5	10^6	7×10^6
Experimental	0·0057	0·0047	0·0035
Kármán	0·0058	0·0045	0·0031
Blasius	0·0024	0·0013	0·0005

16.5. Variation of Pressure within Boundary Layer. The variation of pressure within the boundary layer is important as a positive pressure gradient‡ may be the cause of a break-away. The velocities and pressures within the layer have been measured in a wind tunnel§ for the smooth streamline model shown in Fig. 202. The velocity curves for various sections are shown plotted in the figure; from these curves the boundary-layer thicknesses were obtained and are shown, drawn to scale, on the profile of the model.

The intensity of pressure within the layer is shown plotted on a base representing the length of the model. It will be noticed that there is a high positive pressure at the nose, but this soon changes to a negative pressure. This causes a local flow of air past the model, from nose to tail, which increases the relative velocity of the air. At the tail the pressure becomes positive again. It is found that break-away is liable to occur within a portion of the length where a

* See also § 16.7.

† See *Aerofoil and Airscrew Theory*, by Glauert.

‡ See Fig. 177 for definition of positive and negative pressure gradients.

§ These results are from "Aerodynamical research and hydraulic practice," by A. Fage, *Proc. Instn. Mech. Engrs.*, **130** (1935).

positive pressure gradient exists, and it may occur during a laminated or turbulent type of flow. A turbulent boundary layer can move up

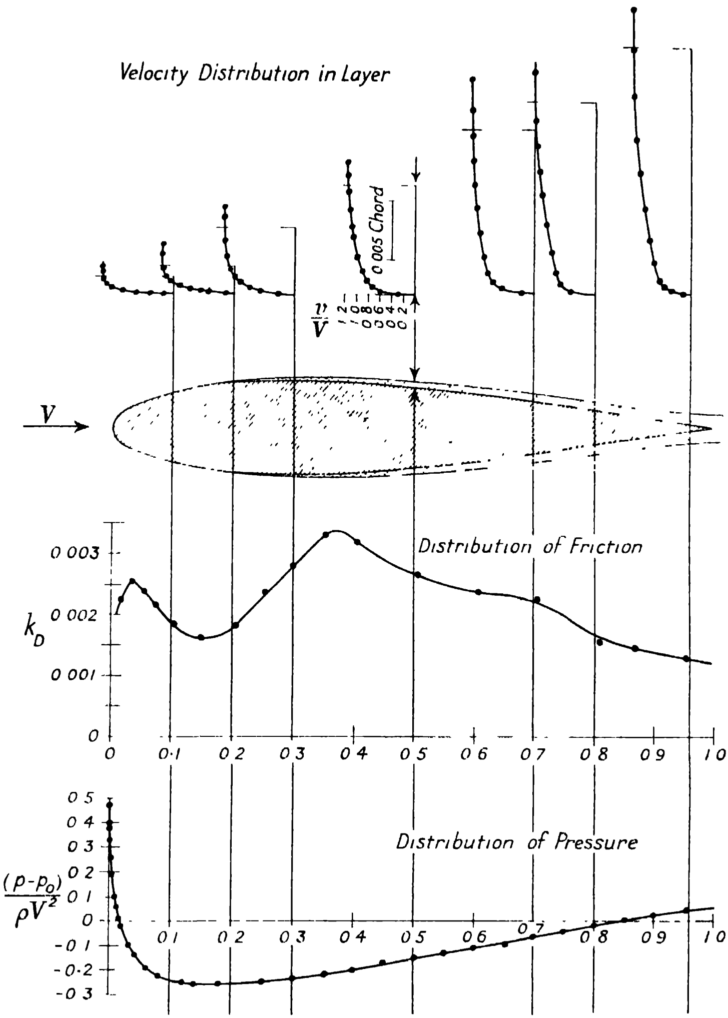


Fig. 202

a steeper pressure gradient, without break-away occurring, than a laminar layer.

The variation of k_D throughout the length of the model is also shown in Fig. 202.

Break-away of the boundary layer on an aeroplane wing causes a sudden reduction in the value of the lift coefficient, which may cause the aeroplane to stall. This is illustrated by the shape of the lift coefficient curve shown in Fig. 203. This break-away can be delayed by the use of slotted wings.*

Within a laminar flow layer, the transmission of momentum from the faster moving bands to the slower is carried out by viscosity. In a turbulent layer the transmission of momentum is made by particles of higher velocity moving inwards and giving up their momentum by collisions. Break-away at any section is due to the fact that the energy of the resistance at this section cannot be absorbed fast enough by the layer. Under such conditions break-away will occur, and the energy will then be dissipated in the vortices formed. This is more liable to occur with short bodies.†

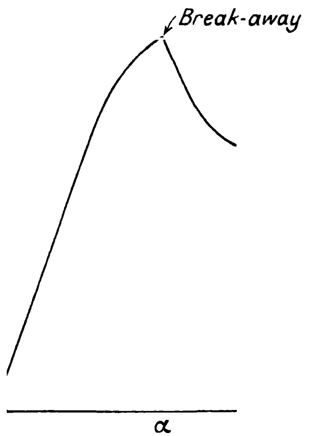


FIG. 203

Fig. 204 shows the velocity curves within the boundary layer, on several sections, which were obtained during a test on a surface in order to examine the cause of break-away. It will be noticed that break-away has commenced to the right of the fourth section. In the break-away area the velocity diagrams show a reversed current of fluid occurring near the surface. This is the commencement of the formation of the vortex, and as the flow is from right to left it denotes the existence of a positive pressure gradient.

16.6. Drag Coefficient for Laminar Flow. The drag coefficient k_D for a flat plate, due to laminar flow, can be obtained by using Pohlhausen's value for the thickness of the laminar boundary layer (eq. (4)). The velocity distribution within the layer may be assumed to follow the Prandtl-Blasius law which is approximately represented by eq. (1).

* See § 16.9 for methods of controlling boundary-layer flow.

† For further details of the boundary-layer theory the reader is referred to the following works—

H. B. Squire, M.A., "Notes on boundary layer flow," *Reports and Memoranda of the Aeronautical Research Committee*, No. 1664.

Aerodynamic Theory, Vol. I–VI (Springer, Berlin; printed in English). See Vol. III and Vol. VI.

Referring to Fig. 205, consider a section through the layer at x from the leading edge. The frictional drag occurring between the leading edge and this section is equal to the rate of change of momentum of the layer up to this section.

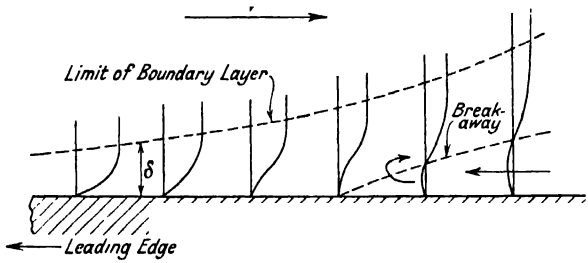


FIG. 204. VELOCITY ACROSS BOUNDARY LAYER

Consider a thin band of the layer at a distance y from the surface; let dy be the thickness of this band and u its velocity. Owing to the disturbing effect of the surface, the velocity of this band has been

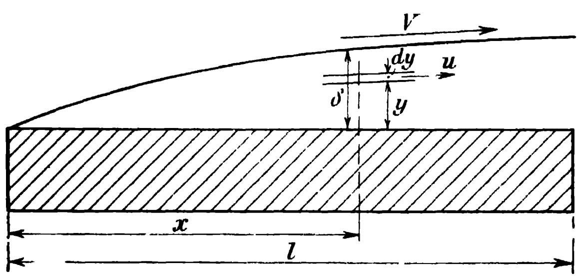


FIG. 205

reduced from V to u; this loss of momentum is caused by the drag on the surface over the length x. By imagining the boundary layer to consist of similar bands, the total change of momentum throughout the layer can be obtained by integration.

Let b = breadth of plate,

l = length of plate.

Drag per side over length x due to thin band

$$= \text{mass per second} \times \text{change of velocity}$$
$$= \rho b\, \mathrm{d}y u(V - u)$$

Total drag on length x due to both sides of plate

$$= D_x = 2\rho b \int_0^\delta u(V - u)\mathrm{d}y$$

Multiplying throughout by V^2/V^2,

$$D_x = 2\rho b V^2 \int_0^\delta \left[\frac{u}{V} - \left(\frac{u}{V}\right)^2\right] dy \quad . \quad . \quad . \quad . \quad (11)$$

Substituting the value of u/V for laminar flow from eq. (1),

$$\frac{u}{V} = 2\frac{y}{\delta} - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4$$

Then

$$\frac{u}{V} - \left(\frac{u}{V}\right)^2 = 2\frac{y}{\delta} - 4\frac{y^2}{\delta^2} - 2\frac{y^3}{\delta^3} + 9\frac{y^4}{\delta^4} - 4\frac{y^5}{\delta^5} - 4\frac{y^6}{\delta^6} + 4\frac{y^7}{\delta^7} - \frac{y^8}{\delta^8}$$

Substituting in eq. (11) and integrating in terms of y between the values of $y = \delta$ and $y = 0$,

$$\begin{aligned} D_x &= 2\rho b V^2(\delta - \tfrac{4}{3}\delta - \tfrac{1}{2}\delta + \tfrac{9}{5}\delta - \tfrac{2}{3}\delta - \tfrac{4}{7}\delta + \tfrac{1}{2}\delta - \tfrac{1}{9}\delta) \\ &= \frac{74}{315}\rho b V^2 \delta \end{aligned}$$

Substituting for δ from Pohlhausen's value for laminar flow (eq. (4)),

$$\begin{aligned} D_x &= \frac{74}{315}\rho b V^2 \times \frac{5 \cdot 83\sqrt{lx}}{\sqrt{R_e}} \\ &= 1 \cdot 369 \rho b V^2 \frac{\sqrt{lx}}{\sqrt{R_e}} \end{aligned}$$

The drag for the whole plate is when $x = l$; then,

total drag for whole plate

$$= D = 1 \cdot 369 \frac{\rho b l V^2}{R_e^{1/2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

But, from eq. (3), Chapter 15,

$$D = k_D \rho A V^2$$

where A = projected area of chord = bl.

Hence, from eq. (12),

$$k_D = 1 \cdot 369 \left(\frac{1}{R_e}\right)^{1/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

which is in close agreement with Blasius's equation (eq. (8)).

16.7. Drag for Turbulent Flow. The drag on a flat plate due to a turbulent boundary layer can be calculated from the rate of change of momentum within the layer. It will be assumed that the velocity distribution within the layer is given by eq. (2), and the thickness of the layer is given by eq. (7).

Referring to Fig. 205, consider a section through the layer at a distance x from the leading edge. The frictional drag occurring between the leading edge and this section will equal the rate of change of momentum of the layer up to this section. Consider a thin band of the layer at a distance y from the surface of the plate; let dy be the thickness of this band and u its velocity. Owing to the disturbing effect of the surface, the velocity of this band has been reduced from V to u. This loss of momentum is caused by the drag over the whole length x. By imagining the boundary layer to consist of similar bands, the total change of momentum throughout the layer can be obtained by integration.

Let b = breadth of plate,

l = length of plate.

Drag per side over length x due to thin band

$$= \text{mass per second} \times \text{change of velocity}$$

$$= \rho b\, dy\, u(V - u)$$

Total drag on length x due to both sides of plate

$$= D_x = 2\rho b \int_0^\delta u(V-u)\, dy$$

$$= 2\rho b V^2 \int_0^\delta \left[\frac{u}{V} - \left(\frac{u}{V}\right)^2\right] dy$$

But, from eq. (2),

$$\frac{u}{V} = \left(\frac{y}{\delta}\right)^{1/7}$$

Hence $$D_x = 2\rho b V^2 \int_0^\delta \left[\left(\frac{y}{\delta}\right)^{1/7} - \left(\frac{y}{\delta}\right)^{2/7}\right] dy \quad . \quad . \quad (14)$$

$$= 2\rho b V^2 \left[\frac{7}{8} y^{8/7} \delta^{-1/7} - \frac{7}{9} y^{9/7} \delta^{-2/7}\right]_0^\delta$$

$$= \frac{7}{36} \rho b V^2 \delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

But. from eq. (7),

$$\delta = 0{\cdot}18 \left(\frac{1}{R_e}\right)^{1/7} \sqrt{lx}$$

Substituting this value of δ in eq. (15),

$$D_x = 0{\cdot}035 \left(\frac{1}{R_e}\right)^{1/7} \rho b V^2 \sqrt{lx}$$

The drag for the whole plate is when $x = l$; then,

total drag on whole plate

$$= 0{\cdot}035 \left(\frac{1}{R_e}\right)^{1/7} \rho l b V^2 \quad . \quad . \quad . \quad . \quad (16)$$

But, from eq. (3), Chapter 15,

$$D = k_D \rho A V^2$$

where $A = bl$ = projected area of chord.

Hence, from eq. (16),

$$k_D = 0{\cdot}035 \left(\frac{1}{R_e}\right)^{1/7} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

It will be noticed that eq. (16) is of the same form as eq. (10), and agrees with the experimental results given in the table of § 16.4.

If the velocity distribution within the layer had been assumed to follow the "$\frac{1}{8}$ power law" instead of $\frac{1}{7}$ power, an equation the same as eq. (9) is obtained.

EXAMPLE 1

What is meant by "boundary layer"?

A roughened thin board 1 ft wide and 8 ft long moves at 10 ft/sec in water. Each boundary layer is 3 in. thick at the rear end of the board, and $u/V = (y/\delta)^{1/4}$.

Find the resistance and express it (*a*) in pounds, (*b*) as a pure number independent of δ. (*Lond. Univ.*)

(*a*) Using eq. (14) and substituting the power of $\frac{1}{4}$ in place of $\frac{1}{7}$,

$$\begin{aligned} \text{drag} &= 2\rho b V^2 \int_0^\delta \left\{\left(\frac{y}{\delta}\right)^{1/4} - \left(\frac{y}{\delta}\right)^{1/2}\right\} dy \\ &= 2\rho b V^2 \left[\frac{4}{5} y^{5/4}\,\delta^{-1/4} - \frac{2}{3} y^{3/2}\,\delta^{-1/2}\right]_0^\delta \\ &= \frac{4}{15}\rho b V^2 \delta \\ &= \frac{4}{15} \times \frac{62{\cdot}4}{32{\cdot}2} \times 1 \times (10)^2 \times \frac{1}{4} \\ &= 12{\cdot}93 \text{ Lb} \end{aligned}$$

(*b*) Use the equation

$$\text{drag} = k_D \rho A V^2$$

where $A = bl$. Then

$$\begin{aligned} k_D &= \frac{\text{drag}}{\rho b l V^2} \\ &= \frac{12{\cdot}93 \times 32{\cdot}2}{62{\cdot}4 \times 1 \times 8 \times (10)^2} \\ &= 0{\cdot}00834 \end{aligned}$$

16.8. The Laminar Sub-layer. In a turbulent boundary layer a narrow portion of the layer, adjacent to the surface of the body, is found to be of a laminar type of flow and is a continuation of the laminar boundary layer commencing at the leading edge (Fig. 206).

This thin laminar layer occurring within a turbulent layer is known as a *laminar sub-layer*. It has an important effect in problems dealing with the roughness of the surface, and with the flow of heat between the surface and a fluid in contact with it.

A surface is said to be *aerodynamically smooth* when the projections due to its roughness do not penetrate beyond this laminar sub-layer. If the surface is sufficiently rough that its projections extend beyond the laminar sub-layer into the turbulent layer, there

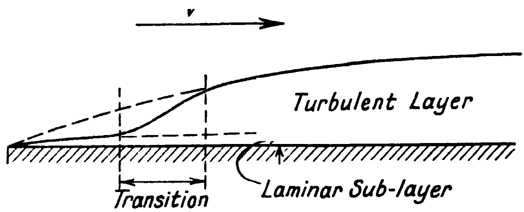


FIG. 206. THE LAMINAR SUB-LAYER

is caused a thickening of the turbulent layer and a corresponding increase in the frictional resistance. The drag due to a turbulent layer is, therefore, unaffected by the roughness of the surface, providing projections due to roughness do not protrude above the laminar sub-layer. The permissible roughness of surface is thus limited by the thickness of the laminar sub-layer.

The effect of the surface roughness on the frictional drag has been investigated experimentally by Nikuradse and Prandtl* (§ 9.1). They have deduced the following empirical law—

$$\frac{1}{k} = 200V\sqrt{C_f}$$

where k represents a scale of roughness in inches, V is the fluid velocity in m.p.h., and C_f is the frictional drag coefficient. The term C_f is the same coefficient as used in the continental drag formula*—

$$\text{frictional drag} = C_f \cdot \frac{A\rho V^2}{2}$$

and is consequently equal to twice the value of k_D.

The variation of the frictional drag of a flat plate with the roughness of the surface is shown by the experimental curve of Fig. 207. In this curve the drag per square foot has been plotted on a base representing the roughness scale k in inches. The fluid velocity of this test was 200 m.p.h. and the plate had a length of 10 ft. The

* See also "Profile drag" by Melville Jones, *Proc. Roy. Aero. Soc.* (1937); and *Aerodynamic Theory*, Vol. III.

surface of the plate was varied during the tests to conform with different degrees of roughness.

In this test the limiting value for aerodynamic smoothness was when $k = 0{\cdot}0005$ in. It will be noticed from the curve that the rapid rise in the value of the drag did not commence until $k = 0{\cdot}0007$ in.

For low values of k the frictional drag remained constant, as denoted by the horizontal portion of the curve; evidently the

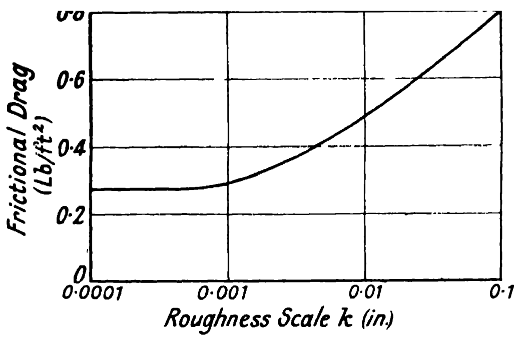


FIG. 207. EFFECT OF ROUGHNESS ON FRICTIONAL DRAG

surface was aerodynamically smooth over the range represented by this horizontal portion.

The laminar sub-layer acts as an insulating medium thus resisting the transfer of heat. It tends to reduce the heat flow in tubular condensers, boilers and heat exchanges. Heat flow takes place more readily through the turbulent portion of the layer on account of the transverse movement of the eddies. Methods have been devised to remove the laminar sub-layer from the surface.

16.9. Methods of Controlling Boundary-layer Flow. It is possible to control the boundary-layer flow of an aerofoil by external means so that an increased lift and a reduced drag are produced. The break-away, or separation, can also be prevented by this method, thus giving greater manœuvring ability to an aeroplane. Many experiments have been carried out showing the various effects of boundary-layer control on aerofoils; these effects can be observed by studying the shape of the streambands produced when the aerofoil is tested in a wind channel, the air stream being made visible by smoke bands. The shapes of the smoke bands are shown in Figs. 208, 210 and 211; these have been reproduced from photographs.

The following five methods have been used to control the boundary-layer flow over the surface of an aerofoil.

1. Wing Slots. The insertion of a slot through the aerofoil causes either suction or increased pressure on the upper surface, according to the position of the slot. The pressure in the boundary layer can thus be controlled and break-away prevented. The effect of the slot can be observed by comparing the shape of the smoke bands shown in Figs. 208 (*a*) and 208 (*c*).

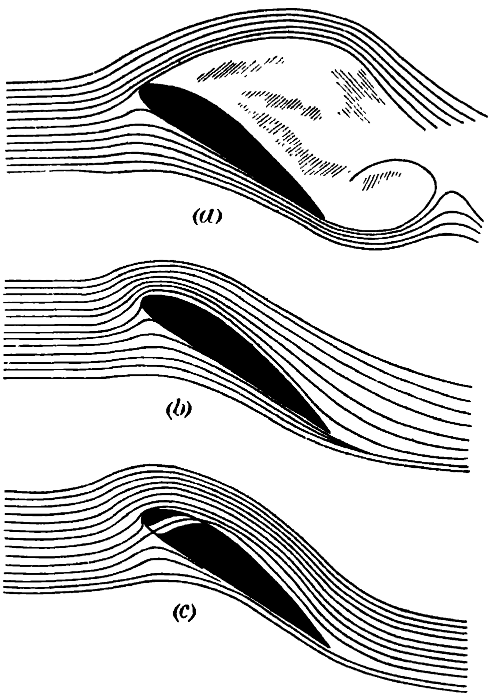


Fig. 208. Wing-slot Method of Controlling Boundary-layer Flow

(*a*) A stalled aerofoil, showing break-away.
(*b*) The same section after boundary-layer suction has been applied.
(*c*) The same section fitted with a slot.

In Fig. 208 (*a*) is shown a stalled aerofoil; the break-away of the boundary layer is clearly demonstrated by the shape of the smoke bands. In Fig. 208 (*c*) is shown the same aerofoil under identical conditions, but in this case a slot has been opened near the leading edge. It will be noticed that the opening of the slot has prevented the break-away of the boundary layer. The smoke bands now adhere to the upper surface of the aerofoil for the whole of its length; the stalling of the aerofoil has thus been prevented. This shows that by fitting a slot in an aeroplane wing an increased manœuvring power is given to the aircraft, as the condition of break-away of the boundary layer can be prevented and stalling is thus delayed. The

principle is used in the Handley-Page slotted wing, and is adopted in many types of aircraft.

The effect of a wing slot on the lift coefficient of a Handley-Page aeroplane wing is shown in the lift-coefficient curve of Fig. 209. In this figure the lift coefficient C_L is plotted on a base representing the angle of incidence. The dotted-line graph is the lift-coefficient curve for the wing when the slot is closed. It will be noticed that it then stalls at an angle of incidence of 17½ degrees; this corre-

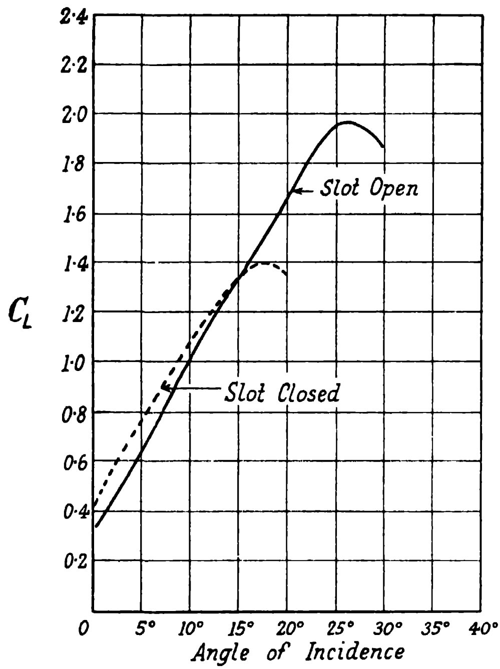


FIG. 209

sponds to a break-away of the boundary layer. Also, the greatest lift coefficient is 1·4 under this condition. The full-line graph is the lift-coefficient curve when the slot is fully open. The wing now reaches an angle of incidence of 26 degrees before stalling, and the greatest lift coefficient is increased to 1·98. From a comparison of these two curves it will be seen that an increased manœuvring power is obtained by the use of the slotted wing. It will be noticed that the slot causes a supply of additional energy to the boundary layer and thus prevents the formation of a reversal of the pressure gradient (§ 16.5).

2. SPOILER FLAPS. The lift coefficient of an aerofoil can be increased by fixing spoiler flaps, or slats, on to its surface. The

effect of a flap on the pressure distribution on the surface can be seen from a comparison of the smoke bands of Figs. 210 (*a*) and 210 (*b*). In Fig. 210 (*a*) is shown an aerofoil without the flap raised. The effect of raising the flap is shown in Fig. 210 (*b*), which is the same aerofoil under identical conditions. The negative pressure in the wake of the flap causes an increase of lift. Thus, variation in the pressure within the boundary layer can be caused by the fixing of a flap to the aerofoil surface.

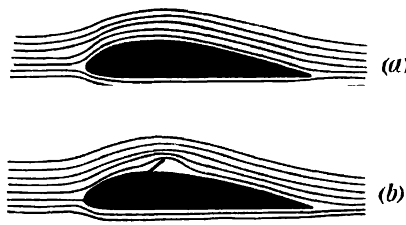


Fig. 210

3. Re-energizing Compressed-air Stream. In this method a stream of compressed air is passed into the boundary layer through passages in the wing; the pressure distribution on the surface of the aerofoil, and within the boundary layer, is thus affected. This prevents the formation of the positive pressure gradient which is the cause of separation of the layer (§ 16.5). It also causes alteration in the Reynolds number of the flow; consequently, break-away conditions are affected. An air compressor, or blower, is required to produce the energizing air stream.

4. Boundary-layer Suction. The performance of an aerofoil can be affected by sucking away the boundary layer of the upper surface. This has been performed on the wings of an aeroplane, whilst in flight, by fitting a perforated surface on to the upper surface of the wings. The air of the boundary-layer flow is sucked through the wing by the action of a Venturi fitted on to the lower surface, or by a blower driven on a separate engine. The suction can also be provided by allowing the exhaust gases from the engines to flow through an ejector and thus produce a suitable suction effect.

In Fig. 208 (*a*) are shown the smoke bands flowing past a stalled aerofoil, break-away of boundary layer having occurred. In Fig. 208 (*b*) is shown the same aerofoil under the same conditions, but in this case boundary-layer suction is being applied. It will be noticed that the smoke bands now pass along the surface, showing that the break-away condition has been prevented by the suction. The suction draws away the accumulation of fluid due to the reversal of the pressure gradient (§ 16.5); separation is thus prevented.

Phillips and Powis* obtained a 26 per cent reduction in profile drag, and an increased lift, by sucking away the boundary layer from the upper surface of an aeroplane's wings by means of a Venturi fitted under each wing. This was done whilst the aircraft was in flight. In another test by the same experimenters, the boundary layer was sucked away from the upper surface of the wings by means of a blower driven by an 8 h.p. engine. It was found that the rate of climb was increased by 29 per cent by this

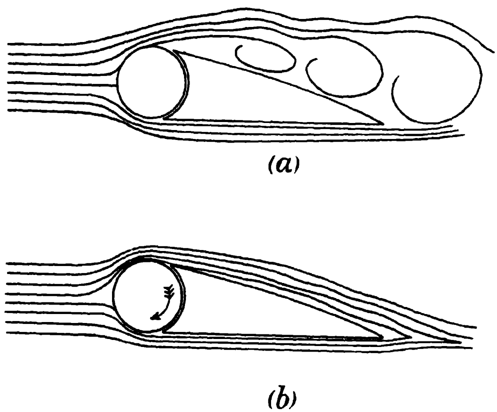


FIG. 211. EFFECT ON BOUNDARY-LAYER FLOW OF ROTATING CYLINDER AS LEADING EDGE

(*a*) Rotor at rest: separation has occurred.
(*b*) Rotor revolving: separation prevented.

latter method, which was equivalent to another 17 h.p. on the aircraft engines. It was also found that a decrease of 22 per cent in profile drag was obtained when the plane was travelling at top speed.

5. ROTATING CYLINDER AS LEADING EDGE. The boundary-layer flow is affected by incorporating a rotating cylinder as the leading edge of the aerofoil, thus making use of the Magnus effect (§ 15.10). This method causes a variation in the pressure distribution on the upper and lower edges of the aerofoil; an increased lift and the prevention of break-away can be accomplished by this method. Power of some type must be provided for the rotation of the cylinder. Lippisch† has shown experimentally, on a model, that by this means the angle of incidence of an aerofoil can be greatly increased without break-away occurring. Views of his aerofoil, fitted with a rotating cylinder for its leading edge, are shown in Fig. 211. In Fig. 211 (*a*) the cylinder is at rest; the smoke bands show that break-away has occurred at the angle of incidence adopted. In Fig.

* See article by F. G. Miles in *Flight* (26th January, 1939).
† A. Lippisch in *Jour. Roy. Aero. Soc.*, **43** (1939), p. 653.

211 (*b*) is shown the same aerofoil under the same conditions, but the cylinder is now being rotated. From the new positions taken by the smoke bands it will be seen that the break-away of the boundary layer has been prevented.

The effect of the rotating cylinder is to cause a stream of fluid of high velocity to enter the boundary layer which is thus re-energized; this prevents the formation of a reversal of pressure gradient and so prevents the occurrence of separation (§ 16.5).*

16.10. Application of Boundary-layer Control to Diffuser. An improvement of the efficiency of diffusers can be obtained by the application of boundary-layer suction to the diverging passage. If the throat diameter of a Venturi meter is very small compared with that of the mouth, there is too rapid a rate of expansion of the jet area for the diverging cone to run full; this causes a loss of pressure. In the same way, the diffuser of a centrifugal pump may have a reduced efficiency due to this cause. By applying boundary-layer suction at the periphery of the diverging passage of a diffuser, the full expansion of the jet can be brought about.

The effect of this suction is shown in the photographs of Figs. 212 and 213. These photographs are due to Prandtl; they show a fluid stream flowing through a diffuser which first contracts to a throat and then diverges rapidly. Two slots, which can be opened or closed at will, have been fitted to each side of the diverging portion; by opening the slots boundary-layer suction is applied to the jet. In the view shown in Fig. 212 the slots are closed; it will be noticed that the jet has not expanded sufficiently to fill the whole width of the diffuser passage. The same jet is shown in Fig. 213; the slots are now open and suction is applied. It will be noticed that the boundary layer is now drawn to the sides of the diffuser and a full expansion of the jet is obtained.

Ackeret† found that a particular diffuser had an efficiency of 50 per cent when no boundary-layer suction was applied. This was increased to 81 per cent by applying suction to one annular slot situated between the throat and outlet. A quantity of water equal to 5 per cent of the total flow was withdrawn through the slot by the suction. On the assumption that the efficiency of the suction pump was 75 per cent, the work done in removing this quantity of water was 3·4 per cent of the kinetic energy of the jet at the throat. Ackeret suggests that by the application of this method the efficiency of diffusers can be increased, and its use may produce important improvements in hydraulic and ventilating engineering.

* For a detailed account of boundary-layer control, see *Modern Developments in Fluid Mechanics*, by Goldstein (Oxford University Press).

† See Ackeret, *Zeitschr. des Vereines deutscher Ingenieure*, **70** (1926), pp. 1155–1157.

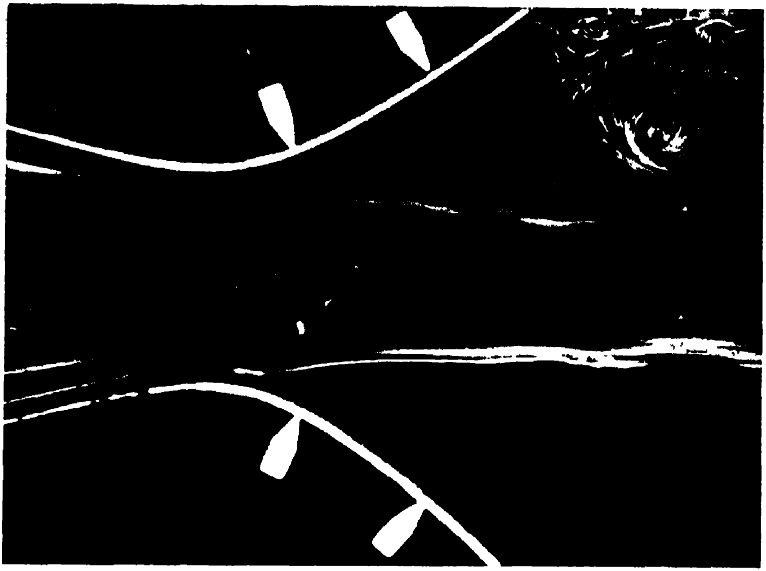


Fig. 212

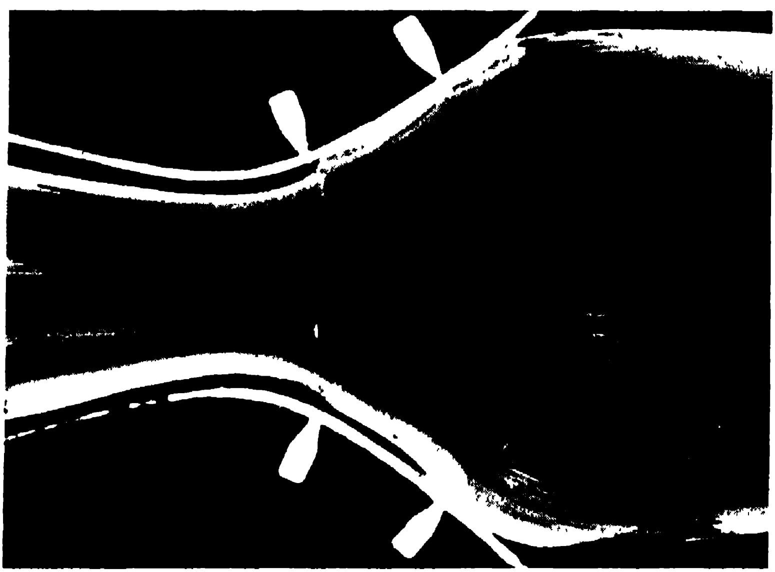


Fig. 213

16.11. Comparison of Fluid-flow Theories. A comparison may now be made of the fluid-flow theories which have been dealt with in previous chapters. This involves a comparison of the following equations. Let the symbol V represent velocity.

(1) *Froude*

$$\text{Resistance} = f'AV^2 \quad (\S\ 7.2)$$

(2) *Darcy*

$$h_f = \frac{4flV^2}{2gd} \text{ [eq. (6), § 7.6]}$$

(3) *Viscosity Formula*

$$\frac{R}{\rho V^2} = C\left(\frac{Vd}{\nu}\right)^n \quad \text{[eq. (6), § 8.2]}$$

(4) *Boundary-layer Theory Combined with Drag Formula*

$$D = k_D \rho A V^2 \text{ [eq. (3), § 15.2]}$$

where $$k_D = k\left(\frac{1}{R_e}\right)^n \quad \text{[eqs. (8), (9) and (10), § 16.4]}$$

k being an experimental constant.

It will be noticed that the drag formula of (4) is the same equation as Froude's if

$$f' = k_D \rho$$

The Darcy formula was built up from Froude's assumptions and is merely another form of Froude's equation.

It was shown in eq. (9), § 8.2, that the viscosity formula of (3) is the same as the Darcy formula if the Darcy constant f is equal to

$$2C\left(\frac{Vd}{\nu}\right)^n$$

Substituting for $R_e = (Vd/\nu)$ in the above equation,

$$f = 2C(R_e)^n$$

which is of the same form as the equation for k_D in the boundary-layer theory.

The viscosity formula (3) may be written—

$$\frac{\text{Resistance per square foot}}{\rho V^2} = C(R_e)^n \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

But $$D = k_D \rho A V^2$$

Hence, resistance per square foot $= \dfrac{D}{A} = k_D \rho V^2$

Substituting this in eq. (18),

$$\frac{k_D \rho V^2}{\rho V^2} = C(R_e)^n$$

from which $$k_D = C(R_e)^n \quad . \quad . \quad . \quad (19)$$

Hence $$\text{Darcy's } f = k_D \times 2$$
$$= C_D \qquad . \quad . \quad . \quad . \quad . \quad (20)$$

Hence, it follows that the viscosity equation of (3) is of the same form as the boundary-layer equation, except for the values of the constants C and n. The constants of the viscosity equation are given in § 8.2. These were obtained from analytical results and experiments on pipe flow respectively, and were found to be—

for laminar flow, $C = 8,\ n = -1$
for turbulent flow, $C = 0{\cdot}032,\ n = -0{\cdot}23$

Substituting these values in eq. (19)—

(*a*) for laminar flow, $k_D = 8\left(\dfrac{1}{R_e}\right)$

but the boundary-layer theory gives this value for a flat plate as

$$k_D = 1{\cdot}327\left(\frac{1}{R_e}\right)^{1/2} \quad \text{[eq. (8), § 16.4]}$$

which is not of the same form;

(*b*) for turbulent flow, substituting the values of C and n in eq. (19),

$$k_D = 0{\cdot}032\left(\frac{1}{R_e}\right)^{0{\cdot}23}$$

which compares favourably with the boundary-layer equations for a flat plate, namely—

$$k_D = 0{\cdot}072\left(\frac{1}{R_e}\right)^{1/5} \quad \text{[eq. (9), § 16.4]}$$

and $$k_D = 0{\cdot}0375\left(\frac{1}{R_e}\right)^{1/7} \quad \text{[eq. (10), § 16.4]}$$

A summary of these comparisons is given in the following table—

Type of Flow	Equivalent Values of k_D in Terms of R_e		
	Froude and Darcy	Poiseuille and Stanton-Pannell	Boundary-layer Theory
Laminar	a constant	$8\left(\frac{1}{R_e}\right)$	$1{\cdot}327\left(\frac{1}{R_e}\right)^{1/2}$
Turbulent	a constant	$0{\cdot}032\left(\frac{1}{R_e}\right)^{0{\cdot}23}$	$0{\cdot}072\left(\frac{1}{R_e}\right)^{1/5}$ to $0{\cdot}0375\left(\frac{1}{R_e}\right)^{1/7}$

It will be noticed from this article that the following are the relative values of the four frictional constants in use—

$$\text{Froude} = f'$$

Drag coefficient $\quad C_D = \frac{2f'}{\rho}$

Drag coefficient $\quad k_D = \frac{C_D}{2} = \frac{f'}{\rho}$

Darcy's coefficient $\quad f = C_D = 2k_D = \frac{2f'}{\rho}$

From this comparison it will be seen that the correct fluid resistance, which depends on the Reynolds number and on the thickness of the boundary layer, is given only by the boundary-layer theory. The Froude and Darcy formulae use the same constant for all types of flow and for all values of R_e; consequently, they hold over a very limited range. The viscosity equation takes into account the variation of R_e and the type of flow, but does not allow for the varying thickness of the boundary layer; hence, the range of its constants is also limited. The boundary-layer theory, as it allows for the variation of R_e and of the thickness of the layer, gives accurate results over a large range.

For the practical design problems of civil, mechanical and shipbuilding engineering, the results obtained from the Froude and Darcy equations are usually of sufficient accuracy. It is impossible to calculate the exact resistance of water pipes, when the lining is of varying roughness when new, contains irregularities at the joints, and is usually covered after use with encrustation and fungus of varying amounts. The resistance of a ship will also vary; its surfaces may be newly painted or they may be covered with variable amounts of fungus or other organic growths. These unavoidable changes make it impossible to apply any equation with close accuracy.

In aeronautical and other more scientific design problems the surface conditions are more constant; also, it is important that more exact calculations be made on account of the high degree of accuracy required.

EXERCISES 16

The following values of ρ and ν are to be used in these examples—

for air, $\quad \rho = \frac{0.075}{g}$ engineers' units; $\quad \nu = 0.000167$ engineers' units

for water, $\quad \rho = \frac{62.4}{g}$ engineers' units; $\quad \nu = 0.0000108$ engineers' units

1. A fluid is flowing past a flat surface with a velocity of 30 ft/sec. If the thickness of the boundary layer at a certain section is 2·5 in., find the velocity within the layer at a distance of 1·5 in. from the surface. Assume the flow is turbulent and that the velocity variation within the layer follows the "$\frac{1}{7}$ power law." (Eq. (2), § 16.2.) *Ans.* 27·9 ft/sec.

2. Calculate the Reynolds number and the thickness of the boundary layer of a flat plate, moving through water with a speed of 20 ft/sec, at a section 15 ft from the leading edge. The length of the plate is 60 ft. Assume the "$\frac{1}{7}$ power law"! (Eq. (7), § 16.3.) *Ans.* $1{\cdot}11 \times 10^8$; 4·6 in.

3. Calculate, from the boundary-layer theory, the Reynolds number and the drag coefficient for a flat plate 3 ft long immersed in an air stream of 60 m.p.h. Assume the "$\frac{1}{2}$ power law" to hold. (Eq. (9), § 16.4.) If the plate is 6 in. broad, find the total resistance at this speed.
Ans. $R_e = 1{\cdot}58 \times 10^6$; $k_D = 0{\cdot}00417$; $D = 0{\cdot}133$ Lb.

4. Using the boundary-layer theory, calculate the Reynolds number and the drag of the hull of a large airship when travelling at a speed of 60 m.p.h. The length is 812·7 ft and the diameter 135·4 ft, giving an area on chord of 91,000 ft². Use the "$\frac{1}{7}$ power law" for the value of k_D. (Eq. (10), § 16.4.) Find also the horse-power required to overcome the hull drag at this speed.
Ans. $R_e = 4{\cdot}28 \times 10^8$; $k_D = 0{\cdot}00219$; 577 h.p.

5. During a series of tests on a model in a wind tunnel the following values of k_D were obtained for the given values of R_e, in order to obtain the scale effect.

R_e	126,000	224,000	631,000	1,728,000
k_D	0·00398	0·00251	0·00158	0·00100

By plotting these values and their logarithms, find, by extrapolating, the value of k_D for the full-size object, the value of its Reynolds number being $4{\cdot}467 \times 10^6$. *Ans.* 0·000631.

6. Define "boundary layer." A thin plate 6 ft long and 1 ft wide moves at 10 ft/sec through water. The boundary layer is 2 in. thick at the rear end of the plate and the velocity distribution is approximately

$$\frac{u}{V} = \left(\frac{y}{\delta}\right)^{1/4}$$

Find the resistance of the plate and express it (*a*) in pounds, (*b*) as a pure number independent of δ. (*Lond. Univ.*) *Ans.* 8·62 Lb; 0·0074.

7. What is meant by "boundary layer"?
Water flows at 15 ft/sec past a thin rough board 1 ft 6 in. wide and 12 ft long. The boundary layer is 4 in. thick at the downstream end of the board. Neglecting pressure variations and assuming

$$\frac{u}{V} = \left(\frac{y}{\delta}\right)^{0}$$

within the boundary layer, find the drag and express it (*a*) in pounds; (*b*) as a pure number independent of δ. (*Lond. Univ.*) *Ans.* 51·8 Lb; 0·00662.

8. A ship-shaped body is immersed in a current. Sketch the general outline of the lines of flow past the body (*a*) at very low speeds, and (*b*) at high speeds. State briefly how the resistance depends upon the velocity, density, and viscosity of the fluid in each case. (*I.Mech.E.*)

9. Give a description of the boundary-layer theory and explain, with sketches, the following phenomena—(1) break-away of layer; (2) Kármán street; (3) the laminar sub-layer and its importance in practice; (4) three methods of boundary-layer control and their effects on practical problems. (*Lond. Univ.*)

CHAPTER 17

EFFECT OF SUPERSONIC SPEED

17.1. Introduction. When the relative velocity between a fluid and a surface in contact is equal to the pressure-wave velocity in the fluid, the relative velocity is said to be *sonic*. If the relative velocity is less than this amount it is termed *subsonic*, and if greater, *supersonic*. As the relative velocity approaches sonic speed there is a great increase in the surface-profile resistance due to the large increase in energy required for the formation of the pressure waves it causes. This resistance reaches its maximum value in the region of the sonic speed. After sonic speed has been exceeded the resistance begins to decline gradually. The range of relative velocity over which these changes take place is known as *transonic*; this range appears to commence when the Mach number of the flow is about 0·6.

The phenomena due to sonic and supersonic relative speed consist of the formation of pressure waves of great intensity which are called *shock waves*. The formation of these shock waves causes great increase in resistance, and their intensity can be so great as to produce a noise equal to that of an explosion, usually termed a supersonic bang. The effect of shock waves must be taken into account in the design of aeroplanes, pipe-flow and nozzle-flow problems for gases, and in the design of turbine and compressor blading. Shock waves may also form locally within the boundary layer in the transonic range.

17.2. Sonic Speed. The velocity of a pressure wave in a fluid was dealt with in § 13.10–§ 13.12, and it was shown that for an adiabatic process

$$\text{sonic velocity} = v_s = \sqrt{\frac{\gamma p}{\rho}} \quad \text{(eq. (26), Chapter 13)}$$

where $\rho = w/g$ and p = pressure in pounds per square foot.

But $w = 1/V$; hence

$$\rho = \frac{1}{Vg}$$

Substituting this value in the above equation,

$$v_s = \sqrt{\gamma g p V} \text{ ft/sec}$$

But $$pV = RT$$

Hence $$v_s = \sqrt{\gamma g R T} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence, for a given gas, v_s is a function of $\sqrt{T}$.

For atmospheric air, $\gamma = 1{\cdot}4$ and

$$R = 96 \text{ ft-Lb in Centigrade units}$$
$$= 53{\cdot}3 \text{ ft-Lb in Fahrenheit units}$$

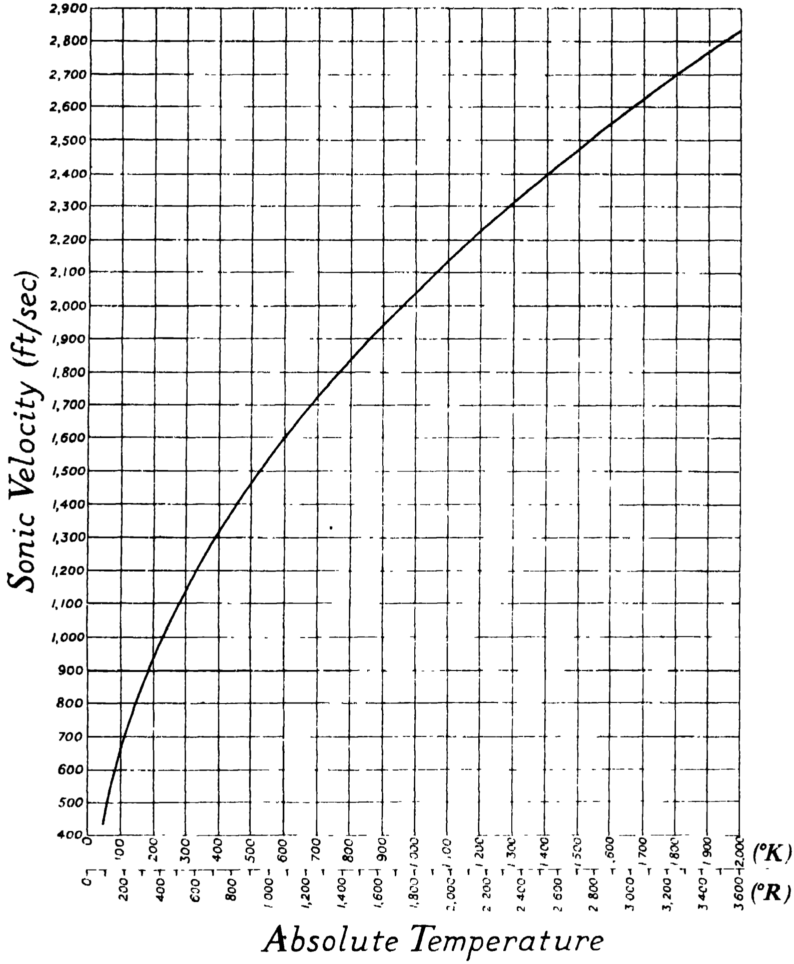


FIG. 214. VELOCITY OF SOUND IN AIR

Substituting these values in eq. (1),

$$v_s = \sqrt{1{\cdot}4 \times 32{\cdot}2 \times 96T}$$
$$= 65{\cdot}8\sqrt{T}, \text{ where } T \text{ is in °K} \quad . \quad . \quad (2)$$
$$= \sqrt{1{\cdot}4 \times 32{\cdot}2 \times 53{\cdot}3T}$$
$$= 49\sqrt{T}, \text{ where } T \text{ is in °R} \quad . \quad . \quad (3)$$

Eqs. (2) and (3) are of sufficient accuracy for engineering problems; they are approximate only, as they do not allow for the small variations of γ and R with temperature.

Fig. 214 gives the variation of v_s with absolute temperature for atmospheric air. This was plotted from eq. (1), and takes into account the variations of γ and R with temperature.

17.3. Shock-wave Photography. Shock waves in a gas can be photographed, or viewed on a ground-glass screen, by a method

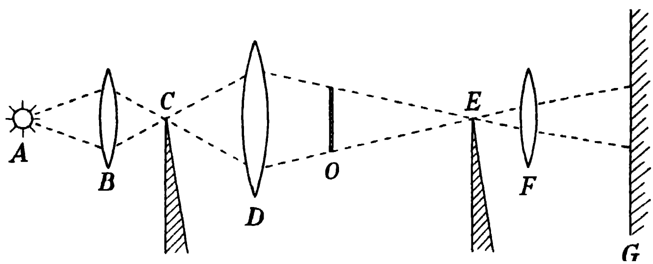


FIG. 215

known as the *Schlieren* method. This method makes use of the principle that the refraction of a ray of light passing through a gas is a function of the density of the gas. Thus, a shock wave, which varies in density across its width in proportion to its pressure, will cause the rays of light to deviate at various angles according to the density of the gas through which they pass.

The method is shown in the diagram of light rays of Fig. 215; this particular method is due to Toepler and is known as the Toepler-Schlieren method. A is a source of light which is concentrated by the lens B on to a sharp-edged screen C which cuts off part of the light. An adjustable convex lens D throws a sharp image of the spot of light at the edge of the screen C on to a second screen E having a sharp edge parallel to the edge of the first screen C. The sharp edge of E is placed so that a narrow ray of light passes over the edge; the remainder of the light being cut off.

An object O is now placed in the path of the rays between D and E. The object O may be a gas flowing past a surface which causes pressure waves of varying density in the gas. Then the rays of light passing through the gas will be deviated, the angle of deviation being proportional to the density gradient in a direction at right-angles to the sharp edge. If the flow of the gas is from left to right, a decreasing density from left to right gives, say, brightness; then brightness represents an expansion of the gas. Increasing density from left to right will then produce darkness; hence, darkness denotes a compressing of the gas. Thus, darkness on a Schlieren

photograph represents an area of increasing pressure and brightness an area of decreasing pressure.

Referring again to Fig. 215, the rays from E are next concentrated by the convex lens F on to the photographic plate.

A simpler method of photographing shock waves and other disturbances in the fluid is to throw an intense light through the gas and to photograph the shadow thus formed on a screen or photographic plate. The screen or plate is situated on the opposite side

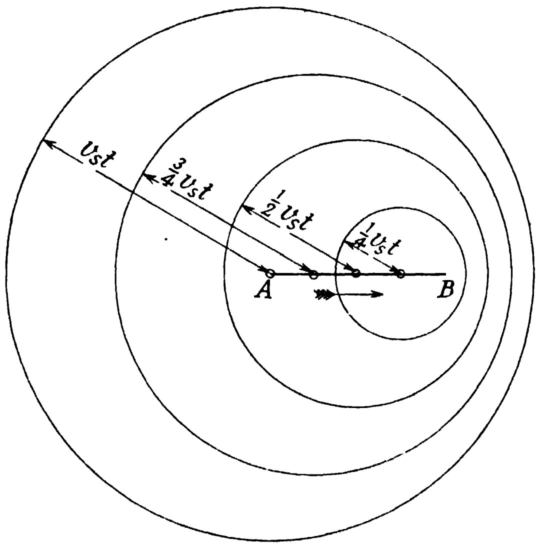


Fig. 216

of the gas to that of the source of light, so that the rays of light penetrating the disturbed gas are deviated in proportion to the refractive index of the gas through which they pass. This, in turn, depends on the pressure of the gas. Hence, the shadow formed varies in intensity with the pressure of the gas penetrated by the light rays. The photograph obtained from this method is known as a *shadowgraph*.

Schlieren photographs and shadowgraphs of shock waves and of weaker pressure waves are shown in Figs. 220 to 223 and in Figs. 232 and 233.

17.4. The Mach Wave. The Mach wave is the name given to the shock wave in a fluid caused by the profile resistance of a surface in contact with the fluid, which occurs when the relative Mach number of the flow reaches unity. The Mach wave from the nose of a body travelling at supersonic speed is analogous to the bow wave of a surface ship; it is named after Professor Ernst Mach who was

the first to provide the explanation of its causes. He produced the following explanation of this phenomenon.

1. Body Moving at Subsonic Speed in a Fluid. Let

v = velocity of body in fluid,

v_s = velocity of sound in fluid.

Referring to Fig. 216, let the point A represent the position of the nose of a body moving through a fluid, from left to right, with a

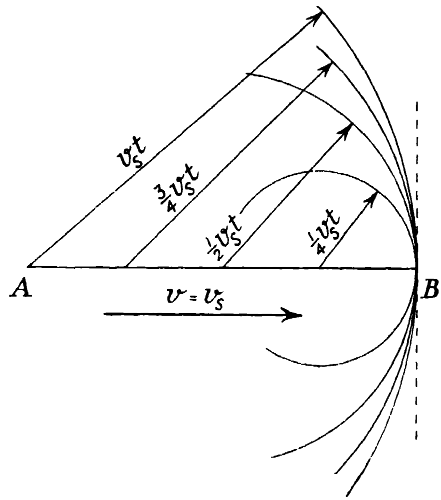


Fig. 217

subsonic velocity. The nose of the body impinging on the fluid causes a disturbance in the form of a spherical pressure wave which travels radially outwards with the velocity of sound. After a short interval of time t the nose of the body reaches B and has caused other pressure waves to radiate outwards as it travelled along the path AB.

The pressure wave from A reached a radius of $v_s \times t$ when the body is at B. When the nose of the body had traversed one-quarter of the distance AB a pressure wave was radiated which reaches a radius of $\frac{3}{4} v_s t$ when the nose is at B. In the same way, other pressure waves have reached radii of $\frac{1}{2} v_s t$ and $\frac{1}{4} v_s t$ respectively when the nose of the body has traversed one-half and three-quarters of the distance AB. The pressure waves are thus in the positions shown in the figure when the nose of the body reaches B.

It will be seen from the figure that the nose of the body is always behind the wave front of the pressure waves it has caused. The body is thus always penetrating an area of disturbed fluid; this has an effect on the fluid resistance to the body's motion.

2. **Body Moving at Sonic Speed in a Fluid.** If the body moves with sonic velocity from A to B the position of the pressure waves when the nose reaches B is as shown in Fig. 217. The construction of this figure is the same as that of Fig. 216 except that the body's velocity v is now equal to v_s. In this case it will be noticed that all the pressure waves propagated during the body's passage from A to B have a common tangent at B. This concentration of the pressure waves at B causes a great increase in the

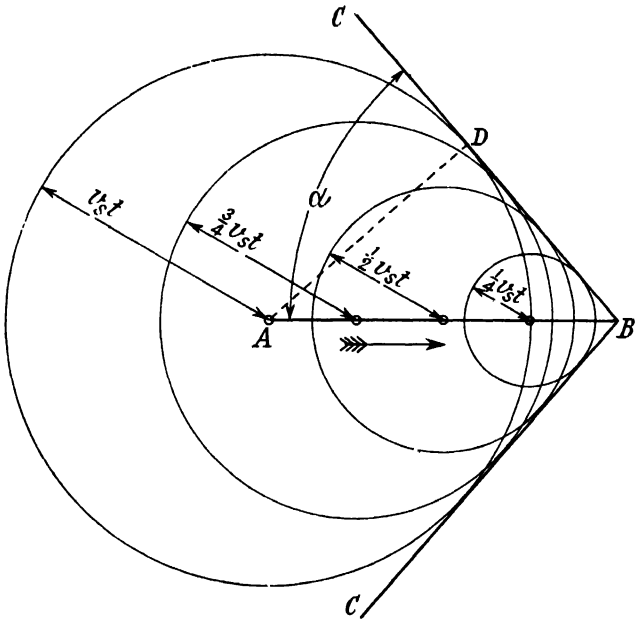


Fig. 218

intensity of pressure at that point. Hence, the nose of the body pushes against this wave of intense pressure which moves along with it and is known as a shock wave. This causes a great increase in the head resistance. It should be noted that in this case the Mach number of the relative velocity equals v/v_s, which is unity.

3. **Body Moving at Supersonic Speed in a Fluid.** Now consider the body when its velocity is supersonic; the wave disturbance it now causes in the fluid is shown in Fig. 218. Consider the instant when the nose of the body is at A; its velocity v is now greater than the wave velocity v_s. Let the nose reach the point B in the time t. Then,

$$AB = vt$$

When the nose of the body was at A a spherical pressure wave was caused which moved radially outwards from A with a velocity v_s;

this wave has travelled a distance of $v_s t$ by the time the nose reached B. When the nose reached distances of $\frac{1}{4}AB$, $\frac{1}{2}AB$ and $\frac{3}{4}AB$ from A, other pressure waves were caused which, at the time the nose reached B, had radiated to distances of $\frac{3}{4}v_s t$, $\frac{1}{2}v_s t$ and $\frac{1}{4}v_s t$ respectively. Hence, these spherical wave fronts are in the position shown in the figure when the nose is at B. It will be noticed that the common tangent to all these spherical waves is a straight line in the elevation view, and actually forms a conical surface with the apex at the nose of the body.

Draw the straight line CB as a common tangent to all the pressure waves shown; let α be the inclination of this line to the line AB. The line CB forms the common wave front advancing with the nose of the body into the undisturbed fluid and is known as the *Mach wave*; it is also classified as a shock wave. This shock wave is a wave of very intense pressure which moves along with the nose of the body, and it may cause a sound equal to that of an explosion when striking the human ear.

Let D be the tangent point on CB of the pressure wave radiating from A (Fig. 218); then,

$$\sin\alpha = \frac{AD}{AB}$$

$$= \frac{v_s t}{vt}$$

$$= \frac{v_s}{v}$$

The angle α is known as the Mach angle. It will be noticed that $\sin\alpha$ is the inverse of the Mach number.

A body moving through a fluid at a supersonic speed will have a similar Mach wave emanating from any projection of its surface; even the projections of the surface roughness may cause weak Mach waves [Figs. 220 (*b*) and 220 (*c*)]. Any abrupt change in the surface shape, any abrupt projection, or a blunt trailing edge, will each cause its own Mach wave, but they are of weaker intensity than that due to the nose; these can be seen in the photographs of Fig. 220 (*a*), 220 (*b*) and 220 (*c*).

Consider the projectile shown in Fig. 219 travelling at a supersonic speed v. AB is the Mach wave at the nose inclined at the Mach angle α to the direction of motion, such that

$$\sin\alpha = \frac{v_s}{v} = \frac{1}{M_a}$$

The projectile has a shoulder projecting at C. The motion of the body will tend to leave a vacuum pressure in the wake of this shoulder. This causes a local increase in the relative velocity to v_1 at C due to the fluid flowing into the vacuum space in the wake of

C. Hence, there is an increase in the local Mach number and a corresponding decrease in the Mach angle α_1 at *C*. From this it follows that the Mach wave *CD* emanating at *C* has a smaller slope than that at the nose *A*.

A similar decrease of slope occurs at the stern *E*, where the local relative velocity v_2 is increased by the fluid flowing into the vacuum left behind in the wake of the projectile. Hence, the Mach wave at *E* has a smaller Mach angle α_2. The Mach wave *EF* from *E* can be

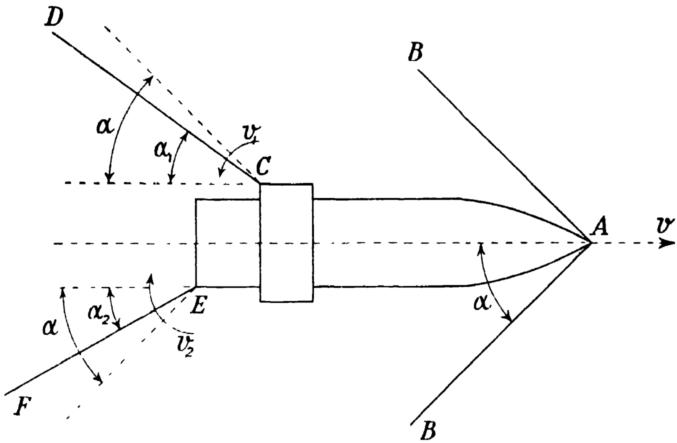


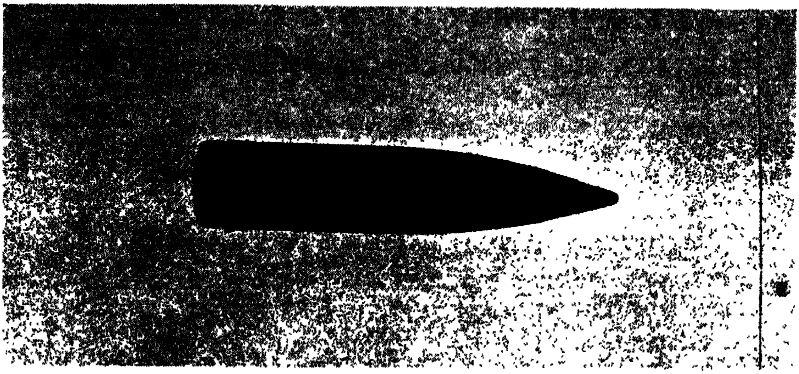
FIG. 219

very intense and may cause the sound of a second explosion. This fact is considered by many authorities to be the cause of the double bang which is heard when an aeroplane travelling at supersonic speed passes overhead at a certain altitude. The first bang is caused by the Mach wave from *A*, the second by the Mach wave from *E*; the time interval between them being due to the difference of their Mach angles and the altitude. The difference between the Mach angles of the nose and tail of a body in supersonic flight appears to vary between 10° and 30°, depending on the shape of the tail.

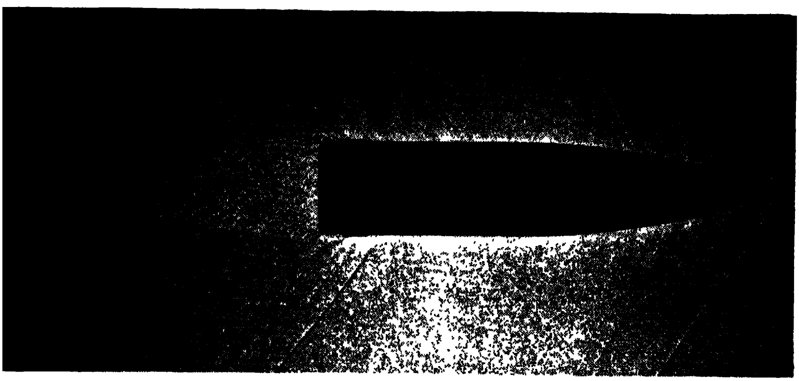
A photograph of a projectile travelling with a Mach number of 1·26 is shown in Fig. 220 (*b*). Mach waves of varying intensity can be seen in the photograph, which shows the difference in their values of α. The eddies shown in the wake of the body represent the disturbance caused by the air flowing into the space left behind by the projectile.

In Fig. 221 is shown the photograph* of the shock waves from a blade of a two-bladed aeroplane propeller when moving at such a speed that the blade-tip velocity was supersonic, having a Mach

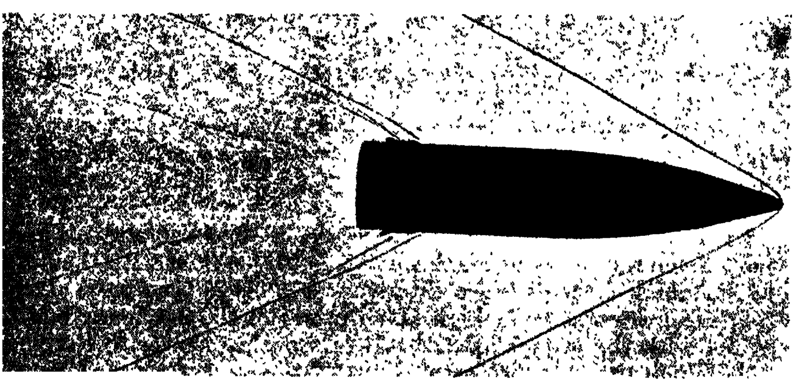
* See paper by Dr. W. F. Hilton, *Proc. Roy. Soc.*, **169**, p. 174.



(*a*) Velocity 1,145 ft/sec: Mach number 1·01.



(*b*) Shadowgraph of 1 in calibre projectile in flight. Velocity 1,425 ft/sec: Mach number 1·26.



(*c*) Velocity 2,830 ft/sec: Mach number 2·51.

FIG. 220. PROJECTILES IN FLIGHT

(*Courtesy of The Institution of Mechanical Engineers*)

number of 1·21. The direction of rotation is anti-clockwise. S is the supersonic shock wave from the leading edge and T is the trailing vortex wave. V_1 is the vortex from the blade tip, V_2 is the blade-tip vortex from the other blade occurring half of a revolution previously, and V_3 is the blade-tip vortex of the blade shown, due to the

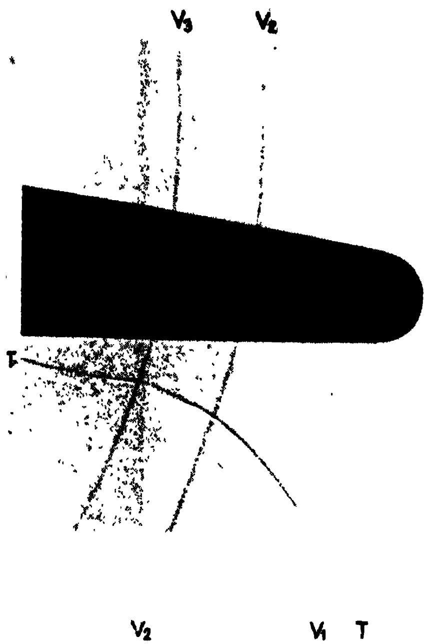


FIG. 221

previous revolution. The eddies in the wake of the blade can be clearly seen.

Oblique shock waves, inclined at the Mach angle, also occur in nozzles through which a jet of fluid is flowing with a supersonic velocity. These are shown in Figs. 233 (*b*) and 233 (*c*), which are Toepler-Schlieren photographs.

It will be noticed from the photographs of Figs. 220 and 221 that at low Mach numbers the Mach wave is not in contact with the nose, but travels a short distance in advance. This is true only at the lower Mach numbers; at high Mach numbers the Mach wave is in contact with the nose (Fig. 222). It approaches nearer to the nose as the Mach number increases. Prandtl states that the reason for the Mach wave being in advance of the nose is because the nose pushes a volume of fluid ahead of it, thus forming a false nose of compressed fluid travelling at the same speed as the projectile; the

Mach wave then occurs at the tip of the false nose. With high Mach numbers the compressed fluid is swept away and the Mach wave is then in contact with the actual nose.

It will be seen from Fig. 220 (*a*) that at a Mach number of 1·01 the Mach angle is 90° and the nose Mach wave is well in advance of

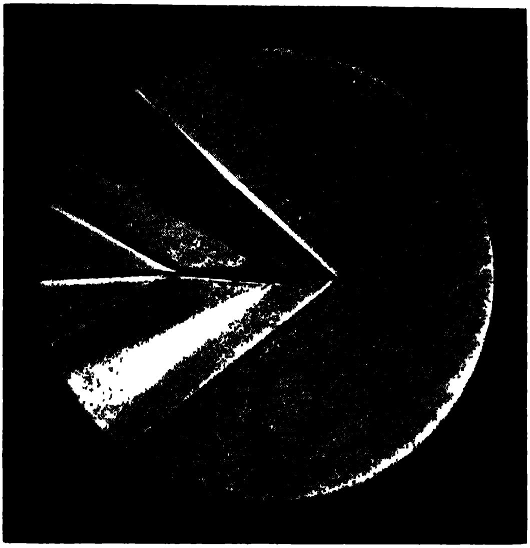


FIG. 222

the nose. In Fig. 220 (*c*) the Mach number of the flight is 2·51 and the Mach wave is almost in contact with the nose.*

In Fig. 222 is shown a Schlieren photograph of the flow past a model double-wedge aerofoil at a Mach number of 1·8. Notice the difference in the Mach angle between the Mach waves from the nose and the tail.†

17.5. The Pressure Jump. This is another type of shock wave which occurs in pipes and nozzles (§ 19.8) when the flow abruptly

* Figs. 220 (*a*) and 232 are reproduced by courtesy of Mr. K. W. Todd of the Gas Turbine Engineering Dept., Metropolitan-Vickers Electrical Co., Ltd. For details see paper "Some developments of instrumentation for air-flow analysis" by K. W. Todd, *Proc. Instn. Mech. Engrs.*, **161** (1949).

Figs. 220 (*b*) and 220 (*c*) are reproduced by courtesy of Mr. W. F. Cope and the Director of the National Physical Laboratory. For details see paper "The flow of gases at sonic and supersonic speeds" by G. A. Hankine and W. F. Cope, *Proc. Instn. Mech. Engrs.*, **155** (1946).

† By courtesy of Mr. F. G. Irving, of the Imperial College of Science and Technology. From paper "A small supersonic wind tunnel" by F. G. Irving, *The Guilds Engineer* (1953), p. 57.

changes from a supersonic flow to a subsonic flow, the change being accompanied by a sudden rise in the pressure of the fluid. The jump is a stationary wave, perpendicular to the direction of the fluid's flow, and is primarily due to the choking effect of a relatively high back pressure. It is analogous to the hydraulic jump in the

FIG. 223

channel flow of liquids (§ 10.11). A Schlieren photograph of a pressure jump* in the diverging cone of a nozzle is shown in Figs. 223 and 233 (*a*).

The pressure jump is accompanied by a relatively large increase in entropy, thus showing it to be an irreversible process. A relatively large amount of frictional reheating occurs during the jump, owing to the disturbance it causes (§ 19.10).

Pressure jumps may also occur within the boundary layer owing to a local supersonic speed; these jumps are approximately perpendicular to the surface (Fig. 232).

17.6. Effect on Lift Coefficient of Aerofoils. The Mach number over the transonic range has a great influence on the lift coefficient of aerofoils. A large reduction in the value of the lift coefficient C_L occurs after M_a reaches the value of about 0·6. When the sonic speed is reached the value of C_L may have been reduced to about one-quarter of its low-speed value, and for some aerofoils it may even become negative.

In Fig. 224 is shown an aerofoil section known as a bi-convex, or knife-edge, aerofoil; this type of aerofoil has been developed for high-speed aeroplane wings and gives good results for high values of the Mach number. The values of C_L for this aerofoil are shown plotted on a Mach-number base for angles of incidence of 7° and 0·5°. It will be noticed that for an angle of incidence of 7° the value of C_L does not fall below 0·22 for a Mach number of 1·8.

In Fig. 225 is shown the value of C_L plotted on a Mach-number base for a high-speed aeroplane's swept-back wing of the supersonic

* By courtesy of Dr. R. D. Tylor, of the Imperial College of Science and Technology, South Kensington.

type; it is plotted over a Mach-number range of 0 to 1·6. The value of C_L plotted is the maximum value for all angles of incidence below

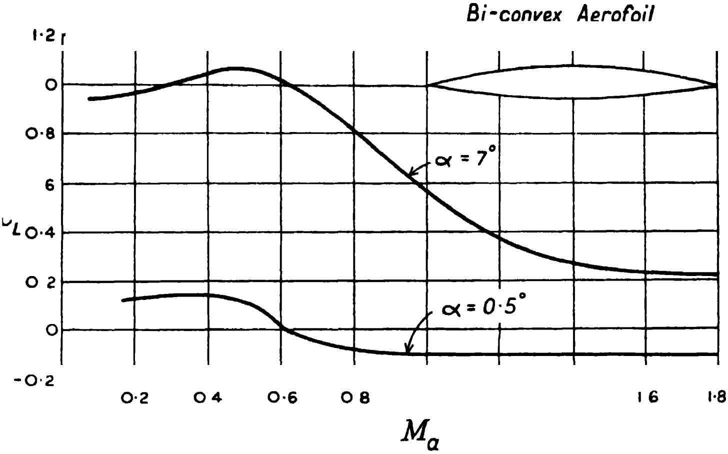
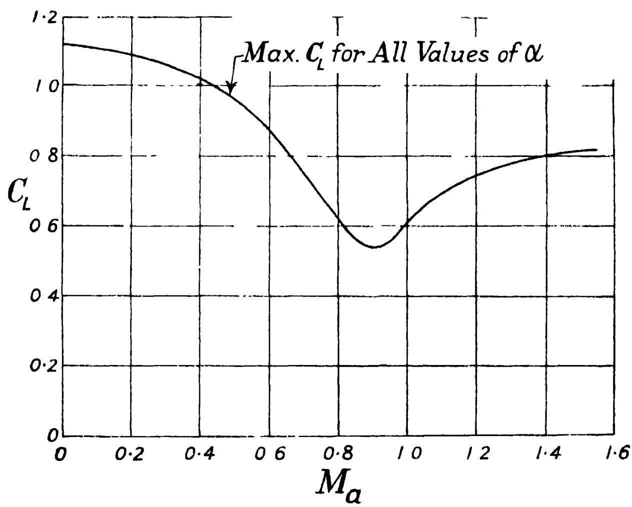


FIG. 224

the stalling angle. It will be noticed from the curve that the lowest value of C_L is about 0·55. This is about one-half of the maximum



SWEPT-BACK WINGS

low-speed value, thus showing the improvement obtained by the modern supersonic swept-back wings. It will also be noticed that the minimum lift occurs when M_a is about 0·9, after which the value

increases with the speed, but it has not recovered its low-speed value when M_a is 1·6.*

In Fig. 226 is shown the ratio of the actual lift coefficient to its low-speed value, over the transonic range, for an ordinary aerofoil.

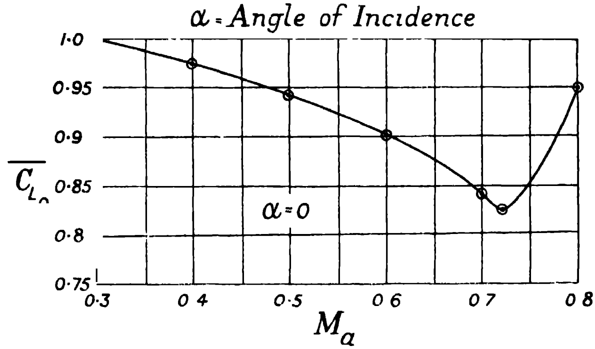


FIG. 226

Let C_{L_0} = lift coefficient for low speed,

C_L = actual lift coefficient at a given value of M_a.

It will be noticed that the ratio plotted, C_L/C_{L_0}, is non-dimensional, so that the curve obtained should hold for the whole class of

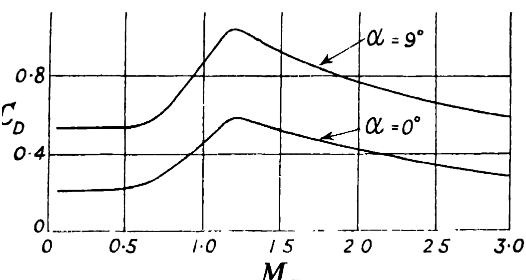


FIG. 227. VALUES OF C_D FOR GERMAN ROCKET PROJECTILE "WASSERFALL"

aerofoils used in the tests. From the curve the lowest value of C_L occurs when $M_a = 0{\cdot}72$ and its value is about 0·85 of the low-speed value.

17.7. Effect on Drag Coefficient of Aerofoils. The drag coefficient of an aerofoil may increase to as much as nine times its slow-speed value, over the transonic range; it then reduces slowly as the supersonic speed increases.

* The results shown in Figs. 225 and 229 were obtained by the research department of the Lockheed Corporation (U.S.A.).

The variation in the value of C_D with the Mach number, for the German rocket-driven projectile "Wasserfall," is shown plotted in Fig. 227 for two values of the angle of incidence α. The "Wasserfall" was a guided missile fitted with aerofoil-section wings. It will be

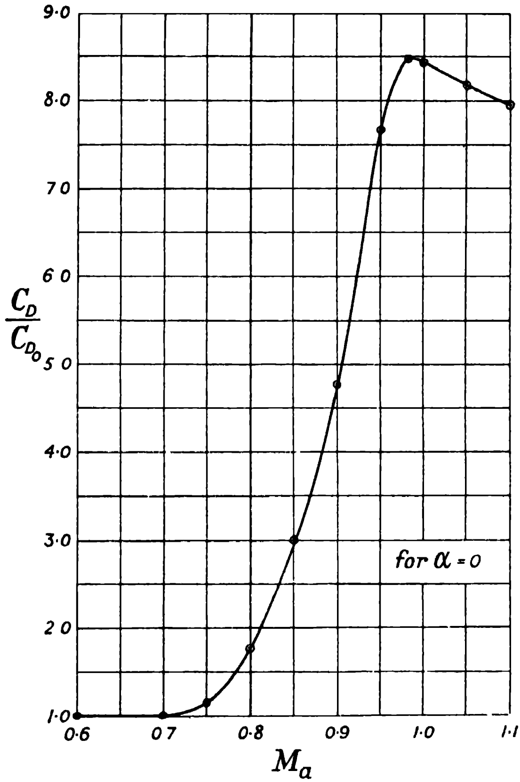


FIG. 228

noticed that the two curves remain almost parallel and that the maximum value of C_D occurs when $M_a = 1{\cdot}2$, after which it begins to decrease as M_a is increased. Starting from rest, the value of C_D was approximately constant up to $M_a = 0{\cdot}6$ and its maximum value, at $M_a = 1{\cdot}2$, was five times its slow-speed value.

This increase in the drag coefficient causes a large increase in the bending moment on the wings of an aeroplane and has been the cause of many structural failures of aeroplanes reaching sonic speed.

Starting from rest, the drag coefficient of an aerofoil falls slightly up to a Mach number of 0·6. After passing this value of M_a the drag coefficient then increases rapidly, reaching its maximum value at approximately the sonic speed of the fluid. This is because of the

large compression wave resistance due to their wave fronts closing up to a common tangent (Fig. 217). When $M_a = 1$, the Mach wave is perpendicular to the direction of motion; hence, the Mach angle is 90° and it thus exerts its maximum resistance. In the supersonic range, as the value of M_a increases the Mach angle gets less, and the Mach wave is in the shape of a cone travelling ahead of the nose of

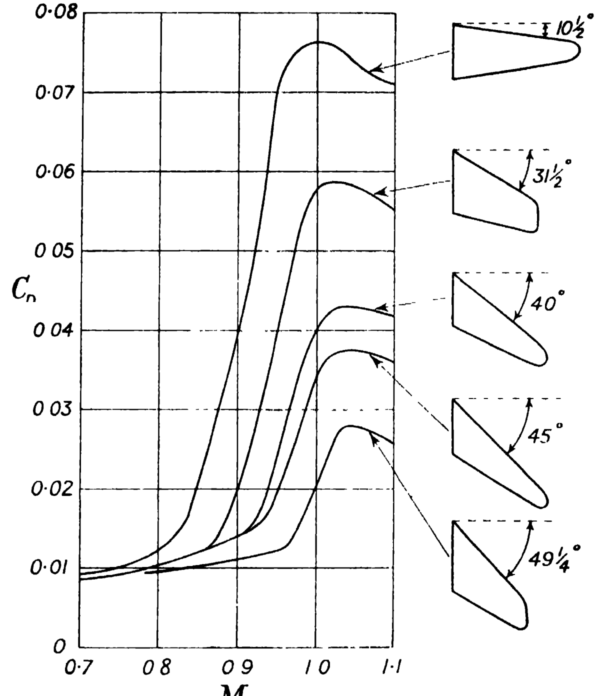


FIG. 229. EFFECT OF LEADING-EDGE SWEPT-BACK ANGLE ON C_D FOR VARIOUS VALUES OF M_a

the body. It thus offers less relative resistance as the angle of the cone decreases.

Let C_D = drag coefficient at a given Mach number,

C_{D_0} = drag coefficient at slow speed ($M_a < 0·6$).

In Fig. 228 is plotted the non-dimensional ratio C_D/C_{D_0} on a base of M_a. These results are from a test on a modern supersonic aerofoil. It will be noticed that the drag has its maximum value when $M_a = 0·98$; it has then been increased in magnitude to 8·5 times its slow-speed value.

Fig. 229 shows the variation of C_D with M_a for aeroplane wings swept backwards at various angles. It will be noticed that the largest

swept-back angle produces the smallest increase in C_D at the sonic speed.

17.8. Effect on Turbine and Compressor Blading. The blading of turbines and compressors can be regarded as a series of aerofoils, acting in parallel, if the pitch/chord ratio is greater than 0·6. If this

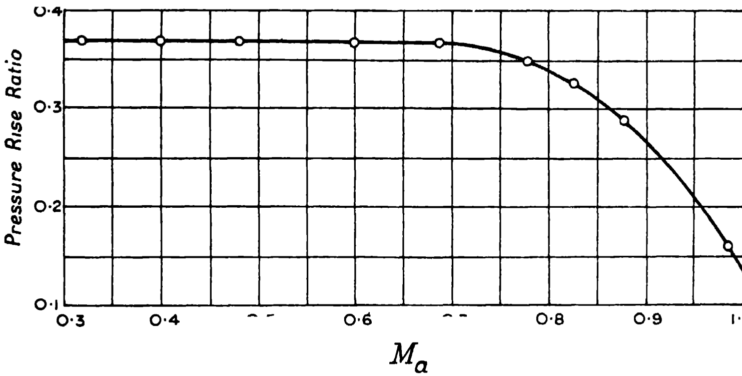


FIG. 230

ratio is less than 0·6 the blading is best designed on the momentum equation [eq. (4), Chapter 6]; it is then known as momentum blading. If the ratio is between 0·6 and 0·8, the momentum equation or the aerofoil equation [eq. (5), Chapter 15] can be used with equal accuracy, as they both give the same results. But if the ratio is greater than 0·8 the blades are too far apart for the fluid passing between them to be regarded as a curved jet flowing parallel to the blade surfaces. In this case the momentum equation does not give a good result; but the blades can be regarded as aerofoils in parallel, as demonstrated in § 15.6–§ 15.9, and a satisfactory solution obtained. A four-bladed Kaplan water turbine (§ 22.16) can be solved by this method.

If the Mach number of the fluid flow relative to the blades of a turbine or compressor is greater than 0·6, it is in the transonic range and there is a great increase in the drag coefficient (§ 17.7) and a large reduction in the lift coefficient (§ 17.6). These changes reduce the efficiency of the blading by a considerable amount.

The effect of the influence of the Mach number on the output of an axial-flow air compressor is shown in Figs. 230 and 231. These tests were carried out in a wind tunnel on a grid of the blades used in the compressor. In Fig. 230 is shown the measured pressure rise ratio plotted on a base representing the Mach number of the air flow through the blades. It will be noticed that the output commences to fall rapidly after a value of $M_a = 0·6$ is passed. The

loss of energy ratio for the same test is shown plotted, on a Mach-number base, in Fig. 231; a large reduction in efficiency is noticeable after a value of $M_a = 0{\cdot}6$ is reached. These changes can be explained by the fall in the lift coefficient and the increase in the drag coefficient which occurs after a Mach number of about 0·6 has been reached. On this account it is desirable in designing high-speed

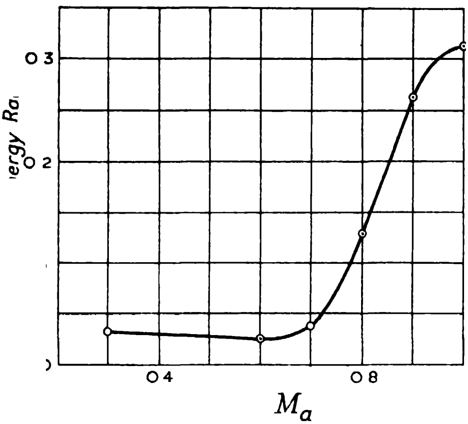


FIG. 231

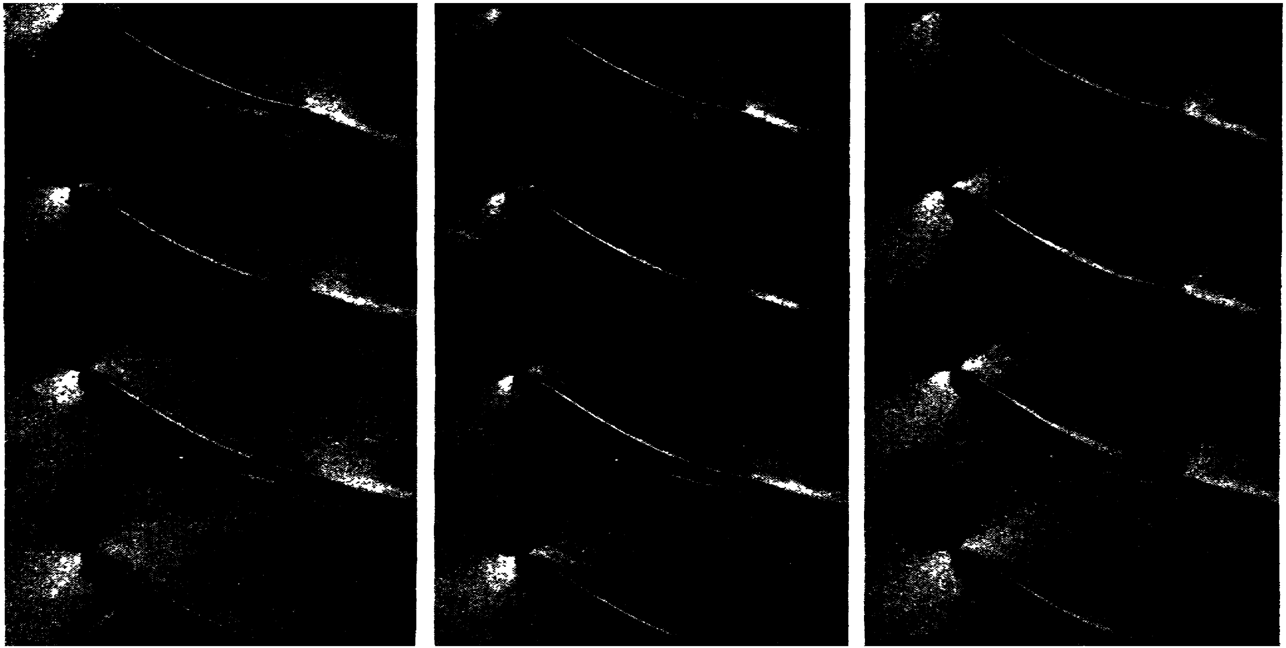
axial-flow compressors and gas turbines to keep the Mach number of the flow within this limit.

In Figs. 232 (*a*), (*b*) and (*c*) are shown Schlieren photographs of the shock waves occurring in the fluid flowing between the blades of a grid of blading during a test in a wind tunnel.* In Fig. 232 (*a*) the Mach number of the flow at inlet was 0·63, in Fig. 232 (*b*) $M_a = 0{\cdot}76$, and in Fig. 232 (*c*) M_a at inlet was 0·84. The photographs show the formation of shock waves and fluid pressure variations between the blades within this transonic range. Within the blade passages the local Mach numbers of the flow may be greater than unity, thus making it possible for shock waves to form; this causes losses in efficiency and output.

If the angle of incidence of the blades is too large, the blades may be acting as stalled aerofoils (Fig. 203) owing to the separation of the boundary layer (§ 16.5). Once the blading has stalled the machine will no longer work efficiently; in fact, it may not work at all.

17.9. Effect on Pipe Flow and Nozzles. It will be shown in Chapter 18 that the fluid flow through a pipe cannot exceed a Mach number of unity at the outlet end on account of choking (§ 18.4).

* See first footnote on page 445.



(*a*) Inlet Mach number = 0 63. (*b*) Inlet Mach number = 0 76. (*c*) Inlet Mach number = 0 84.

FIG. 232. TOEPLER-SCHLIEREN PHOTOGRAPHS OF SHOCK WAVES

(*Courtesy of The Institution of Mechanical Engineers*)

(a)

(b)

(c)

FIG. 233. TOEPLER-SCHLIEREN PHOTOGRAPHS OF JET OF AIR DISCHARGING FROM A CONVERGING-DIVERGING NOZZLE

(Courtesy of The Director of the National Physical Laboratory)

When gases are expanded through nozzles the velocity at the throat cannot exceed the sonic velocity of the gas (§ 14.7). In the converging-diverging nozzle, if the flow is supersonic beyond the throat, oblique shock waves will occur in the diverging cone owing to the change of slope of the internal surface at the throat. These are Mach waves and are inclined at the Mach angle; they cause other oblique and reflected shock waves along the length of the diverging cone, which are reflected from the sides of the nozzle. The formation of these waves is explained by the Prandtl-Meyer equation.*

Toepler-Schlieren photographs† of a jet of air discharging from the mouth of a converging-diverging nozzle are shown in Fig. 233. Let p_1 be the pressure in the mouth of the nozzle and p_2 the pressure of the atmosphere into which the jet is discharging.

In Fig. 233 (*a*) $p_2 > p_1$. The flow in the diverging cone is supersonic. It will be noticed that a pressure jump occurs just inside the nozzle mouth, after which the flow is subsonic.

In Figs. 233 (*b*) and 233 (*c*) $p_2 < p_1$. Oblique shock waves and reflected shock waves can be seen in the discharging jet. The Mach number of the flow was 1·43 for views (*a*), (*b*) and (*c*). For view (*b*), $p_1/p_2 = 1{\cdot}09$; and for view (*c*), $p_1/p_2 = 1{\cdot}50$.

EXERCISES 17

1. An aeroplane is flying at 550 m.p.h. at a high altitude where the atmospheric temperature is $-70°$C. Calculate the sonic velocity at this altitude and the Mach number of the aeroplane's flight.

Ans. $v_s = 938$ ft/sec; $M_a = 0{\cdot}86$.

2. In a cold air expansion turbine used for producing liquid oxygen, the cold air leaves the guide vanes at a temperature of $-242°$F and with a velocity of 896 ft/sec. Calculate the sonic velocity of the air and the Mach number of its flow.

Ans. $v_s = 725$ ft/sec; $M_a = 1{\cdot}23$.

3. Gases issue from the guide vanes of a gas turbine at a temperature of 1,136°F and with a velocity of 1,770 ft/sec. Calculate the sonic velocity of the gas leaving the guide vanes and the Mach number of its flow.

Ans. $v_s = 1{,}960$ ft/sec; $M_a = 0{\cdot}902$.

4. Calculate the lift obtained and the h.p. required to drive an aeroplane at a speed of 350 m.p.h. The chord area of each of the two wings is 80 ft², and the air resistance of all parts other than the two wings is equal to 30 per cent of the total wing resistance. The propulsion efficiency of the propeller is 80 per cent.

The lift and drag coefficients, C_L and C_D, at the angle of incidence of the flight are 0·36 and 0·02 respectively. The atmospheric pressure and temperature at the altitude of the flight are 10 Lb/in.² and 240°K respectively.

What is the Mach number of the plane's motion at this speed?

($pV = 96wT$, $\gamma = 1{\cdot}4$, $v_s = \sqrt{(\gamma p/\rho)}$). (*Lond. Univ.*)

Ans. $L = 6{\cdot}56$ tons; 1,240 h.p.; $M_a = 0{\cdot}503$.

* See *Modern Developments in Fluid Mechanics*, Vol. I.

† From National Physical Laboratory Note "Preliminary experiments with two-dimensional supersonic jets" by D. W. Holder and A. Chinneck.

5. A rocket has a speed of 3,000 m.p.h. and passes through the atmosphere where the pressure is 0·5 Lb/in.^{2} and the temperature 220°K. Calculate the Mach number of the rocket's flight and show by a graphical construction, drawn to scale, the shock wave from the nose of the rocket. What is the Mach angle of the shock wave? *Ans.* $M_a = 4{\cdot}505$; $\alpha = 12°\ 50'$.

6. A streamline body moves through the atmosphere with a velocity of 1,500 ft/sec; the pressure of the atmosphere is 15 Lb/in.^{2}, the temperature 20°C, and γ is 1·4. If the velocity of sound in a gas is given by the equation $v_s = \sqrt{(\gamma p/\rho)}$, and the characteristic equation for air is $pV = 96wT$, calculate the Mach number of the body's motion.

Show, by a graphical construction carefully drawn to scale, the shock wave caused by the nose of the body, and state the value of the Mach angle in degrees. (*Lond. Univ.*) *Ans.* $M_a = 1{\cdot}33$; $\alpha = 48{\cdot}85°$.

7. Obtain by dimensional analysis an expression for the compression wave resistance of a body of length L moving with a velocity v in a gaseous fluid of absolute density ρ and having a bulk elastic modulus of K.

A rocket travels at 3,600 m.p.h. through the stratosphere where the absolute pressure is 0·4 Lb/in.^{2} and the temperature -70°F. What is the Mach number of the rocket's flight?

Draw to scale, the shock wave at the nose of the body assuming the nose to be knife-edged, and state the value, in degrees, of the Mach angle. $pV = 53{\cdot}3T$ in Fahrenheit units and $\gamma = 1{\cdot}4$. Velocity of sound $= \sqrt{(\gamma p/\rho)}$. (*Lond. Univ.*) *Ans.* $M_a = 5{\cdot}46$; $\alpha = 10{\cdot}55°$.

CHAPTER 18

FLOW OF GASES THROUGH PIPES

18.1. General Case for Frictional Flow through Lagged Pipe. The adiabatic flow of gases through lagged pipes, due to a considerable pressure drop and subjected to frictional resistance, presents a very complex problem. There is a large increase in specific volume of the gas as it flows along the pipe, due to the pressure drop and to the frictional reheating. The increase in specific volume is accompanied by a corresponding decrease in density; this causes a large increase in velocity towards the low-pressure end of the pipe because the weight of flow per second through any given section must remain constant. The pressure drop is absorbed in overcoming the frictional resistance and in providing the accelerating force necessary to cause the increase of velocity.

If the pressures at the two ends of the pipe and the temperature at inlet are known, it is possible to calculate the velocity, temperature, and pressure at any part of the pipe and the weight of flow per second. This can be done by the application of the four fundamental equations governing the flow of gases, namely, the momentum equation, the energy equation, the characteristic equation of a gas, and the law of continuity of flow.

Let a known gas flow through a pipe of length l, diameter d, and cross-sectional area a, as shown in Fig. 234. Let the suffixes 1 and 2 apply to the inlet and outlet ends respectively. The pressures p_1 and p_2 and the absolute temperature T_1 are known.

Consider a short length dx of the pipe at a distance x from the inlet end. Let p, T, v, V, and w be the pressure, absolute temperature, velocity, specific volume, and density at the left of the short length of pipe. Let these change to $p + dp$, $T + dT$, $v + dv$, and $V + dV$ over the length dx.

The frictional drag, D_f, on the inside surface of the pipe can be obtained from the equation of § 15.2,

$$D_f = \frac{C_f \rho A v^2}{2}$$

Let f = Darcy's frictional coefficient used in the equation $h_f = 4flv^2/2gd$. It was shown in § 16.11 that $C_f = f$; then,

$$D_f = \frac{fw\pi dlv^2}{2g} \text{ for whole pipe}$$

$$= \frac{f2\pi d^2}{4} \times \frac{lv^2}{Vgd}, \text{ as } w = \frac{1}{V}$$

$$= \frac{2falv^2}{gVd}$$

Applying this equation to the short length of pipe dx,

$$dD_f = \frac{2fav^2}{gVd} \times dx$$

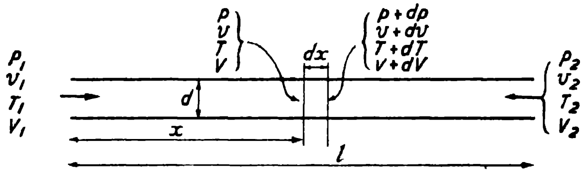


FIG. 234

Applying the momentum equation to the short length of pipe dx,

force due to pressure difference

= rate of change of momentum

+ frictional drag

Then

$$[p - (p + dp)]a = \frac{wav}{g}[(v + dv) - v] + \frac{2fav^2}{gVd} \times dx$$

Hence
$$- dp = \frac{v\,dv}{Vg} + \frac{2fv^2}{gVd} \times dx$$

or
$$-\frac{dp}{dx} = \frac{v}{Vg}\left(\frac{dv}{dx} + \frac{2fv}{d}\right) \quad . \quad . \quad . \quad . \quad (1)$$

Another equation can be obtained by applying the energy equation [eq. (7), Chapter 14] to any length of the pipe. This equation may be written

$$J(H_1 - H_2) = \frac{v_2^2 - v_1^2}{2g}$$

or
$$c_p(T_1 - T_2) = \frac{v_2^2 - v_1^2}{2gJ} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

if constant specific heats are assumed, and c_p is in heat units.

Applying eq. (2) to the short length of pipe dx,

$$c_p[T - (T + dT)] = \frac{(v + dv)^2 - v^2}{2gJ}$$

Hence
$$- c_p\,dT = \frac{v^2 + 2v\,dv + (dv)^2 - v^2}{2gJ}$$

$$= \frac{v\,\mathrm{d}v}{gJ}$$

Dividing throughout by dx,

$$-c_p\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{v}{g}\frac{\mathrm{d}v}{\mathrm{d}x\,J} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

A further equation can be obtained by differentiating the characteristic equation for a gas. For 1 Lb of gas,

$$pV = RT$$

Differentiating, $p\,\mathrm{d}V + V\,\mathrm{d}p = R\,\mathrm{d}T$

Dividing throughout by dx,

$$p\frac{\mathrm{d}V}{\mathrm{d}x} + V\frac{\mathrm{d}p}{\mathrm{d}x} = R\frac{\mathrm{d}T}{\mathrm{d}x} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The final equation is obtained from the law of continuity of flow,

$$wav = W$$

As W and a are constant throughout the whole length of the pipe, and $w = 1/V$,

$$\frac{v}{V} = \frac{W}{a} = \text{constant}$$

Differentiating,

$$V\,\mathrm{d}v - v\,\mathrm{d}V = 0$$

Hence $$\mathrm{d}V = \frac{V}{v}\,\mathrm{d}v$$

Dividing throughout by dx,

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{V}{v}\frac{\mathrm{d}v}{\mathrm{d}x} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Substituting the value of dV/dx from eq. (5) in eq. (4),

$$p\frac{V}{v}\frac{\mathrm{d}v}{\mathrm{d}x} + V\frac{\mathrm{d}p}{\mathrm{d}x} = R\frac{\mathrm{d}T}{\mathrm{d}x} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Substituting in eq. (6) the values of dp/dx from eq. (1) and dT/dx. from eq. (3),

$$p\frac{V}{v}\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{v}{g}\left(\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{2fv}{d}\right) = -\frac{R}{Jc_p}\frac{v}{g}\frac{\mathrm{d}v}{\mathrm{d}x} \quad . \quad . \quad . \quad (7)$$

Substituting for $pV = RT$ and $R/c_p = (\gamma - 1)J/\gamma$

$$\frac{\mathrm{d}v}{\mathrm{d}x}\left[\frac{RT}{v} - \frac{v}{g} + \left(\frac{\gamma-1}{\gamma}\right)\frac{v}{g}\right] = \frac{2fv^2}{gd}$$

or $$\frac{\mathrm{d}v}{dx}\left[\frac{RT}{v} - \frac{v}{\gamma g}\right] = \frac{2fv^2}{gd} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Applying eq. (2) to the conditions at the entrance of the pipe and at the section x ft from the entrance,

$$c_pT = c_pT_1 - \frac{v^2}{2gJ} + \frac{v_1^2}{2gJ}$$

from which $$T = T_1 - \frac{v^2}{2gc_pJ} + \frac{v_1^2}{2gc_pJ}$$

Substituting this value of T in eq. (8), and dividing throughout by v^2,

$$\frac{dv}{dx}\left[R\left(\frac{gT_1}{v^3} - \frac{1}{2Jc_pv} + \frac{v_1^2}{2Jc_pv^3}\right) - \frac{1}{\gamma v}\right] = \frac{2f}{d}$$

Hence $$dv\left[\frac{RgT_1}{v^3} - \frac{R}{2Jc_pv} + \frac{Rv_1^2}{2Jc_pv^3} - \frac{1}{\gamma v}\right] = \frac{2f\,dx}{d}$$

Integrating between the limits $x = l$ and 0, and $v = v_2$ and v_1,

$$\int_{v_1}^{v_2}\left(\frac{gRT_1\,dv}{v^3} - \frac{R\,dv}{2Jc_pv} + \frac{Rv_1^2\,dv}{2Jc_pv^3} - \frac{dv}{\gamma v}\right) = \int_0^l \frac{2f\,dx}{d}$$

Hence $$\left[-\frac{RT_1g}{4v^2} - \frac{R}{4Jc_p}\log_e v - \frac{Rv_1^2}{8Jc_pv^2} - \frac{1}{2\gamma}\log_e v\right]_{v_1}^{v_2} = \left[\frac{fx}{d}\right]_0^l$$

Inserting the limits and substituting for $R/Jc_p = (\gamma - 1)/\gamma$,

$$-\frac{RT_1g}{4}\left(\frac{1}{v_2^2} - \frac{1}{v_1^2}\right) - \left(\frac{\gamma + 1}{4\gamma}\right)\log_e \frac{v_2}{v_1} - \left(\frac{\gamma - 1}{8\gamma}\right)v_1^2\left(\frac{1}{v_2^2} - \frac{1}{v_1^2}\right) = \frac{fl}{d} \quad . \quad (9)$$

Let r = ratio of velocities at ends of pipe; then

$$r = \frac{v_2}{v_1} = \frac{V_2}{V_1} = \frac{w_1}{w_2}$$

$$= \left(\frac{p_1}{p_2}\right) \times \left(\frac{T_2}{T_1}\right) \quad . \quad . \quad . \quad . \quad (10)$$

Substituting $v_2 = rv_1$ in eq. (9),

$$\frac{RT_1g(r^2 - 1)}{4r^2v_1^2} - \left(\frac{\gamma + 1}{4\gamma}\right)\log_e r + \left(\frac{\gamma - 1}{8\gamma}\right)\left(\frac{r^2 - 1}{r^2}\right) = \frac{fl}{d} \quad . \quad (11)$$

This is a general equation for all gases; the values of R and γ can be substituted when applied to any given gas. It should be noticed that R is in foot-pound units. If applied to air, $\gamma = 1{\cdot}4$ and $R = 96$ in C.H.U. or 53·3 in B.Th.U. Eq. (11) then becomes

$$\frac{RT_1g(r^2 - 1)}{4r^2v_1^2} - 0{\cdot}429\log_e r + 0{\cdot}035\left(\frac{r^2 - 1}{r^2}\right) = \frac{fl}{d} \quad . \quad (12)$$

From eq. (11),

$$v_1 = \sqrt{\frac{RT_1g(r^2 - 1)}{4r^2\left[\frac{fl}{d} + \left(\frac{\gamma+1}{4\gamma}\right)\log_e r - \frac{\gamma-1}{8\gamma}\left(\frac{r^2-1}{r^2}\right)\right]}} \quad . \quad (13)$$

Applying this equation to air, $\gamma = 1{\cdot}4$, then

$$v_1 = \sqrt{\frac{RT_1g(r^2 - 1)}{4r^2\left[\frac{fl}{d} + 0{\cdot}429\log_e r - 0{\cdot}035\,\frac{(r^2-1)}{r^2}\right]}} \quad . \quad . \quad (14)$$

Eqs. (11) to (14) may be termed the momentum equation, as they were evolved from the application of the fundamental momentum equation.

Substituting $v_2 = rv_1$ in eq. (2),

$$c_p(T_1 - T_2) = \frac{v_1^2}{2gJ}(r^2 - 1)$$

from which $$v_1 = \sqrt{\frac{2gc_p(T_1 - T_2)J}{(r^2 - 1)}} \quad . \quad . \quad . \quad (15)$$

where c_p is in heat units.

If constant specific heats are not assumed, this equation becomes

$$v_1 = \sqrt{\frac{2gJ(H_1 - H_2)}{(r^2 - 1)}} \quad . \quad . \quad . \quad (16)$$

Eqs. (15) and (16) may be termed the energy equation.

The adiabatic flow through a given pipe can now be solved by the application of the momentum and energy equations; the solution can be obtained only by trial or by plotting. If the values of p_1, p_2, and T_1 are known, the two equations (13) and (15) can be plotted on a base representing T_2 and ordinate representing v_1. Values of T_2 should be assumed and the corresponding value r for each value of T_2 can then be calculated from the equation

$$r = \frac{p_1}{p_2} \times \frac{T_2}{T_1}$$

The values of v_1 from the two equations (13) and (15) can then be calculated for each equation and plotted. The point of intersection of the two curves gives the correct value of v_1 and T_2. This has been done for air, and eqs. (14) and (15) are shown plotted in Fig. 235.

It will be shown later (§ 18.4 and § 18.6) that the maximum possible value of v_1 is when v_2 reaches its sonic velocity; that is, when the Mach number at outlet is unity. This fact limits the weight of flow through the pipe.

Having found the value of v_1, the weight of flow through the pipe can be calculated from the equation

$$W = w_1 a v_1 \text{ Lb/sec}$$

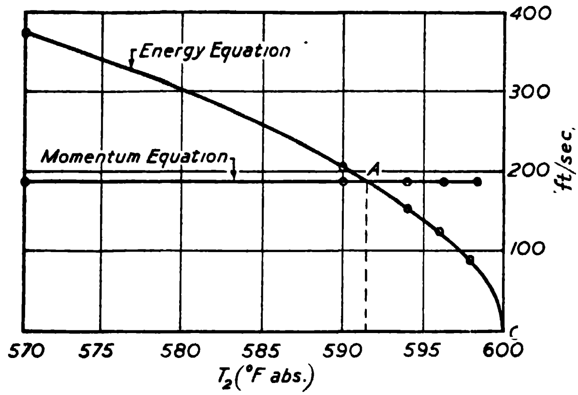


FIG. 235

18.2. Method of Solution of Momentum and Energy Equations. The application of the energy and momentum equations of § 18.1 to a given problem of adiabatic flow of a gas through a pipe is illustrated by the following worked-out example.

Consider a 6 in. diameter lagged pipe, 1,000 ft long, through which air is flowing. The pressure and temperature of the entering air are 200 Lb/in.² and 600°F abs. and the pressure at the outlet end of the pipe is 100 Lb/in.² Darcy's frictional coefficient f is 0·0025. It is required to find the temperature of the air at outlet and the weight of air flowing per second. Then $p_1 = 200$, $p_2 = 100$, and $T_1 = 600$.

Hence
$$r = \frac{p_1}{p_2} \times \frac{T_2}{T_1}$$
$$= \frac{200}{100} \times \frac{T_2}{600} = \frac{T_2}{300}$$
$$R = 53{\cdot}3 \text{ ft-Lb units, F degrees}$$
$$c_p = 0{\cdot}24 \text{ heat units}$$
$$\frac{fl}{d} = \frac{0{\cdot}0025 \times 1{,}000}{0{\cdot}5} = 5$$

From the energy equation [eq. (15)],

$$v_1 = \sqrt{\frac{64{\cdot}4 \times 0{\cdot}24 \times 778(T_1 - T_2)}{(r^2 - 1)}} \text{ ft/sec}$$

$$= \sqrt{\frac{12{,}000(T_1 - T_2)}{(r^2 - 1)}} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

From the momentum equation [eq. (14)],

$$v_1 = \sqrt{\frac{53{\cdot}3 \times 600 \times 32{\cdot}2(r^2 - 1)}{4r^2\left[5 + 0{\cdot}429 \log_e r - 0{\cdot}035\,\frac{(r^2 - 1)}{r^2}\right]}} \text{ ft/sec}$$

$$= \sqrt{\frac{257{,}300(r^2 - 1)}{r^2\left[5 + 0{\cdot}429 \log_e r - 0{\cdot}035\,\frac{(r^2 - 1)}{r^2}\right]}} \quad . \quad (18)$$

These equations are best solved by assuming various values for T_2 and calculating the corresponding values of r from the above equation

$$r = \frac{T_2}{300}$$

Then, by substituting these values of r in the above energy and momentum equations, the corresponding values of v_1 can be calculated. These values of v_1 can then be plotted for each equation on a base representing T_2; the point of intersection of the two curves gives the correct value of v_1 and T_2. The method is demonstrated in the following table.

T_2 (°F abs.)	$r = \frac{T_2}{300}$	r^2	$\log_e r$	Energy Equation [eq. (17)] v_1	Momentum Equation [eq. (18)] v_1
598	1·991	3·97	0·6886	90·0	191·0
596	1·985	3·94	0·6856	127·6	190·9
594	1·980	3·92	0·6831	157·0	190·8
590	1·965	3·87	0·6755	208·5	190·4
570	1·900	3·61	0·6418	372·0	188·4

The values of v_1 for each equation are shown plotted in Fig. 235. From point of intersection of curves, A,

$$v_1 = 190{\cdot}5 \text{ ft/sec}$$
$$T_2 = 591{\cdot}5\,°\text{F abs.}$$

Then

$$r = \frac{p_1}{p_2} \times \frac{T_2}{T_1}$$
$$= \frac{200}{100} \times \frac{591{\cdot}5}{600} = 1{\cdot}975$$

Hence

$$v_2 = r \times v_1$$
$$= 1{\cdot}975 \times 190{\cdot}5 = 376{\cdot}2 \text{ ft/sec}$$

At inlet end of pipe,

$$144 p_1 V_1 = R w_1 T_1$$

For one cubic foot, $w_1 = \dfrac{144 \times 200 \times 1}{53{\cdot}3 \times 600}$

$$= 0{\cdot}9 \text{ Lb/ft}^3$$

$$\text{Weight of air flowing} = W = w_1 \times \frac{\pi}{4} d^2 \times v_1$$

$$= 0{\cdot}9 \times 0{\cdot}785 \times (\tfrac{1}{2})^2 \times 190{\cdot}5$$

$$= 33{\cdot}6 \text{ Lb/sec}$$

18.3. Variation of Pressure, Temperature and Velocity throughout Length of Pipe. The variation of pressure, temperature and velocity throughout the length of a given pipe can be obtained from the equations of § 18.1 for a given inlet pressure p_1, temperature T_1, and outlet pressure p_2. From these values the inlet velocity v_1 can be calculated by the method shown in § 18.2.

Consider a section of the pipe at a distance of x from the inlet end. Let p_x, T_x and v_x be the pressure, absolute temperature and velocity at the section. As the inlet velocity v_1 is now known, and the frictional resistance term at the section is now fx/d, the value of the velocity ratio r_x at the section can be calculated, for any assumed value of x, from the momentum equation of § 18.1. Then $v_x = r_x v_1$.

First apply the momentum equation [eq. (13)];

$$v_1 = \sqrt{\frac{RT_1 g(r_x^2 - 1)}{4r_x^2 \left[\dfrac{fx}{d} + \left(\dfrac{\gamma + 1}{4\gamma}\right) \log_e r_x - \dfrac{\gamma - 1}{8\gamma}\left(\dfrac{r_x^2 - 1}{r_x^2}\right)\right]}}$$

For an assumed value of x, this equation will give the value of r_x as v_1 and T_1 are already known.

Next obtain v_x from the equation

$$v_x = r_x v_1$$

Next calculate the value of T_x from the energy equation [eqs. (15) or (16)];

$$v_1 = \sqrt{\frac{2gc_p(T_1 - T_x)J}{(r_x^2 - 1)}}$$

Then, from eq. (10),

$$r_x = \frac{p_1}{p_x} \times \frac{T_x}{T_1}$$

from which the value of p_x can be calculated.

By repeating this for different values of x, the values of p_x, T_x and v_x are obtained for the whole length of the pipe.

This method has been applied to a given pipe problem and the values of p_x, T_x and v_x so obtained are shown plotted to scale in Fig. 236. These results are for air flowing through a 6 in. diameter pipe, 1,000 ft long, the value of fl/d being 5. It will be noticed from the curves that there is a rapid drop in pressure and temperature in the last 5 per cent of the pipe's length. This is accompanied by a very rapid increase in velocity, the velocity curve being almost vertical at the outlet end of the pipe. The final point on the velocity

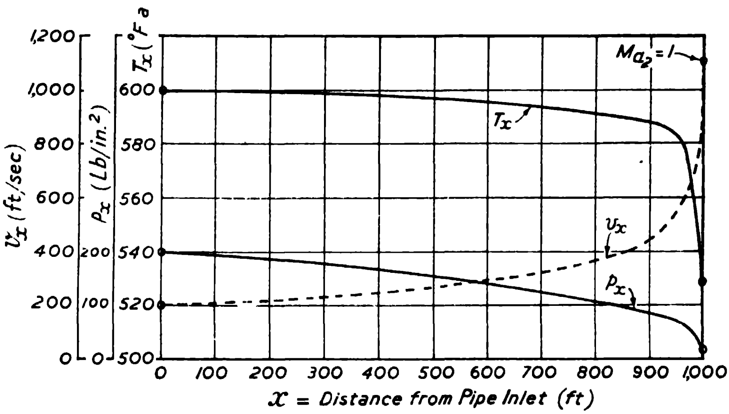


FIG. 236

curve shows that the Mach number of the flow was unity at outlet. It will be shown in § 18.4 and § 18.6 that the velocity in any pipe cannot exceed that producing a Mach number of unity.

The rapid increase in velocity near the outlet end of the pipe is mainly due to the large increase of specific volume; this causes a choking effect on the flowing gas and limits the outlet velocity to its sonic value.

18.4. Line of Condition on the $T - S$ Chart. The pressure and temperature of the gas as it flows along the pipe can be plotted on the temperature-entropy chart; the points obtained will represent the varying condition of the gas for the whole length of pipe. This has been done for the air-flow problem of § 18.3, and the pressures and temperatures have been plotted on the heat-entropy chart given on the folding inset, Fig. 155, facing page 330. By joining the points by a smooth curve the line of condition is obtained for the air as it flows along the whole length of the pipe. This diagram is reproduced, to scale, in Fig. 237. The line of condition so obtained is called a Fanno curve.

On examining the line of condition it will be noticed that it becomes vertical at a point A. It is found that this point corresponds to an outlet velocity having a Mach number of unity.* Beyond this point the curve inclines to the left, thus denoting a decrease in entropy. But the velocity of the points of the curve below A is supersonic, as it is continuously increasing as the pressure falls along the pipe. This means that the entropy is decreasing, although there is an increase in the frictional reheating. This is impossible

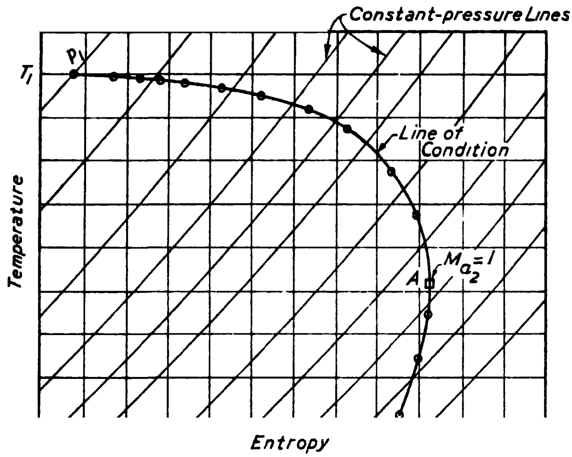


FIG. 237

in nature as frictional reheating causes an increase in entropy. Hence, it follows that the expansion of the gas cannot occur beyond the point A.

It will be seen from this that the velocity of a gas in a pipe cannot exceed its sonic velocity, and the Mach number of its flow is limited to unity.† The line of condition can hold only up to the point A, at which point the curve is vertical. Any further pressure drop at the outlet of the pipe will cause the flow to choke, and no further increase in velocity or weight of flow can occur. Hence, the maximum weight of flow of gas through a pipe for a given pressure drop occurs when the Mach number of the flow at outlet is unity.

18.5. Variation of Outlet Pressure for a Given Pipe. The effect of the variation of the outlet pressure p_2 on the flow through a given pipe is demonstrated in the curves of Fig. 238. It is assumed that

* For a mathematical proof of this statement see § 18.14.

† This is analogous to the maximum flow over a broad-crested weir in which the velocity over the sill cannot exceed the value for a Froude number of unity. (§ 5.10).

the inlet condition of the gas remains constant; then the values of p_1 and T_1 are fixed. The value of fl/d for the pipe is also constant. By applying the momentum and energy equations of § 18.1 [eqs. (13) and (15)] the values of v_1 can be calculated for a complete range of assumed values for p_2. The values of v_1 so obtained are shown plotted, on a base representing p_2, in Fig. 238 for two different pipes, one having a value of $fl/d = 0{\cdot}5$, the other having a value of $fl/d = 5$. The value of p_2, which produced sonic flow at outlet, was also

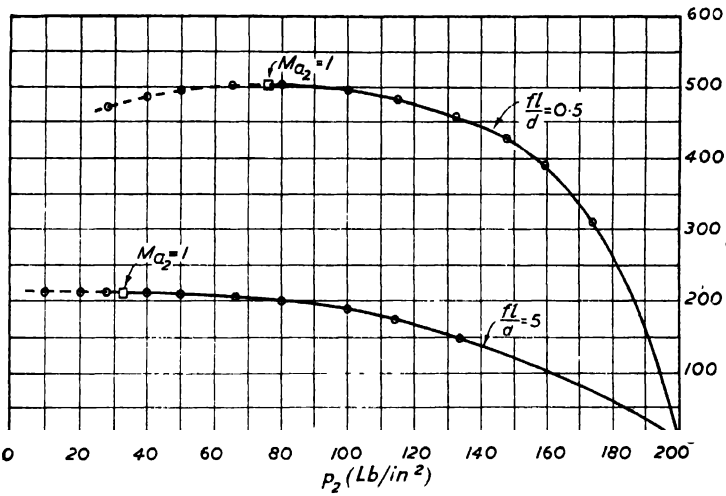


FIG. 238

calculated, for each pipe, and their positions are shown plotted on each curve. These points represent the outlet pressure and value of v_1 when $M_{a_2} = 1$.

On examining these velocity curves it will be noticed that the entrance velocity increases as the outlet pressure is reduced and reaches its maximum value when $M_{a_2} = 1$. The portion of the curves beyond this point is shown dotted, because any further reduction of the outlet pressure causes the flow to choke and no further increase of v_1 and weight of discharge is possible (§ 18.4). Thus, the energy and momentum equations of § 18.1 hold only up to the condition when $M_{a_2} = 1$.

The outlet condition of the gas for each pipe is shown plotted on the heat-entropy chart of Fig. 239 for each assumed outlet pressure. A smooth curve has been shown drawn through the points obtained, and the point representing the condition of $M_{a_2} = 1$ is shown plotted on each curve. It will be noticed that the curves are vertical when the sonic speed at outlet is reached; below this point the

curves are shown dotted as they represent a condition of flow which cannot be attained.

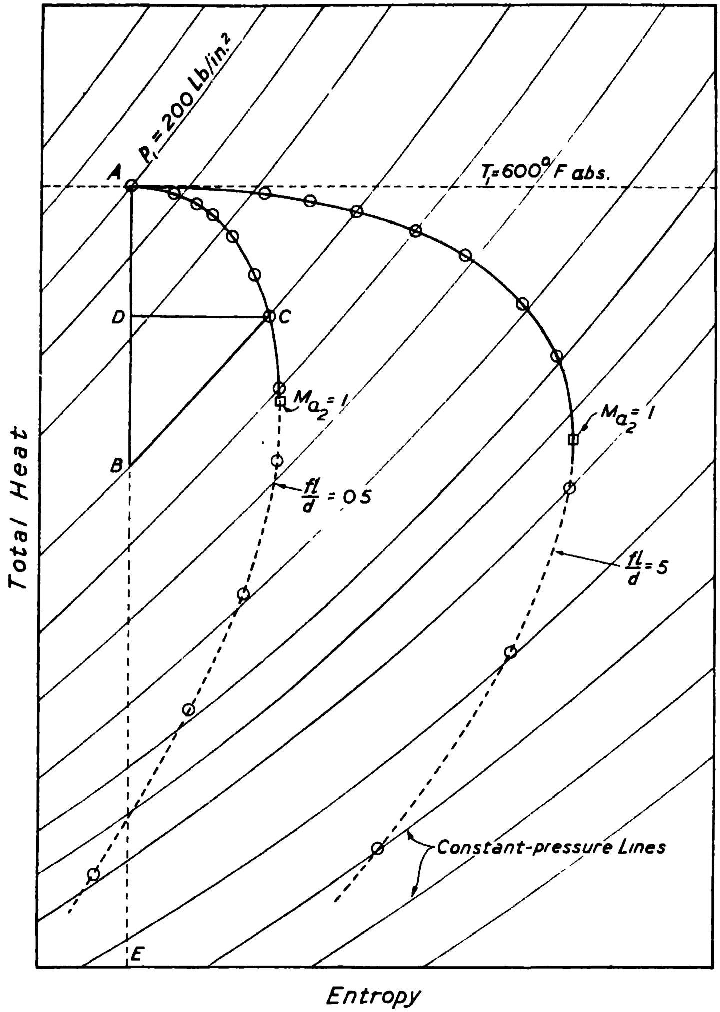


FIG. 239

It should also be noticed that the inlet velocity curves of Fig. 238 also represent the weight of discharge per second to a different scale, as

$$W = w_1 a v_1$$

and w_1 and a are constant for each pipe.

18.6. Effect of Outlet Pressure on the Frictional Reheating. The fact that the outlet velocity v_2 cannot exceed its sonic speed is also demonstrated by an examination of the frictional reheating. Referring to the lines of condition at the outlet end of the pipe for various values of p_2 shown in Fig. 239, the amount of frictional reheating at any point can be obtained from the method given in

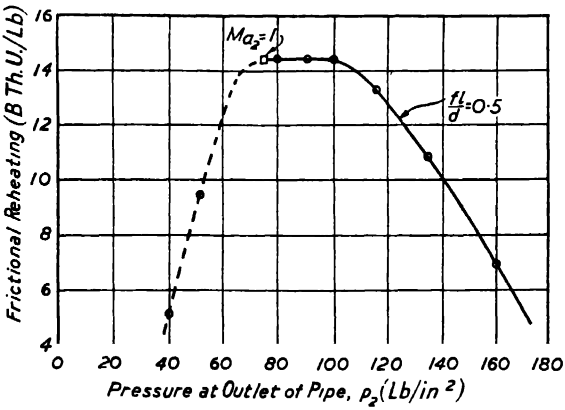


FIG. 240

§ 14.4. This can be repeated for all values of p_2 and the results plotted on a base representing p_2.

Consider the line of condition of the pipe having a value of $fl/d = 0{\cdot}5$, and consider any point C at a known value of p_2. Had the expansion in the pipe been a pure adiabatic without friction it would have been represented by the vertical line AB, the point B being on the same constant-pressure line as C. Let the horizontal through C cut the adiabatic line AB at D. Then the frictional reheating is represented by the constant-pressure process BC and the total heat, or enthalpy, produced by the reheating is represented by DB. This can be scaled off from the heat-entropy chart given on the folding inset of Fig. 155, or it can be calculated from the specific heat.

$$\text{Hence,}\quad \left.\begin{array}{c}\text{frictional reheating at point}\\ C \text{ in B.Th.U./Lb of gas}\end{array}\right\} = H_D - H_B$$
$$= c_p(T_D - T_B)$$

This has been done for all the points of the line of condition and the results are shown plotted, on a base representing p_2, in Fig. 240.

It will be seen from the curve obtained that the amount of frictional reheating increases as p_2 is reduced and reaches a maximum when the Mach number at outlet is unity. For values of p_2

lower than this critical value, the amount of reheating is reduced. This is impossible in practice because the lowering of p_2 would produce an increase in velocity which must produce an increase in frictional reheating. Hence, it follows that the value of p_2 cannot fall below that which produces sonic speed in the pipe at its outlet end. This confirms the result obtained in § 18.4.

It will be noticed in Fig. 240 that the portion of the reheat curve beyond the value of $M_{a_2} = 1$ is shown dotted. This is because this portion of the curve is unattainable in practice because the value of the Mach number at outlet is greater than unity.

It follows that, as the velocity at the outlet end of a pipe cannot exceed its sonic velocity, the pressure in the pipe at outlet cannot be less than that required to produce a value of $M_{a_2} = 1$. If the pipe is exhausting into a chamber having a lower pressure than this, there will be a sudden pressure drop between the pipe outlet and the chamber. This will cause shock waves to occur in the jet exhausting into the chamber and a corresponding loss of energy.

18.7. Chart for Flow of Air in Pipes. The application of the momentum and energy equations given in § 18.1 to a given pipe-flow problem causes a considerable amount of labour in obtaining the solution. In order to reduce the work involved, the author has prepared a chart from which the solution can be readily obtained for a known pressure-drop ratio and a known value of fl/d. The chart has been reproduced to scale in Fig. 241, facing this page.

The base represents the temperature ratio T_2/T_1 for the inlet and outlet ends of the pipe, and the ordinate represents the value of $v_1/\sqrt{T_1}$ which is proportional to the Mach number at inlet. The curves shown as full lines are lines of constant-pressure ratio, p_1/p_2, for the inlet and outlet ends of the pipe. The dotted-line curves represent constant values of fl/d. The boundary curve represents the condition when $M_{a_2} = 1$ at the outlet end of the pipe; this is the limiting condition, and no values to the left of this curve are attainable.

At the extreme right of the diagram, between values of T_2/T_1 of 0·995 and 1·0, the chart is too congested to give accurate results; the temperature change over this range is too small to be taken into account; hence the flow is approximately isothermal and the problem can be solved by the application of the simple methods given in § 14.11 and § 14.12.

In the solution of a pipe problem by the chart, if the values of p_1/p_2, T_1, a and fl/d are known, from the point of intersection of the p_1/p_2 curve with the fl/d curve the values of T_2/T_1 and $v_1/\sqrt{T_1}$ can be read off the chart. From these ratios the values of v_1 and T_2 can be calculated. Then

$$W = w_1 a v_1 \text{ Lb/sec}$$

EXAMPLE 1

Find the maximum discharge from a $1\frac{1}{2}$ in. diameter pipe, 10 ft long, which exhausts into a chamber at atmospheric pressure. The pressure and temperature at the inlet end of the pipe are 40 Lb/in.² and 80°F. Assume $f = 0{\cdot}0025$ for the whole length of the pipe.

$$\frac{fl}{d} = \frac{0{\cdot}0025 \times 10}{\frac{1}{8}} = 0{\cdot}2$$

$$T_1 = 460 + 80 = 540°\text{F abs.}$$

From heat-entropy chart, $w_1 = 0{\cdot}208$ Lb/ft³.

From pipe-flow chart, the outlet end condition is given by the point of intersection of the $fl/d = 0{\cdot}2$ line with the curve representing $M_{a_2} = 1$. This gives $v_1/\sqrt{T_1} = 27{\cdot}15$. Hence,

$$\begin{aligned} \text{maximum } v_1 &= 27{\cdot}15 \times \sqrt{540} \\ &= 631 \text{ ft/sec} \end{aligned}$$

Then,

$$\begin{aligned} \text{maximum } W &= w_1 a v_1 \\ &= 0{\cdot}208 \times 0{\cdot}785 \times (\tfrac{1}{8})^2 \times 631 \\ &= 1{\cdot}548 \text{ Lb/sec} \end{aligned}$$

18.8. Value of Darcy's f by Successive Approximations. The value of Darcy's frictional coefficient f must first be assumed for finding the value of fl/d. This can be taken as 0·0025 as a first approximation. From the results obtained by this assumed value the mean value of R_e can be estimated; from this a more exact value of f can be obtained from the Prandtl curve of Fig. 118. Then by using this new value of f a new value of fl/d is obtained. The problem can now be solved from the chart by using this second value of fl/d and a more accurate result obtained. No further attempts are necessary as the value by this second approximation is of sufficient accuracy.

The method of using the chart and a solution by the above method of successive approximations are demonstrated in the following worked-out example.

EXAMPLE 2

Air flows through a 6 in. pipe, 100 ft long. The pressure and temperature at the inlet end are 200 Lb/in.² and 600°F abs; the pressure at the outlet end is 120 Lb/in.² Calculate from the chart the discharge in pounds per second and the final temperature and velocity of the air.

Assume, for a first approximation, Darcy's $f = 0{\cdot}0025$. Then,

$$\frac{fl}{d} = \frac{0{\cdot}0025 \times 100}{0{\cdot}5} = 0{\cdot}5$$

$$\frac{p_1}{p_2} = \frac{200}{120} = 1{\cdot}67$$

The point of intersection of the curves representing these values can now be obtained from the pipe-flow chart, Fig. 241. From chart,

$$\frac{v_1}{\sqrt{T_1}} = 19{\cdot}5 \text{ and } \frac{T_2}{T_1} = 0{\cdot}9513$$

Hence $\qquad v_1 = 19{\cdot}5 \times \sqrt{600} = 478$ ft/sec

Also $\qquad T_2 = 0{\cdot}9513 \times 600 = 571$°F abs.

$$r = \frac{p_1}{p_2} \times \frac{T_2}{T_1}$$

$$= 1{\cdot}67 \times 0{\cdot}9513 = 1{\cdot}58$$

$$v_2 = rv_1$$

$$= 1{\cdot}58 \times 478 = 756 \text{ ft/sec}$$

Next find the approximate mean value for R_e from the approximate mean velocity on the assumption that the velocity, density, pressure and temperature variation along the length of the pipe follow a straight-line law.

$$\text{Mean velocity} = \frac{v_1 + v_2}{2}$$

$$= \frac{478 + 756}{2} = 617 \text{ ft/sec}$$

$$\text{Mean temperature} = \frac{T_1 + T_2}{2}$$

$$= \frac{600 + 571}{2} = 585\text{°F abs.}$$

$$\text{Mean pressure} = \frac{p_1 + p_2}{2}$$

$$= \frac{200 + 120}{2} = 160 \text{ Lb/in.}^2$$

From total heat-entropy diagram, at $p = 160$ and $T = 585$,
mean density in pipe

$$= 0{\cdot}77 \text{ Lb/ft}^3$$

From viscosity curve for air given in Appendix 2,
mean value of coefficient of viscosity

$$= 0{\cdot}412 \times 10^{-6}$$

Then $\qquad \text{mean } R_e = \dfrac{\rho v d}{\eta}$

$$= \frac{0{\cdot}77 \times 617 \times 0{\cdot}5}{32{\cdot}2 \times 0{\cdot}412 \times 10^{-6}} = 17{\cdot}9 \times 10^6$$

$\log 17{\cdot}9 \times 10^6 = 7{\cdot}253$

From Prandtl curve of Fig. 118,

$$f = 0{\cdot}00185$$

Second Approximation. Using above new value for f,

$$\frac{fl}{d} = \frac{0{\cdot}00185 \times 100}{0{\cdot}5} = 0{\cdot}37$$

From pipe-flow chart, using this new value of fl/d,

$$\frac{v_1}{\sqrt{T_1}} = 21{\cdot}4 \text{ and } \frac{T_2}{T_1} = 0{\cdot}944$$

Hence $\qquad v_1 = 21{\cdot}4 \times \sqrt{600} = 524$ ft/sec

and $\qquad T_2 = 0{\cdot}944 \times 600 = 566°$F abs.

$$r = 1{\cdot}67 \times 0{\cdot}944 = 1{\cdot}573$$

$$v_2 = 524 \times 1{\cdot}573 = 824 \text{ ft/sec}$$

These results may be taken as the final values; it is found that any further approximations do not materially affect them.

From heat-entropy diagram, $w_1 = 0{\cdot}95$ Lb/ft³. Then

$$\begin{aligned} W &= w_1 a v_1 \\ &= 0{\cdot}95 \times 0{\cdot}785 \times (\tfrac{1}{2})^2 \times 524 \\ &= 97{\cdot}5 \text{ Lb/sec} \end{aligned}$$

18.9. Pipe Flow from One Chamber to Another. The problem of the flow of gas between two containers by means of a long pipe can be solved by the method of successive approximations. It will be found that a result of sufficient accuracy is given by the second attempt.

Consider the problem shown in Fig. 242. Let the left-hand chamber be the high-pressure supply and be maintained at a constant pressure and temperature. Let the gas flow into the right-hand chamber which is maintained at a constant pressure p_2. A vena contracta is liable to form at the entrance of the pipe. Let Section 1 represent the commencement of the normal full flow in the pipe after the vena contracta, and let Section 2 be the outlet section of the pipe where the pressure reaches p_2.

Now the energy and momentum equations obtained in § 18.1 will apply to the pipe only between Sections 1 and 2, but the pressure at Section 1 will be less than p_1 by an amount due to the velocity head, $v_1^2/2g$. As a first approximation, apply the equations of § 18.1

to the pipe for a pressure range of p_1 to p_2, thus ignoring the velocity head. This will give an approximate value for v_1. Then,

actual value of pressure at Section 1

$$= p_1 \quad \frac{w_1 v_1^2}{2g} \text{ Lb/ft}^2$$

Using this new pressure, the calculations can be repeated as a second approximation, and a new and more accurate value of v_1

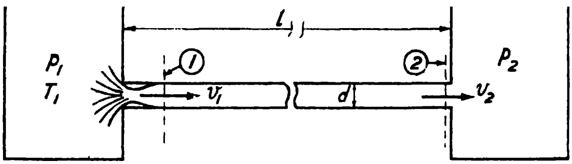


FIG. 242

obtained. This second value will be found to be of sufficient accuracy and no further attempts are necessary.

The new temperature and density at Section 1 can be found from the heat-entropy diagram by assuming the expansion between the high-pressure chamber and Section 1 to be isentropic.

Let w_1 = density at Section 1 after isentropic expansion. Then,

$$W = w_1 a v_1 \text{ Lb/sec}$$

This problem can also be solved for air by using the pipe-flow chart of Fig. 241 instead of the energy and momentum equations. The application of the above method is shown in the following worked-out example.

EXAMPLE 3

Air at a pressure of 200 Lb/in.² and a temperature of 600°F abs. flows from a chamber through a 6 in. diameter pipe, 100 ft long, into another chamber in which a constant pressure of 120 Lb/in.² is maintained (Fig. 242). The high-pressure chamber is maintained at constant pressure and temperature. Calculate the weight of discharge and velocity at outlet assuming a constant value of 0·0025 for f.

First Approximation. Neglect velocity head at Section 1.

$$\frac{fl}{d} = \frac{0{\cdot}0025 \times 100}{0{\cdot}5} = 0{\cdot}5$$

$$\frac{p_1}{p_2} = \frac{200}{120} = 1{\cdot}67$$

From pipe-flow chart, Fig. 241,

$$\frac{v_1}{\sqrt{T_1}} = 19{\cdot}5$$

Hence $\qquad v_1 = 19{\cdot}5 \times \sqrt{600} = 478$ ft/sec

From heat-entropy diagram,

$$w_1 = 0{\cdot}95 \text{ Lb/ft}^3$$

$$\text{Pressure drop at Section 1} = \frac{w_1 v_1^2}{144 \times 2g}$$

$$= \frac{0{\cdot}95 \times 478^2}{144 \times 64{\cdot}4}$$

$$= 23{\cdot}3 \text{ Lb/in.}^2$$

$$\text{New value of } p_1 = 200 - 23{\cdot}3 = 176{\cdot}7 \text{ Lb/in.}^2$$

From heat-entropy diagram, assuming isentropic expansion from 200 Lb/in.² to 176·7 Lb/in.²,

$$\text{new value of } T_1 = 580°\text{F abs.}$$

$$\text{new value of } w_1 = 0{\cdot}87 \text{ Lb/ft}^3$$

Second Approximation. From pipe-flow chart, using the new values of p_1, w_1, and T_1,

$$\frac{p_1}{p_2} = \frac{176{\cdot}7}{120} = 1{\cdot}472$$

Then
$$\frac{v_1}{\sqrt{T_1}} = 18{\cdot}5$$

Hence
$$v_1 = 18{\cdot}5 \times \sqrt{580} = 446 \text{ ft/sec}$$

Also
$$\frac{T_2}{T_1} = 0{\cdot}97$$

Hence
$$T_2 = 0{\cdot}97 \times 580 = 563°\text{F abs.}$$

$$W = w_1 a v_1$$

$$= 0{\cdot}87 \times \frac{\pi}{4} \times (\tfrac{1}{2})^2 \times 446$$

$$= 76{\cdot}2 \text{ Lb/sec}$$

$$r = \frac{p_1}{p_2} \times \frac{T_2}{T_1}$$

$$= 1{\cdot}472 \times 0{\cdot}97 = 1{\cdot}43$$

$$v_2 = v_1 \times r$$

$$= 446 \times 1{\cdot}43 = 638 \text{ ft/sec}$$

18.10. Equations of Flow in Terms of Mach Numbers. The momentum and energy equations of § 18.1 can be expressed in terms of the Mach number by substituting for the velocity of sound.

From eq. (26), Chapter 13,

$$\text{velocity of sound} = v_s = \sqrt{\frac{\gamma p}{\rho}}$$

$$\sqrt{\frac{\gamma g p}{w}}, \quad \text{as } \rho = \frac{w}{g}$$

$$= \sqrt{pV\gamma g}, \text{ as } w = \frac{1}{V}$$

$$= \sqrt{RT\gamma g}, \text{ as } pV = RT$$

Hence, at inlet and outlet ends of the pipe,

$$v_{s_1} = \sqrt{RT_1\gamma g} \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$v_{s_2} = \sqrt{RT_2\gamma g} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Also $$M_{a_1} = \frac{v_1}{v_{s_1}} \text{ and } M_{a_2} = \frac{v_2}{v_{s_2}}$$

From the energy equation (eq. (15)),

$$v_1 = \sqrt{\frac{2gc_p(T_1 - T_2)J}{(r^2 - 1)}}$$

$$= \sqrt{\frac{2g\gamma R(T_1 - T_2)}{(\gamma - 1)(r^2 - 1)}} \text{ as } c_p = \frac{\gamma R}{(\gamma - 1)J}$$

Substituting for v_{s_1} and v_{s_2} from eqs. (19) and (20),

$$v_1 = \sqrt{\frac{2(v_{s_1}{}^2 - v_{s_2}{}^2)}{(\gamma - 1)(r^2 - 1)}} \quad . \quad . \quad . \quad (21)$$

But $$r = \frac{v_2}{v_1}$$

$$= \frac{v_{s_2}M_{a_2}}{v_{s_1}M_{a_1}}$$

Hence $$v_{s_2} = \frac{M_{a_1}}{M_{a_2}} r v_{s_1} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Substituting eq. (22) in eq. (21),

$$\frac{(\gamma - 1)(r^2 - 1)}{2} = \frac{1}{M_{a_1}{}^2} - \frac{M_{a_1}{}^2 r^2 v_{s_1}{}^2}{M_{a_2}{}^2 v_1{}^2}$$

from which

$$\frac{(\gamma - 1)(r^2 - 1)}{2} = \frac{1}{M_{a_1}{}^2} - \frac{r^2}{M_{a_2}{}^2} \quad . \quad . \quad . \quad . \quad (23)$$

Eq. (23) is the energy equation in terms of the Mach numbers at inlet and outlet.

From the momentum equation [eq. (13)],

$$v_1 = \frac{RT_1 g(r^2 - 1)}{4r^2\left[\frac{fl}{d} + \left(\frac{\gamma + 1}{4\gamma}\right)\log_e r - \frac{\gamma - 1}{8\gamma}\left(\frac{r^2 - 1}{r^2}\right)\right]}$$

Substituting from eq. (19),

$$v_1 = \sqrt{\frac{v_{s_1}^2(r^2-1)}{4r^2\gamma\left[\frac{fl}{d} + \left(\frac{\gamma+1}{4\gamma}\right)\log_e r - \frac{\gamma-1}{8\gamma}\left(\frac{r^2-1}{r^2}\right)\right]}}$$

But $v_1/v_{s_1} = M_{a_1}$; hence

$$M_{a_1} = \sqrt{\frac{(r^2-1)}{4r^2\gamma\left[\frac{fl}{d} + \left(\frac{\gamma+1}{4\gamma}\right)\log_e r - \frac{\gamma-1}{8\gamma}\left(\frac{r^2-1}{r^2}\right)\right]}} \quad . \quad (24)$$

Eq. (24) is the momentum equation in terms of the Mach number at inlet.

18.11. Isothermal Flow of Gas through Pipes. This is a general case for isothermal flow; it occurs in pipes which are unlagged, and gives results which hold for larger pressure drops and which are more accurate than the approximate methods given in § 14.11 and § 14.12.

Consider the pipe shown in Fig. 243 through which gas is flowing isothermally from left to right against a frictional resistance which

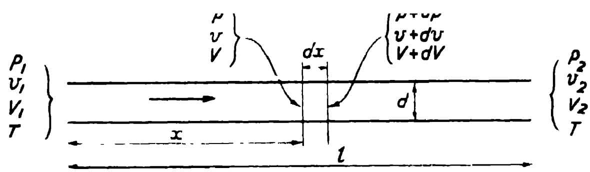


FIG. 243

reheats the gas. The temperature is maintained constant throughout its length by an interchange of heat through the walls of the pipe. Let suffix 1 apply to the inlet and suffix 2 to the outlet end. Consider a section at x from the inlet end and having a length dx. Let p, v and V be the pressure, velocity and specific volume at the left of the section, and let these increase to $p + dp$, $v + dv$ and $V + dV$ at the right of the section.

The momentum equation for this pipe can be deduced in the same way as eq. (1) for adiabatic flow was deduced; the same proof and result apply to isothermal flow also. Hence, the momentum equation for isothermal flow is

$$\frac{dp}{dx} = \frac{v}{gV}\left(\frac{dv}{dx} + \frac{2fv}{d}\right). \quad . \quad . \quad (25)$$

For isothermal expansion,

$$pV = \text{constant}$$

Differentiating, $\quad p\,dV + V\,dp = 0$

Hence $$-dp = \frac{p\,dV}{V} \quad . \quad . \quad . \quad . \quad (26)$$

From the law of continuity of flow,

$$W = wav$$

$$= \frac{av}{V}$$

Hence $$\frac{v}{V} = \frac{W}{a} = \text{constant}$$

Differentiating, $$V\,\mathrm{d}v - v\,\mathrm{d}V = 0$$

or $$\frac{\mathrm{d}V}{V} = \frac{\mathrm{d}v}{v} \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Substituting eq. (27) in eq. (26),

$$-\,\mathrm{d}p = p\,\frac{\mathrm{d}v}{v}$$

Dividing throughout by dx,

$$-\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{p}{v}\frac{\mathrm{d}v}{\mathrm{d}x} \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Substituting eq. (28) in eq. (25),

$$\frac{p}{v}\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v}{gV}\left(\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{2fv}{d}\right)$$

Then $$\frac{gpV}{v^2}\frac{\mathrm{d}v}{\mathrm{d}x} = \left(\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{2fv}{d}\right)$$

Substituting for $pV = RT$,

$$\frac{\mathrm{d}v}{\mathrm{d}x}\left(\frac{gRT}{v^2} - 1\right) = \frac{2fv}{d}$$

or $$gRTv^{-3}\,\mathrm{d}v - \frac{\mathrm{d}v}{v} = \frac{2f\,\mathrm{d}x}{d}$$

Integrating between the inlet and outlet ends of the pipe,

$$gRT\int_{v_1}^{v_2} v^{-3}\,\mathrm{d}v - \int_{v_1}^{v_2}\frac{\mathrm{d}v}{v} = \frac{2f}{d}\int_0^l \mathrm{d}x$$

Then $$-\frac{gRT}{2}\left[\frac{1}{v^2}\right]_{v_1}^{v_2} - \left[\log_e v\right]_{v_1}^{v_2} = \frac{2fl}{d}$$

Hence $$-\frac{gRT}{2}\left(\frac{1}{v_2^2} - \frac{1}{v_1^2}\right) - \log_e\frac{v_2}{v_1} = \frac{2fl}{d}$$

Substituting for $r = v_2/v_1$,

$$\frac{gRT(r^2-1)}{2r^2v_1^2} = \log_e r + \frac{2fl}{d}$$

Hence
$$v_1 = \sqrt{\frac{gRT(r^2 - 1)}{2r^2\left(\log_e r + \dfrac{2fl}{d}\right)}} \quad . \quad . \quad (29)$$

For isothermal flow,
$$r = \frac{v_2}{v_1} = \frac{V_2}{V_1} = \frac{p_1}{p_2}$$

Hence, if the pressure ratio between the inlet and outlet ends of the pipe is known and the temperature T is known, the value of v_1 can be calculated from eq. (29) for any assumed value of Darcy's f. The approximate mean value for f can be obtained by the method of successive approximations given in § 18.8. This is also demonstrated in the following worked-out example.

It is interesting to notice that eq. (29) could have been obtained by substituting $\gamma = 1$ in the equation for adiabatic flow [eq. (13)]. This is because both expansions may be assumed to follow the law $pV^n =$ constant, where $n = \gamma$ for an adiabatic expansion and $n = 1$ for an isothermal process.

If the line of condition for an isothermal flow be plotted on the heat-entropy chart a horizontal line is obtained because the temperature is constant. If the flow had been a frictional adiabatic, the line of condition would have been as shown in Fig. 237. Hence, for the same pipe-flow problem, the vertical distance between these two lines of condition would represent approximately the heat flow through the walls of the pipe. It is obviously impossible for this large heat transfer to take place in such a short interval of time in the case of an isothermal flow having a large pressure drop and a low value of fl/d, and the actual flow would tend to approximate to the adiabatic condition, in which case the Mach number of the flow at outlet cannot exceed unity. From this it appears that an isothermal flow is possible only for small pressure ratios or in very long pipes having large values of fl/d, and that in all cases the value of M_{a_2} cannot exceed unity.

The problem of isothermal flow from one chamber to another by means of a pipe can be solved by a method of successive approximations in the same manner as that of the adiabatic flow given in § 18.9, except that the isothermal equation [eq. (29)] should be used instead of the adiabatic equations.

EXAMPLE 4

Air at 100 Lb/in.^{2} pressure and a temperature of 60°F flows isothermally along a pipe 3 in. in diameter and 60 ft long; the pressure at the outlet end is 50 Lb/in.^{2} Calculate the flow in pounds per second and the velocity at outlet. Assume a value for Darcy's f and correct this value by a second approximation.

Assume $\qquad f = 0{\cdot}002$ as a first approximation

$$T = 460 + 60 = 520°\text{F abs.}$$

$$r = \frac{p_1}{p_2} = \frac{100}{50} = 2$$

Applying eq. (29),

$$v_1 = \sqrt{\frac{gRT(r^2 - 1)}{2r^2\left(\log_e r + \dfrac{2fl}{d}\right)}}$$

$$= \sqrt{\frac{32{\cdot}2 \times 53{\cdot}3 \times 520\,(4 - 1)}{2 \times 4\left(\log_e 2 + \dfrac{2 \times 0{\cdot}002 \times 60}{\frac{1}{4}}\right)}}$$

$$= 450 \text{ ft/sec}$$

Then $$v_2 = rv_1$$

$$= 2 \times 450 = 900 \text{ ft/sec}$$

$$\text{Mean velocity} = \frac{450 + 900}{2} = 675 \text{ ft/sec}$$

$$\text{Mean pressure} = 75 \text{ Lb/in.}^2$$

From Appendix 2,

mean coefficient of viscosity

$$= \eta = 0{\cdot}37 \times 10^{-6} \text{ engineers' units}$$

From heat-entropy diagram,

$$\text{mean } w = 0{\cdot}4 \text{ Lb/ft}^3$$

Then, $$\text{mean } R_e = \frac{\rho vd}{\eta}$$

$$= \frac{0{\cdot}4 \times 675 \times 0{\cdot}25}{32{\cdot}2 \times 0{\cdot}37 \times 10^{-6}}$$

$$= 5{\cdot}66 \times 10^6$$

From the Prandtl curve of Fig. 118, $f = 0{\cdot}0022$.

Second Approximation. Substituting this value of f in eq. (29),

$$v_1 = \sqrt{\frac{32{\cdot}2 \times 53{\cdot}3 \times 520 \times (4 - 1)}{2 \times 4\left(\log_e 2 + \dfrac{2 \times 0{\cdot}0022 \times 60}{\frac{1}{4}}\right)}}$$

$$= 437 \text{ ft/sec}$$

Then $$v_2 = 437 \times 2 = 874 \text{ ft/sec}$$

From heat-entropy diagram,

$$w_1 = 0{\cdot}53$$

Flow per second $= W = w_1 a v_1$

$$= 0{\cdot}53 \times \frac{\pi}{4} \times (\tfrac{1}{2})^2 \times 437$$

$$= 11{\cdot}34 \text{ Lb/sec}$$

18.12. Equation for Isothermal Flow in Terms of the Mach Number. For isothermal flow the velocity of sound remains constant along the whole length of the pipe, as it is a function of the absolute temperature only.

From § 18.10,

$$v_s = \sqrt{\gamma g R T}$$

Then $$M_a = \frac{v_1}{\sqrt{\gamma g R T}}$$

Substituting for v_1 in eq. (29),

$$M_{a_1} = \sqrt{\frac{(r^2 - 1)}{2r^2\gamma\left(\log_e r + \dfrac{2fl}{d}\right)}} \quad . \quad . \quad . \quad (30)$$

As the velocity of sound is constant,

$$r = \frac{v_2}{v_1} = \frac{M_{a_2}}{M_{a_1}}$$

Hence $$M_{a_1} = \frac{M_{a_2}}{r}$$

Substituting this value in eq. (30),

$$M_{a_2} = \sqrt{\frac{(r^2 - 1)}{2\gamma\left(\log_e r + \dfrac{2fl}{d}\right)}} \quad . \quad . \quad . \quad (31)$$

18.13. Stagnation Temperature and Pressure of a Gas.

(*a*) Stagnation, or Total, Temperature. The stagnation temperature of a flowing gas is its equivalent temperature if the gas is brought to rest and all of its kinetic energy thus converted into heat. This is sometimes called the *total temperature.*

Let T = actual temperature of the gas in degrees absolute,

A = cross-sectional area of pipe (square feet),

v = velocity of gas,

p = pressure of gas (pounds per square foot),

T' = stagnation temperature in degrees absolute.

$$\text{Energy of gas} = c_p T + \frac{v^2}{2gJ}$$

Then $$c_pT' = c_pT + \frac{v^2}{2gJ}$$

or $$T' = T + \frac{v^2}{2gJc_p} \qquad . \quad . \quad . \quad . \quad (32)$$

Hence $$v = \sqrt{2gJc_p} \times (T' - T)^{1/2} \qquad . \quad . \quad (33)$$

As $W = Av/V$ and $144pV = RT$, then

$$p = \frac{RWT}{144Av}$$

Substituting for v from eq. (33),

$$p = \frac{RW}{144A\sqrt{2gJc_p}}\left[\frac{T}{(T' - T)^{1/2}}\right]$$

Let $m = \dfrac{RW}{144A\sqrt{2gJc_p}}$ = a constant for a given problem. Then

$$p = m\left[\frac{T}{(T' - T)^{1/2}}\right]. \qquad . \quad . \quad . \quad (34)$$

(*b*) STAGNATION, OR TOTAL, PRESSURE OF A GAS. This is the pressure the gas would attain if brought to rest by a frictionless adiabatic compression in a diffuser. It is also known as the *total pressure.*

Then for a gas at absolute temperature T and pressure p, moving at a velocity v, which is subjected to a frictionless adiabatic compression,

increase of enthalpy = loss of kinetic energy

or $$c_p(T' - T) = \frac{v^2}{2gJ}$$

Hence $$T' - T = \frac{v^2}{2gJc_p}$$

from which T' is obtained. Also, for a frictionless adiabatic compression,

$$\frac{T'}{T} = \left(\frac{p'}{p}\right)^{\frac{\gamma-1}{\gamma}}$$

from which the stagnation pressure p' can be calculated.

18.14. Proof that Velocity in Pipe is Sonic when its Entropy is a Maximum. It was shown in § 18.4 that the velocity in a pipe was sonic when the line of condition on the T–S chart was vertical; this being the point of maximum entropy. This can be proved by

obtaining an expression for the entropy of the gas and differentiating for a maximum; that is when

$$\frac{dS}{dT} = 0$$

Applying this method to the pipe of § 18.1 and using the notation of § 18.1,

$$S_2 - S_1 = c_p \log_e \frac{T_2}{T_1} + \frac{R}{J} \log_e \frac{p_1}{p_2}$$

Expanding and substituting for p_2 from eq. (34),

$$S_2 - S_1 = c_p(\log_e T_2 - \log_e T_1) + \frac{R}{J}\left[\log_e p_1 - \log_e \frac{mT_2}{(T' - T_2)^{1/2}}\right]$$

Expanding,

$$S_2 - S_1 = c_p(\log_e T_2 - \log_e T_1) + \frac{R}{J}[\log_e p_1 - \{\log_e m + \log_e T_2 - \tfrac{1}{2}\log_e (T' - T_2)\}]$$

Differentiating for a maximum,

$$\frac{dS}{dT_2} = \frac{c_p}{T_2} - \frac{R}{J}\left[\frac{1}{T_2} + \frac{1}{2(T' - T_2)}\right] = 0$$

$$= \frac{1}{T_2}\left(c_p - \frac{R}{J}\right) - \frac{R}{2J} \times \frac{1}{(T' - T_2)} = 0 \quad . \quad . \quad . \quad (35)$$

But $R/J = c_p(\gamma - 1)/\gamma$ so that $c_p - R/J = c_p/\gamma$. Hence, substituting in eq. (35),

$$\frac{2Jc_p}{R\gamma} = \frac{T_2}{T' - T_2}$$

Substituting for $T' - T_2$ from eq. (32),

$$\frac{2Jc_p}{R\gamma} = \frac{2gJc_pT_2}{v_2^2}$$

Hence $$v_2^2 = R\gamma g T_2$$

but, from eq. (1), Chapter 17, velocity of sound at temperature $T_2 = v_{s_2} = \sqrt{(R\gamma g T_2)}$.

Hence $$v_2 = v_{s_2}$$

Hence the maximum entropy occurs when the pipe velocity is sonic; that is, when the line of condition of the pipe flow on the T–S chart is vertical. It was shown in § 18.4 that the pipe velocity cannot exceed this amount.

EXERCISES 18

1. Air at a temperature of 60°F and a pressure of 60 Lb/in.² flows through a lagged pipe of 1 in. diameter and 20 ft long; the pressure at the outlet end of the pipe is 24 Lb/in.² Assuming $f = 0{\cdot}0025$ throughout, calculate from the equations of § 18·1 the discharge in pounds per second and the velocity at the outlet. $R = 53{\cdot}3$ ft-Lb Fahrenheit units and $\gamma = 1{\cdot}4$.

Ans. $W = 0{\cdot}74$ Lb/sec; $v_2 = 958$ ft/sec.

2. Hydrogen flows through a lagged pipe $1\frac{1}{2}$ in. in diameter and 30 ft long. The temperature and pressure at the inlet end are 20°C and 19 Lb/in.^{2}; and the pressure at outlet end is 14·7 Lb/in.^{2} Assume $f = 0{\cdot}004$ throughout the length. For hydrogen, $R = 1{,}375$ ft-Lb Centigrade units, $c_p = 3{\cdot}43$ heat units and $\gamma = 1{\cdot}4$. Calculate the weight of flow and the velocity at outlet.
Ans. $W = 0{\cdot}0914$ Lb/sec; $v_2 = 1{,}402$ ft/sec.

3. Find the maximum flow of air along a 2 in. diameter pipe, 20 ft long, if the pressure and temperature at the inlet end are 60 Lb/in.^{2} and 60°F, and if the outlet end exhausts into a chamber in which the pressure is atmospheric. Assume $f = 0{\cdot}002$ throughout the length of pipe. *Ans.* $W = 4{\cdot}04$ Lb/sec.

4. A lagged pipe, 2 in. in diameter and 50 ft long, connects two large chambers A and B. Air flows adiabatically along the pipe from A to B. A constant pressure of 40 Lb/in.^{2} and a temperature of 60°F is maintained in A and a constant pressure of 15 Lb/in.^{2} is maintained in B. Assuming the pipe frictional coefficient f to be 0·002 throughout the length of the pipe, calculate to the second approximation the weight of air flowing per second between the chambers. *Ans.* $W = 1{\cdot}8$ Lb/sec.

5. Find the weight of flow of air through a $1\frac{1}{2}$ in. diameter pipe, 80 ft long, when the pressure in the pipe at the inlet end is 90 Lb/in.^{2} and that at the outlet end 30 Lb/in.^{2} Use a mean value of Darcy's f obtained by a method of successive approximations. The temperature of the air at inlet is 100°F. Find also the temperature and velocity of the air at outlet.
Ans. $W = 1{\cdot}757$ Lb/sec; $T_2 = 44{\cdot}5$°F; $v_2 = 874$ ft/sec.

6. Air flows from a compressed air chamber through a 1 in. diameter pipe. The pressure and temperature of the compressed air in the chamber are maintained constant at 100 Lb/in.^{2} and 70°F; the length of the pipe is 100 ft and the pressure at the outlet end of the pipe is 40 Lb/in.^{2}

Calculate the discharge in pounds per second. Allow for the pressure drop at the pipe inlet due to velocity head and obtain the mean value for the frictional coefficient f from a successive approximation solution.
Ans. $W = 0{\cdot}592$ Lb/sec.

7. Hydrogen flows isothermally through an unlagged pipe, 2 in. in diameter and 60 ft long. The pressure and temperature at the inlet end of the pipe are 35 Lb/in.^{2} and 60°F respectively; the pressure at the outlet end is 15 Lb/in.^{2} Calculate the flow in pounds per second, and the velocity of flow at the outlet end of the pipe. Assume the frictional coefficient f to be 0·003 throughout the whole length of the pipe. For hydrogen, $R = 765$ ft-Lb Fahrenheit units. *Ans.* $W = 0{\cdot}136$ Lb/sec; $v_2 = 2{,}065$ ft/sec.

8. Calculate the flow of compressed air through a $1\frac{1}{2}$ in. diameter pipe, 40 ft long, if the pressure and temperature inside the pipe at inlet are 80 Lb/in.^{2} and 60°F, and the pressure inside the pipe at outlet is 60 Lb/in.^{2} The flow is isothermal throughout and the mean value of Darcy's f is to be obtained by a method of successive approximations. $R = 53{\cdot}3$ ft-Lb Fahrenheit units for air. *Ans.* $W = 1{\cdot}61$ Lb/sec.

9. Two large compressed air chambers are 120 ft apart and are connected by a straight 3 in. diameter pipe. The high-pressure chamber contains air maintained at a constant pressure of 40 Lb/in.^{2} at 65°F; the low-pressure chamber is under a constant pressure of 25 Lb/in.^{2} Assuming an isothermal flow, calculate the rate of flow through the pipe, assuming a constant value for f of 0·003 throughout. Allow for the pressure drop at the pipe entrance due to the velocity head, up to the second approximation. $R = 53{\cdot}3$ ft-Lb Fahrenheit units. *Ans.* $W = 2{\cdot}76$ Lb/sec.

10. Find, using the chart for air flow, the required diameter of a lagged pipe, 2,000 ft long, to deliver the maximum possible weight of air per second. The pressure and temperature at the inlet end are 100 Lb/in.² and 120°F, and the pressure at the outlet end is 20 Lb/in.² The flow may be assumed adiabatic and Darcy's $f = 0{\cdot}0025$ throughout. What is the weight of air delivered?

Ans. $d = 7{\cdot}2$ in.; $W = 34{\cdot}25$ Lb/sec.

11. Superheated steam at a pressure of 68 Lb/in.² and a temperature of 200°C is supplied to the inlet end of a pipe of 6 in. diameter and 1,000 ft long. The outlet end of the pipe discharges into a chamber in which the pressure is constant at 15 Lb/in.² Calculate the temperature and velocity of the steam at the discharge end of the pipe, and the weight of steam discharged per second. Find, also, the sonic velocity and the Mach number of the flow at the outlet end. The following are the mean values of the properties of the superheated steam during the flow: $c_p = 0{\cdot}45$; $\gamma = 1{\cdot}35$; $R = 163$; Darcy's $f = 0{\cdot}0025$; density of steam at outlet $= 0{\cdot}0354$ Lb/ft³. (*Lond. Univ.*)

Ans. $t_2 = 127$°C; $v_2 = 1{,}237$ ft/sec; $W = 8{\cdot}69$ Lb/sec;
$v_{s_2} = 1{,}628$ ft/sec;

CHAPTER 19

FLOW OF GASES THROUGH TAPERING PIPES

19.1. Adiabatic Flow with Frictional Reheating. The complex problem of the flow of gases through a tapering pipe can be solved by the application of the four fundamental equations, as was done in § 18.1 when dealing with uniform pipes; namely—

1. The momentum equation.
2. The energy equation.
3. The gas equation.
4. The equation of continuity of flow.

The resulting equations obtained can be used for the solution of problems dealing with tapering pipes, converging and converging-diverging nozzles, and diffusers. The problem is also complicated by the frictional reheating and by the losses due to shock waves which may occur when the velocity of the gas becomes supersonic (§ 17.5 and § 17.9).

Consider the diverging pipe shown in Fig. 244 and consider an element of length $\mathrm{d}x$ at x from the left. Referring to the vertical section to the left of $\mathrm{d}x$, let

p = pressure of gas, pounds per square foot,
T = absolute temperature of gas,
v = velocity of gas,
d = diameter of pipe,
θ = slope of walls to longitudinal axis,
V = specific volume of gas, cubic feet per pound,
A = area of cross-section of pipe, square feet.

Let these values increase respectively to $p + \mathrm{d}p$, $T + \mathrm{d}T$, $v + \mathrm{d}v$, $d + \mathrm{d}(d)$, $V + \mathrm{d}V$, and $A + \mathrm{d}A$ over the length $\mathrm{d}x$, as shown in Fig. 244. Use suffix 1 for the extreme left-hand section of the pipe, when $x = 0$, and suffix 2 for the extreme right-hand section, when $x = l$.

Then,
$$\tan\theta = \frac{d_2 - d_1}{2l}$$

and
$$d = d_1 + 2x\tan\theta$$

Differentiating,
$$\mathrm{d}(d) = 2\,\mathrm{d}x\tan\theta$$

Also
$$A = \frac{\pi d^2}{4}$$

$$= \frac{\pi}{4}(d_1 + 2x\tan\theta)^2 \quad . \quad . \quad . \quad (1)$$

Differentiating, $\quad \mathrm{d}A = \pi(d_1 + 2x\tan\theta)\mathrm{d}x\tan\theta$

Dividing throughout by eq. (1),

$$\frac{\mathrm{d}A}{A} = \frac{\pi(d_1 + 2x\tan\theta)\mathrm{d}x\tan\theta}{\frac{\pi}{4}(d_1 + 2x\tan\theta)^2}$$

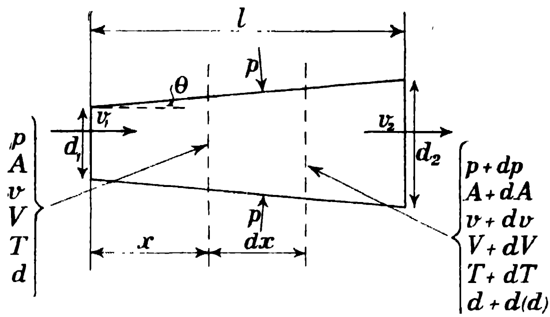


FIG. 244

$$= \frac{4\,\mathrm{d}x\tan\theta}{d_1 + 2x\tan\theta} \qquad (2)$$

The frictional drag on the inside surface of the element can be obtained from the equations of § 15.2.

$$C_f = \text{Darcy's } f \quad (\S\ 16.11)$$

$$\text{Chord area} = A' = \pi d \times \mathrm{d}x$$

$$\text{Frictional drag} = D_f = \frac{C_f A' \rho v^2}{2}$$

$$= \frac{fw(\pi d \times \mathrm{d}x)v^2}{2g} \left(\text{as } \rho = \frac{w}{g}\right)$$

$$= \frac{fv^2\,\mathrm{d}x}{2gV} \times 2A^{1/2}\sqrt{\pi} \quad (\text{as } \pi d = 2A^{1/2}\sqrt{\pi})$$

Dividing throughout by A,

$$\frac{D_f}{A} = \frac{f\sqrt{\pi}v^2\,\mathrm{d}x}{VgA^{1/2}}$$

Substituting for $A^{1/2}$ from eq. (1),

$$\frac{D_f}{A} = \frac{f\sqrt{\pi}v^2\,\mathrm{d}x \times 2}{gV\sqrt{\pi}(d_1 + 2x\tan\theta)}$$

$$= \frac{2fv^2}{gV} \times \frac{\mathrm{d}x}{(d_1 + 2x\tan\theta)} \qquad (3)$$

Applying the momentum equation to the element,

horizontal force = mass/second × change of velocity + frictional resistance

$$[p - (p + \mathrm{d}p)](A + \mathrm{d}A) = \frac{wAv\,\mathrm{d}v}{g} + D_f$$

Then
$$-A\,\mathrm{d}p = \frac{wAv\,\mathrm{d}v}{g} + D_f$$

Dividing throughout by A and substituting for D_f/A from eq. (3),

$$-\mathrm{d}p = \frac{v\,\mathrm{d}v}{Vg} + \frac{2fv^2}{gV} \times \frac{\mathrm{d}x}{(d_1 + 2x\tan\theta)} \quad . \quad (4)$$

Applying the energy equation to the element, from eq. (2), Chapter 18,

$$c_p\,\mathrm{d}T = \frac{v\,\mathrm{d}v}{gJ} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence
$$\mathrm{d}T = -\frac{v\,\mathrm{d}v}{Jc_pg} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where c_p is in heat units.

From the gas equation $pV = RT$,

$$p\,\mathrm{d}V + V\,\mathrm{d}p = R\,\mathrm{d}T \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence
$$-\mathrm{d}p = p\frac{\mathrm{d}V}{V} - R\frac{\mathrm{d}T}{V} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From the equation of continuity of flow,

$$W = \frac{Av}{V} = \text{constant}$$

Hence
$$\frac{\mathrm{d}A}{A} = \frac{\mathrm{d}V}{V} - \frac{\mathrm{d}v}{v}$$

Substituting for $\mathrm{d}A/A$ from eq. (2),

$$\frac{\mathrm{d}V}{V} = \frac{\mathrm{d}v}{v} + \frac{4\,\mathrm{d}x\tan\theta}{(d_1 + 2x\tan\theta)} \quad . \quad . \quad (9)$$

Substituting eq. (9) in eq. (8),

$$\mathrm{d}p = p\frac{\mathrm{d}v}{v} + \frac{4p\,\mathrm{d}x\tan\theta}{(d_1 + 2x\tan\theta)} - \frac{R\,\mathrm{d}T}{V} \quad . \quad (10)$$

Substituting for $\mathrm{d}T$ from eq. (6) in eq. (10),

$$-\mathrm{d}p = p\frac{\mathrm{d}v}{v} + \frac{4p\,\mathrm{d}x\tan\theta}{(d_1 + 2x\tan\theta)} + \frac{Rv\,\mathrm{d}v}{Jc_pVg} \quad . \quad (11)$$

From energy equation,

$$T = T_1 - \frac{v^2}{2gJc_p} + \frac{v_1^2}{2gJc_p} \quad . \quad . \quad . \quad (12)$$

Equating eq. (4) and eq. (11),

$$\frac{p\,dv}{v} + \frac{4p\,dx\tan\theta}{(d_1 + 2x\tan\theta)} + \frac{Rv\,dv}{c_pVgJ} = \frac{v\,dv}{Vg} + \left(\frac{2fv^2}{gV} \times \frac{dx}{(d_1 + 2x\tan\theta)}\right)$$

Multiplying throughout by V and substituting RT for pV,

$$RT\frac{dv}{v} + \frac{RT4\tan\theta\,dx}{(d_1 + 2x\tan\theta)} + \left(\frac{R}{Jc_p} - 1\right)\frac{v\,dv}{g} = \frac{2fv^2\,dx}{g(d_1 + 2x\tan\theta)}$$

Substituting for T from eq. (12),

$$RT_1\frac{dv}{v} - \frac{Rv\,dv}{2gJc_p} + \frac{Rv_1^2\,dv}{2gc_pvJ} + \left(\frac{R}{Jc_p} - 1\right)\frac{v\,dv}{g}$$

$$= \left[\frac{2fv^2}{g} - 4RT_1\tan\theta + \frac{4Rv^2\tan\theta}{2gc_pJ} - \frac{4Rv_1^2\tan\theta}{2gc_pJ}\right]\frac{dx}{(d_1 + 2x\tan\theta)}$$

Substituting for $R/c_p = (\gamma - 1)J/\gamma$,

$$RT_1\frac{dv}{v} - \left(\frac{\gamma + 1}{\gamma}\right)\frac{v\,dv}{2g} + \left(\frac{\gamma - 1}{\gamma}\right)\frac{v_1^2}{2g}\frac{dv}{v} = \left[\frac{2fv^2}{g} - 4RT_1\tan\theta\right.$$

$$\left. + \frac{4\tan\theta}{2g}\left(\frac{\gamma - 1}{\gamma}\right)v^2 - \frac{4\tan\theta}{2g}\left(\frac{\gamma - 1}{\gamma}\right)v_1^2\right]\frac{dx}{(d_1 + 2x\tan\theta)}$$

Hence, re-arranging for integrating, and placing all constants within the square brackets,

$$\frac{\left[RT_1 + \left(\frac{\gamma - 1}{\gamma}\right)\frac{v_1^2}{2g}\right]\frac{dv}{v} + \left[-\left(\frac{\gamma + 1}{2g\gamma}\right)\right]v\,dv}{\left[\frac{2f}{g} + \frac{4\tan\theta}{2g}\left(\frac{\gamma - 1}{\gamma}\right)\right]v^2 + \left[-4RT_1\tan\theta - \frac{4\tan\theta}{2g}\left(\frac{\gamma - 1}{\gamma}\right)v_1^2\right]}$$

$$= \frac{dx}{(d_1 + 2x\tan\theta)}$$

This may be written,

$$\frac{\bar{C}\frac{dv}{v} + \bar{D}\,v\,dv}{\bar{B}v^2 + \bar{A}} = \frac{dx}{(d_1 + 2x\tan\theta)} \quad . \quad . \quad . \quad (13)$$

where $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are the four constant terms within the square brackets.

Then $$\bar{A} = -4\tan\theta\left[RT_1 + \left(\frac{\gamma - 1}{\gamma}\right)\frac{v_1^2}{2g}\right] \quad . \quad (14a)$$

But $(\gamma - 1)/\gamma = R/c_pJ$.

Hence $$\bar{A} = -4\tan\theta\left[R\left(T_1 + \frac{v_1^2}{2gJc_p}\right)\right]$$

But from § 18.13,

$$T_1 + \frac{v_1^2}{2gJc_p} = \text{stagnation temperature } T'$$

Hence $$\bar{A} = -4RT'\tan\theta \quad . \quad . \quad . \quad (14b)$$

and is a constant for any given problem.

Also $$\bar{B} = \frac{2f}{g} + \frac{4\tan\theta}{2g}\left(\frac{\gamma - 1}{\gamma}\right) \quad . \quad . \quad . \quad (15)$$

and is a constant for any given problem;

and $$\bar{C} = RT_1 + \left(\frac{\gamma - 1}{\gamma}\right)\frac{v_1^2}{2g} \quad . \quad . \quad (16a)$$

$$= RT' \quad . \quad . \quad . \quad . \quad . \quad (16b)$$

This is constant for any given problem but is independent of the type of tapering pipe.

$$\bar{D} = -\frac{\gamma + 1}{2g\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

This is a constant for all tapering pipes but depends on the type of gas flowing.

It will be noticed that the value of ratio $\bar{C}/\bar{A} = -1/(4\tan\theta)$.

Integrating* eq. (13) between the limits of v_2 and v_1 and l and 0,

$$\int_{v_1}^{v_2}\left[\frac{\frac{\bar{C}}{\bar{A}}(\bar{A}v^{-1} + \bar{B}v) + \left(\bar{D} - \frac{\bar{B}\bar{C}}{\bar{A}}\right)v}{\bar{A} + \bar{B}v^2}\right]\mathrm{d}v = \int_0^l \frac{}{(d_1 + 2x\tan\theta)}$$

or

$$\frac{\bar{C}}{\bar{A}}\int_{v_1}^{v_2}\frac{\mathrm{d}v}{v} + \frac{\left(\bar{D} - \frac{\bar{B}\bar{C}}{\bar{A}}\right)}{2\bar{B}}\int_{v_1}^{v_2}\frac{2\bar{B}v\,\mathrm{d}v}{\bar{A} + \bar{B}v^2} = \int_0^l \frac{\mathrm{d}x}{(d_1 + 2x\tan\theta)}$$

from which

$$\frac{\bar{C}}{\bar{A}}\Big[\log_e v\Big]_{v_1}^{v_2} + \left(\frac{\bar{D} - \frac{\bar{B}\bar{C}}{\bar{A}}}{2\bar{B}}\right)\Big[\log_e(\bar{A} + \bar{B}v^2)\Big]_{v_1}^{v_2}$$
$$= \frac{1}{2\tan\theta}\Big[\log_e(d_1 + 2x\tan\theta)\Big]_0^l$$

* The author is indebted to his former colleague Mr. R. Bland, late of the Imperial College of Science and Technology, for integrating eq. (13).

Inserting the limits and reducing to common logs,

$$\frac{\bar{C}}{\bar{A}} \log r + \left(\frac{\bar{D}}{2\bar{B}} - \frac{\bar{C}}{2\bar{A}}\right) \log \left(\frac{\bar{A} + \bar{B}v_1{}^2r^2}{\bar{A} + \bar{B}v_1{}^2}\right) = \frac{\log \frac{d_2}{d_1}}{2 \tan \theta} \quad . \quad . \quad (18)$$

where $r = v_2/v_1$ and $d_1 + 2l \tan \theta = d_2$.

Any problem on a tapering pipe can be solved by the application of eqs. (12) and (18); the constants $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ must first be calculated from eqs. (14), (15), (16) and (17). To obtain the solution it is necessary to apply the energy equation [eq. (12)] and the momentum equation [eq. (18)] and to solve by plotting; the method is demonstrated in § 19.2.

The value of the velocity ratio can be obtained from the gas law and equation of continuity of flow. As

$$\frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2} \text{ per pound of gas}$$

Then
$$\frac{V_2}{V_1} = \frac{p_1}{p_2} \times \frac{T_2}{T_1}$$

Also
$$W = \frac{A_1v_1}{V_1} = \frac{A_2v_2}{V_2}$$

Hence
$$r = \frac{v_2}{v_1} = \frac{V_2}{V_1} \times \frac{A_1}{A_2}$$

$$= \frac{p_1T_2A_1}{p_2T_1A_2} \quad . \quad . \quad . \quad . \quad (19)$$

The above equations for a tapering pipe reduce to the same equations obtained in § 18.1 for a uniform pipe if the angle of side slope, θ, is inserted as zero. This must be done before the equations are finally integrated, because, after integrating, eq. (18) is of a logarithmic form and gives no solution when $\theta = 0$.

19.2. Application of Equations to a Diverging Pipe. The equations evolved in § 19.1 will now be applied to the diverging pipe shown in Fig. 245. Let air at a pressure of 100 Lb/in.^{2} and a temperature of 500°R be supplied to the left-hand end of the pipe, and let the back pressure at the right-hand end be constant at 15 Lb/in.^{2} Apply suffix 1 to the left-hand end and suffix 2 to the right-hand end. Assume the frictional coefficient f to be 0·0025.

For the pipe shown in Fig. 245,

$d_1 = 1$ in. $\qquad \tan \theta = \dfrac{1{\cdot}75 - 1{\cdot}0}{2 \times 18} = 0{\cdot}0208$

$d_2 = 1{\cdot}75$ in.

$l = 18$ in. $\qquad \gamma = 1{\cdot}4 \qquad R = 53{\cdot}3$ ft-Lb units

$A_1 = 0{\cdot}00544$ ft$^2 \qquad A_2 = 0{\cdot}0167$ ft^2

First calculate the values of the constants $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ from the equations of § 19.1.

From eq. (14*a*) $\quad \bar{A} = -4\tan\theta\left[RT_1 + \left(\frac{\gamma - 1}{\gamma}\right)\frac{v_1^2}{2g}\right]$

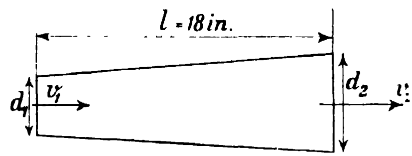


FIG. 245

$$= -4 \times 0{\cdot}0208\left[(53{\cdot}3 \times 500) + \frac{0{\cdot}4}{1{\cdot}4}\frac{v_1^2}{2g}\right]$$

$$= -2{,}218 - 0{\cdot}000369v_1^2$$

From eq. (15) $\quad \bar{B} = \frac{2f}{g} + \frac{4\tan\theta}{2g}\left(\frac{\gamma - 1}{\gamma}\right)$

$$= \frac{2 \times 0{\cdot}0025}{32{\cdot}2} + \frac{4 \times 0{\cdot}0208 \times 0{\cdot}4}{64{\cdot}4 \times 1{\cdot}4}$$

$$= 0{\cdot}0005254$$

From eq. (16*a*) $\quad \bar{C} = RT_1 + \left(\frac{\gamma - 1}{\gamma}\right)\frac{v_1^2}{2g}$

$$= (53{\cdot}3 \times 500) + \frac{0{\cdot}4}{1{\cdot}4}\frac{v_1^2}{64{\cdot}4}$$

$$= 26{,}650 + 0{\cdot}00444v_1^2$$

From eq. (17) $\quad \bar{D} = -\left(\frac{\gamma + 1}{2g\gamma}\right)$

$$= -\left(\frac{1{\cdot}4 + 1}{64{\cdot}4 \times 1{\cdot}4}\right)$$

$$= -0{\cdot}0266$$

$$\frac{\bar{C}}{\bar{A}} = -\frac{1}{4\tan\theta}$$

$$\frac{1}{4 \times 0{\cdot}0208} = -12$$

$$\frac{\bar{D}}{2\bar{B}} \quad \frac{-0{\cdot}0266}{2 \times 0{\cdot}0005254} = -25{\cdot}3$$

Applying eq. (19),

$$r = \frac{p_1 T_2 A_1}{p_2 T_1 A_2}$$

$$\frac{100 \times T_2 \times 0{\cdot}00544}{15 \times 500 \times 0{\cdot}0167} = 0{\cdot}00434 T_2 \qquad . \ (20)$$

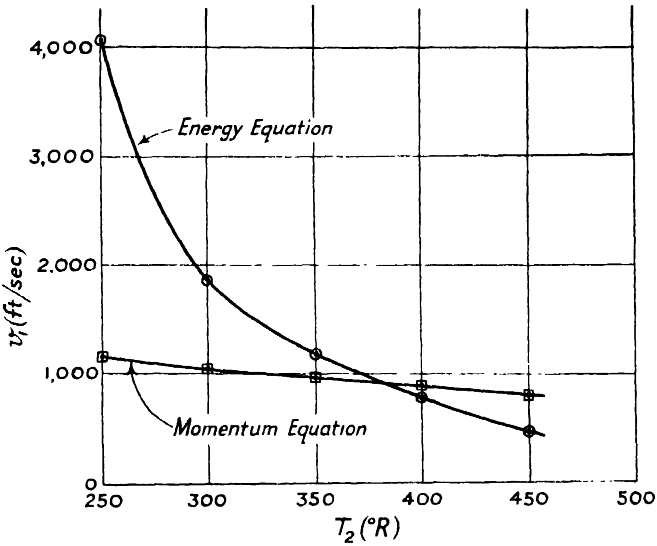


FIG. 246

Applying the momentum equation [eq. (18)],

$$\frac{C}{A}\log r + \left(\frac{D}{2B} - \frac{C}{2A}\right)\log\left(\frac{A + Bv_1^2 r^2}{A + Bv_1^2}\right) = \frac{\log \dfrac{d_2}{d_1}}{2 \tan \theta}$$

Inserting the above values of the constants,

$$-12\log r + (-25{\cdot}3 + 6) \times$$

$$\log\left(\frac{-2{,}218 - 0{\cdot}000369v_1^2 + 0{\cdot}0005254v_1^2r^2}{-2{,}218 - 0{\cdot}000369v_1^2 + 0{\cdot}0005254v_1^2}\right) = \frac{\log 1{\cdot}75}{2 \times 0{\cdot}0208}$$

Then

$$-12\log r \quad 19{\cdot}3\log \left(\frac{2{,}218 - 0{\cdot}000369v_1^2 + 0{\cdot}0005254v_1^2r^2}{-2{,}218 + 0{\cdot}0001564v_1^2}\right)$$

$$= 5{\cdot}84$$

To solve this equation, assume convenient values of T_2, and obtain from eq. (20) the corresponding values of r for each assumed value of T_2. Next substitute these values of r in the above momentum equation and thus obtain the corresponding values of v_1.

The momentum equation can now be represented by a graph plotted with v_1 as the ordinate on a base representing T_2. The results obtained by this method are given in the table below and the momentum equation is shown plotted in Fig. 246.

Next apply the energy equation [eq. (12)],

$$T_2 + \frac{v_2^2}{2gJc_p} = T_1 + \frac{v_1^2}{2gJc_p}$$

Hence, substituting $v_2 = rv_1$,

$$T_2 = T_1 - \frac{v_1^2}{2gJc_p}(r^2 - 1) \quad . \quad . \quad . \quad (21)$$

Using the same assumed values of T_2 and the same corresponding values of r as in the momentum equation, calculate the corresponding values of v_1 from eq. (21). The results so obtained are given in the following table. From these results the graph representing the energy equation can now be plotted with v_1 as ordinate and on a base T_2. This curve is also shown plotted in Fig. 246; the point of intersection of the energy curve with the momentum curve gives the correct value of v_1 and T_2.

T_2 (assumed) (°R)	$r = 0{\cdot}00434T_2$	v_1 from Momentum Equation [eq. (18)]	v_1 from Energy Equation [eq. (12)]
		ft/sec	ft/sec
250	1·087	1,131	4,058
300	1·302	1,034	1,857
350	1·517	950	1,177
400	1·736	870	774
450	1·952	801	462

From point of intersection of curves,

$$v_1 = 900 \text{ ft/sec}$$

$$T_2 = 382\text{°R}$$

From eq. (20), $r = 0{\cdot}00434 \times 382$

$$= 1{\cdot}66$$

hence $v_2 = 1{\cdot}66 \times 900$

$$= 1{,}495 \text{ ft/sec}$$

As $144p_1V_1 = RT_1$, $V_1 = \dfrac{53{\cdot}3 \times 500}{144 \times 100}$

$$= 1{\cdot}85 \text{ ft}^3\text{/Lb}$$

$$\text{Weight of flow} = \frac{A_1 v_1}{V_1}$$

$$\frac{0{\cdot}00544 \times 900}{1{\cdot}85} = 2{\cdot}645 \text{ Lb/sec}$$

19.3. Supersonic Flow in Diverging Pipe. Consider the diverging pipe shown in Fig. 247. Let air be supplied to the left-hand end at a pressure of 52·8 Lb/in.^{2}, a temperature of 500°R, and flowing at

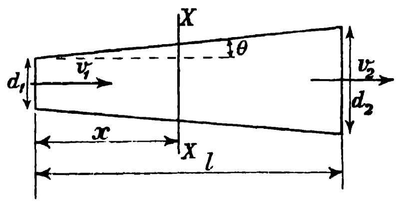


FIG 247

its sonic speed of 1,097 ft/sec. The following dimensions apply to the pipe—

$$d_1 = 1 \text{ in.} \qquad v_1 = 1{,}097 \text{ ft/sec}$$

$$d_2 = 2 \text{ in.} \qquad \tan\theta = 0{\cdot}0208$$

$$l = 24 \text{ in.} \qquad f = 0{\cdot}0025$$

$$V_1 = \frac{53{\cdot}3T_1}{144p_1} = \frac{53{\cdot}3 \times 500}{144 \times 52{\cdot}8} = 3{\cdot}51 \text{ ft}^3/\text{Lb}$$

$$W = \frac{A_1 v_1}{V_1} = \frac{0{\cdot}00544 \times 1{,}097}{3{\cdot}51} \quad 1{\cdot}7 \text{ Lb/sec}$$

The values of p, T, v and V at any section of the pipe can be calculated by applying the momentum equation, the energy equation, the equation of continuity of flow, and the gas equation to various sections of the pipe. Consider a section X at x in. from the left-hand end. Let d_x, A_x, p_x, T_x, v_x and V_x apply to section X.

First calculate the values of the four constants of the momentum equation, $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$, from the equations of § 19.1.
From eq. (16*a*),

$$\bar{C} = RT_1 + \left(\frac{\gamma - 1}{\gamma}\right)\frac{v_1^2}{2g}$$

$$= (53{\cdot}3 \times 500) + \frac{0{\cdot}4}{1{\cdot}4} \times \frac{1{,}097^2}{64{\cdot}4} = 32{,}000$$

From eq. (14a),

$$\bar{A} = -4\bar{C}\tan\theta$$
$$= -4 \times 32{,}000 \times 0{\cdot}0208 = -2{,}666$$

From eq. (17),

$$\bar{D} = -\frac{\gamma + 1}{2g\gamma} = -\frac{2{\cdot}4}{64{\cdot}4 \times 1{\cdot}4} = -0{\cdot}0266$$

From eq. (15),

$$\bar{B} = \frac{2f}{g} + 4\tan\theta\left(\frac{\gamma - 1}{\gamma}\right)$$
$$\left(\frac{2 \times 0{\cdot}0025}{32{\cdot}2}\right) + \frac{4 \times 0{\cdot}0208}{64{\cdot}4}\left(\frac{0{\cdot}4}{1{\cdot}4}\right) = 0{\cdot}0005244$$

Applying the momentum equation [eq. (18)],

$$\frac{\bar{C}}{\bar{A}}\log r + \left(\frac{\bar{D}}{2\bar{B}} - \frac{\bar{C}}{2\bar{A}}\right)\log\left(\frac{\bar{A} + \bar{B}v_1^2r^2}{\bar{A} + \bar{B}v_1^2}\right) = \frac{\log\frac{d_x}{d_1}}{2\tan\theta}$$

Inserting the known values of $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, v_1, d_x, d_1 and $\tan\theta$, the value of r can be obtained by plotting. Then,

$$v_x = rv_1$$

Applying the energy equation [eq. (12)], to the inlet end of the pipe and to section X,

$$H_1 + \frac{v_1^2}{2gJ} = H_x + \frac{v_x^2}{2gJ}$$

Then $$c_p(T_1 - T_x) \quad \frac{v_x^2 - v_1^2}{2gJ}$$

from which equation the value of T_x can be calculated.

Applying the equation of continuity of flow,

$$W = \frac{A_xv_x}{V_x}$$

This equation gives V_x as W is known.

From the gas equation,

$$144p_xV_x = RT_x$$

from which p_x can be calculated.

This process can be repeated for all chosen values of x, for the whole length of the pipe, thus obtaining the values of r, v_x, T_x, V_x and p_x for each chosen value of x. This has been done for $x = 3$, 6, 9, 12, 15, 18, 21 and 24 in., and the results obtained are given in the following table. The Mach number at X is also given in the table. It should be noted that the constants of the momentum equation, $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$, have the same value for all sections of the pipe.

x (in.)	0	3	6	9	12	15	18	21	24
. .	1·0	1·433	1·524	1·598	1·661	1·716	1·764	1·8	1·8225
v_x (ft/sec) . .	1,097	1,572	1,671	1,752	1,822	1,880	1,938	1,970	2,000
T_x (°R) . .	500	394·3	367	344·4	323·5	306	288	276	267
V_x (ft³/Lb) . .	3·51	6·39	8·4	10·62	13·17	15·9	19·03	22·2	25·3
p_x (Lb/in.²) . .	52·8	22·85	16·2	12·0	9·10	7·14	5·61	4·59	3·85
M_x . . .	1·0	1·617	1·78	1·925	2·07	2·19	2·33	2·42	2·5

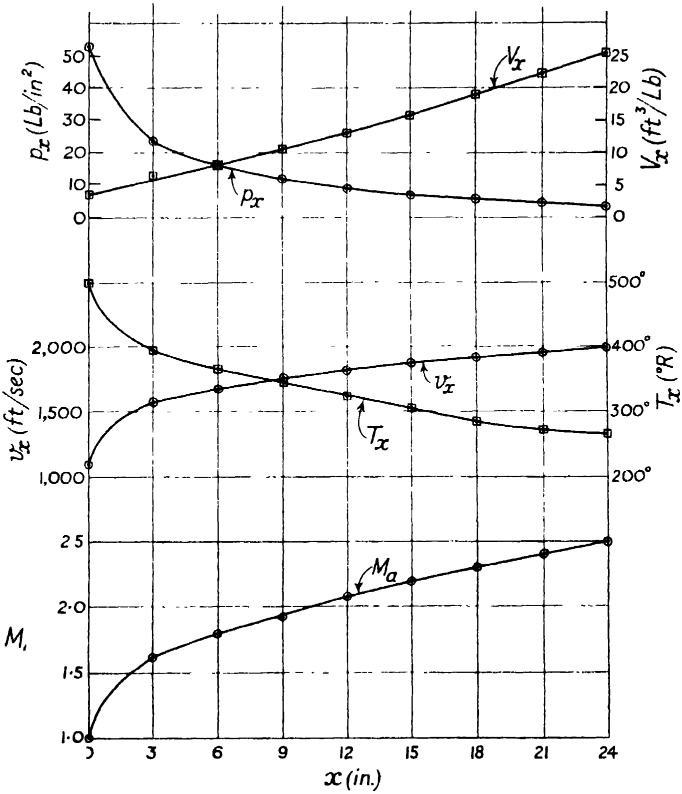


FIG. 248

Curves showing the variation of p_x, v_x, T_x, V_x and M_{a_x} are plotted in Fig. 248 on a base representing the centre-line of the pipe. It will be noticed that the rate of increase in the velocity, and the rate of fall of the pressure, are rapid at the beginning of the pipe but become almost constant after the first 12 in. of the length.

The process is also shown plotted on the temperature-entropy diagram of Fig. 249. It will be seen from this curve that there is a

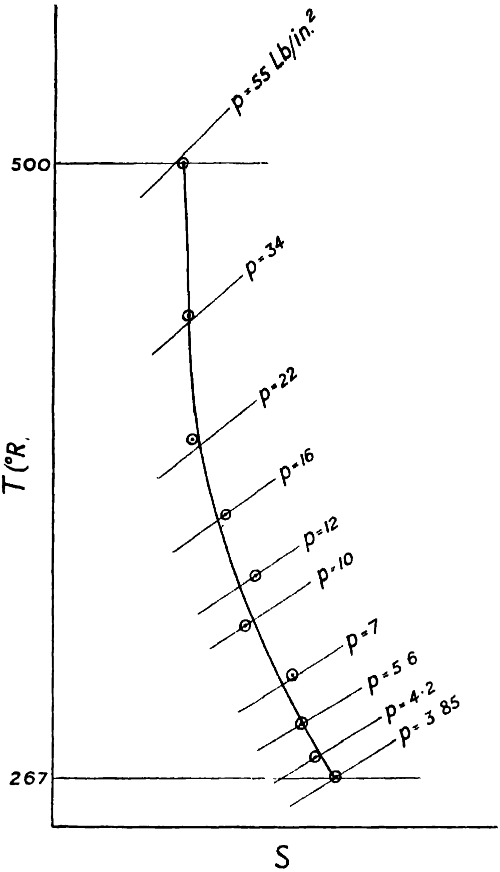


FIG. 249

continuous rate of increase in entropy as the expansion increases, which denotes an increase in frictional reheating.

In the above solution it has been assumed that no pressure jump occurs in the diverging pipe.

19.4. Subsonic Flow through Diverging Pipe. In § 19.3 the problem of supersonic flow through a diverging pipe was considered. Using the same diverging pipe and assuming the same air supply at the inlet end for p_1, T_1 and sonic speed v_1, the problem will now be solved for subsonic flow. The solution is obtained in the same

manner as that for supersonic flow, but in this case the velocity ratio r will be less than unity because the velocity decreases as x increases. The solution will also apply to a diffuser.

Referring to the diverging pipe of Fig. 247, the pipe dimensions are the same and the air is assumed to enter the inlet end with the same pressure, temperature, velocity and mass flow as before; namely, $v_1 = 1{,}097$ ft/sec, $p_1 = 52{\cdot}8$ Lb/in.^{2}, $T_1 = 500°$R, and $W = 1{\cdot}7$ Lb/sec. When applying the momentum equation the constants $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ have the same values as in § 19.3 because v_1 and T_1 are the same. The momentum equation is consequently the same as in § 19.3 and can be solved for various assumed values of r by plotting or by trial. The assumed values of r chosen range from $r = 0{\cdot}1$ to $r = 0{\cdot}9$.

From § 19.3, the momentum equation is

$$\frac{\bar{C}}{\bar{A}}\log r + \left(\frac{\bar{D}}{2\bar{B}} - \frac{\bar{C}}{2\bar{A}}\right)\log\left(\frac{\bar{A} + \bar{B}v_1^2r^2}{\bar{A} + \bar{B}v_1^2}\right) = \frac{\log \dfrac{d_x}{d_1}}{2\tan\theta}$$

where $\bar{A} = 2{,}666$, $\bar{B} = 0{\cdot}0005244$, $\bar{C} = 32{,}000$ and $\bar{D} = -0{\cdot}0266$.

From this equation the value of r at each 3 in. section of the pipe has been calculated. Then

$$v_x = rv_1$$

Next apply the energy equation to each value of v_x:

$$c_p(T_1 - T_x) = \frac{v_x^2 - v_1^2}{2gJ}$$

thus obtaining the value of T_x at each 3 in. length of pipe.

From the equation of continuity of flow,

$$V_x = \frac{A_x v_x}{W}$$

from which the required values of A_x can be calculated.

Next apply the gas equation

$$144 p_x V_x = 53{\cdot}3 T_x$$

which gives the value of p_x at each 3 in. length. The results obtained are given in the following table.

x (in.)	0	3	6	9	12	15	18	21
r_x .	1	0·589	0·452	0·356	0·297	0·250	0·213	0·186
v_x .	1,097	645	495	392	315	274	234	204
T_x .	500°	565·2	579·5	587·3	591·7	593·7	595·4	596·4
A_x .	0·00544	0·0069	0·00853	0·0103	0·01225	0·0144	0·0167	0·0192
p_x .	52·8	79·7	86·3	91·8	92·9	94·5	96·0	95·8

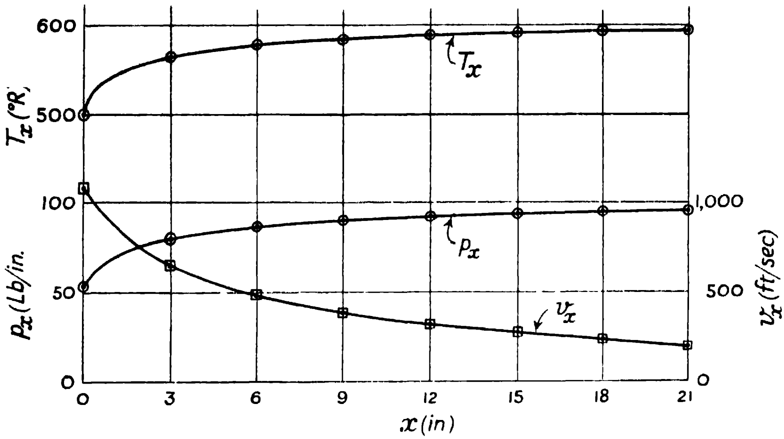


FIG. 250

Curves showing the variation of p_x, T_x and v_x are shown plotted in Fig. 250. It will be noticed that the pressure p_x remains almost constant after the 12 in. section is passed. Hence, if the pipe is being used as a diffuser, there would be no further gain by extending the length beyond 12 in.

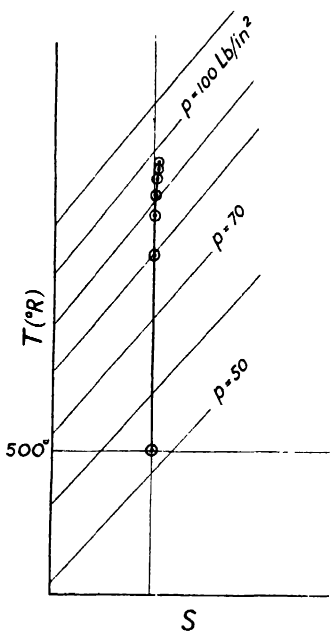


FIG. 251

The flow through the pipe is also shown plotted on the temperature-entropy diagram of Fig. 251. It will be noticed that there is only a slight increase in entropy during the process and that this occurs near the outlet end. This denotes that the frictional reheating of the air is extremely small; hence, the compression approximates to a reversible adiabatic process.

19.5. Application to a Converging Pipe. The equations of § 19.1 can be applied to a converging pipe, but in this case tan θ is negative. The method of solution is demonstrated in the following worked-out example; this method can also be applied to a conical shaped converging nozzle. It will be noticed from the results obtained that the velocity at the discharge end cannot exceed its sonic speed; a

similar conclusion was found to apply to the uniform pipe problem of Chapter 18.

Let air of pressure 100 Lb/in.² and temperature 600°R be supplied at the left-hand end of the converging pipe shown in Fig. 252, and

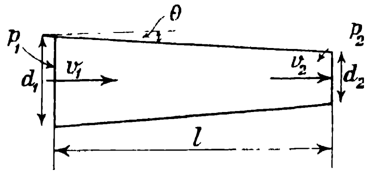


Fig. 252

let the back pressure at the right-hand end be 50 Lb/in.² Use suffix 1 for the left-hand end and suffix 2 for the discharge end. The following are the particulars of the pipe—

$$d_1 = 1{\cdot}5 \text{ in.} \qquad f = 0{\cdot}0025$$

$$d_2 = 1{\cdot}0 \text{ in.} \qquad A_1 = 0{\cdot}01223 \text{ ft}^2$$

$$l = 12 \text{ in.} \qquad A_2 = 0{\cdot}00544 \text{ ft}^2$$

$$\theta = \frac{d_2 - d_1}{2l} = \frac{1 - 1{\cdot}5}{2 \times 12} = -0{\cdot}0208$$

First find the constants $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ from eqs. (14) to (17). From eq. (14*a*)

$$\bar{A} = -4 \tan \theta \left[RT_1 + \left(\frac{\gamma - 1}{\gamma}\right) \frac{v_1^2}{2g} \right]$$

$$= 0{\cdot}0832 \left[(53{\cdot}3 \times 600) + \frac{0{\cdot}4}{1{\cdot}4} \frac{v_1^2}{64{\cdot}4} \right]$$

$$= 2{,}660 + 0{\cdot}00037 v_1^2$$

From eq. (15)

$$\bar{B} = \frac{2f}{g} + \frac{4 \tan \theta}{2g} \left(\frac{\gamma - 1}{\gamma}\right)$$

$$= \frac{2 \times 0{\cdot}0025}{32{\cdot}2} - \frac{0{\cdot}0832}{64{\cdot}4} \left(\frac{0{\cdot}4}{1{\cdot}4}\right)$$

$$= 0{\cdot}0001552 \quad 0{\cdot}00037 = -0{\cdot}000215$$

From eq. (16*a*)

$$\bar{C} = RT_1 \quad \left(\frac{\gamma - 1}{\gamma}\right) \frac{v}{2}$$

$$= (53{\cdot}3 \times 600) + \frac{0{\cdot}4}{1{\cdot}4} \times \frac{v_1^2}{64{\cdot}4}$$

$$= 32{,}000 + 0{\cdot}00444 v_1^2$$

From eq. (17)

$$\bar{D} = -\frac{\gamma + 1}{2g\gamma}$$

$$= -\frac{2{\cdot}4}{64{\cdot}4 \times 1{\cdot}4} = -0{\cdot}0266$$

Then $$\frac{\bar{C}}{\bar{A}} = \frac{1}{-4 \tan \theta} = 12$$

and $$\frac{\bar{D}}{2\bar{B}} = \frac{-0{\cdot}0266}{2 \times (-0{\cdot}000215)} = 62{\cdot}2$$

Applying the momentum equation [eq. (18)],

$$\frac{\bar{C}}{\bar{A}} \log r + \left(\frac{\bar{D}}{2\bar{B}} - \frac{\bar{C}}{2\bar{A}}\right) \log \left(\frac{\bar{A} + \bar{B}v_1{}^2r^2}{\bar{A} + \bar{B}v_1{}^2}\right) = \frac{\log \dfrac{d_2}{d_1}}{2 \tan \theta}$$

Inserting above values,

$$12 \log r + (62{\cdot}2 - 6) \log \left(\frac{2{,}660 + 0{\cdot}00037v_1{}^2 - 0{\cdot}000215v_1{}^2r^2}{2{,}660 + 0{\cdot}00037v_1{}^2 - 0{\cdot}000215v_1{}^2}\right)$$

$$= \frac{\log \dfrac{1}{1{\cdot}5}}{-2 \times 0{\cdot}0208}$$

Then

$$12 \log r + 56{\cdot}2 \log \left[\frac{2{,}660 + 0{\cdot}00037v_1{}^2 - 0{\cdot}000215v_1{}^2r^2}{2{,}660 + 0{\cdot}000155v_1{}^2}\right]$$

$$= 4{\cdot}23. \qquad . \ (22)$$

From eq. (19)

$$r = \frac{p_1T_2A_1}{p_2T_1A_2}$$

$$= \frac{100 \times T_2 \times 0{\cdot}01223}{50 \times 600 \times 0{\cdot}00544} = 0{\cdot}0075T_2 \ . \ (23)$$

The method of solution of eq. (22) is to assume convenient values for T_2 and obtain the corresponding values of r from eq. (23). Then by inserting these values of r in eq. (22), the corresponding values of v_1 can be calculated. This has been done and the values obtained are shown in the table below. The values of v_1 can now be plotted on a base representing T_2; the curve so obtained represents the momentum equation and is shown plotted in Fig. 253.

Next apply the energy equation [eq. (12)] to the problem and obtain another series of values for v_1 for the same assumed values of T_2 used in the solution of the momentum equation.

Applying the energy equation [eq. (12)],

$$T_1 + \frac{v_1^2}{2gJc_p} = T_2 + \frac{(rv_1)^2}{2gJc_p}$$

from which
$$v_1^2 = (T_2 - T_1)\frac{2gJc_p}{(1 - r^2)}$$

Hence
$$v_1 = \sqrt{(T_2 - 600)\frac{12,020}{(1 - r^2)}} \tag{24}$$

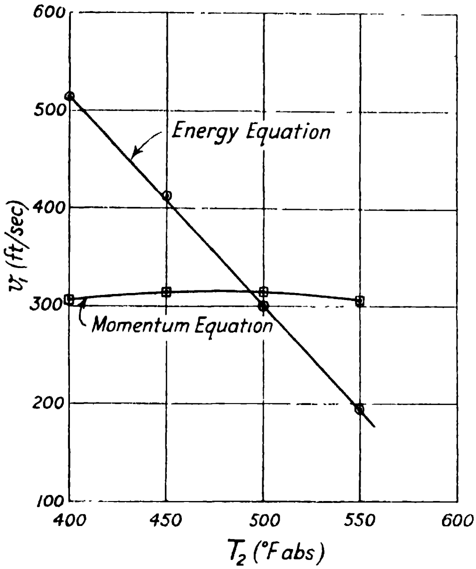


FIG. 253

The values for v_1 obtained from eq. (24) are given in the following table. These are shown plotted on a base T_2 in Fig. 253, the curve so obtained representing the energy equation. The true value of v_1 and T_2 is represented by the point of intersection of the momentum and energy equations.

T_2 (assumed) (°R)	r [from eq. (23)]	v_1 from Momentum Eq. [eq. (22)] (ft/sec)	v_1 from Energy Eq. [eq. (24)] (ft/sec)
550	4·127	305	193·5
500	3·750	316	303·2
450	3·375	315	416·7
400	3·000	306	516·5

From the point of intersection of the curves of Fig. 253,

$$v_1 = 316 \text{ ft/sec}$$

$$T_2 = 493°\text{R}$$

From eq. (23) $$r = 0{\cdot}0075 \times 493 = 3{\cdot}7$$

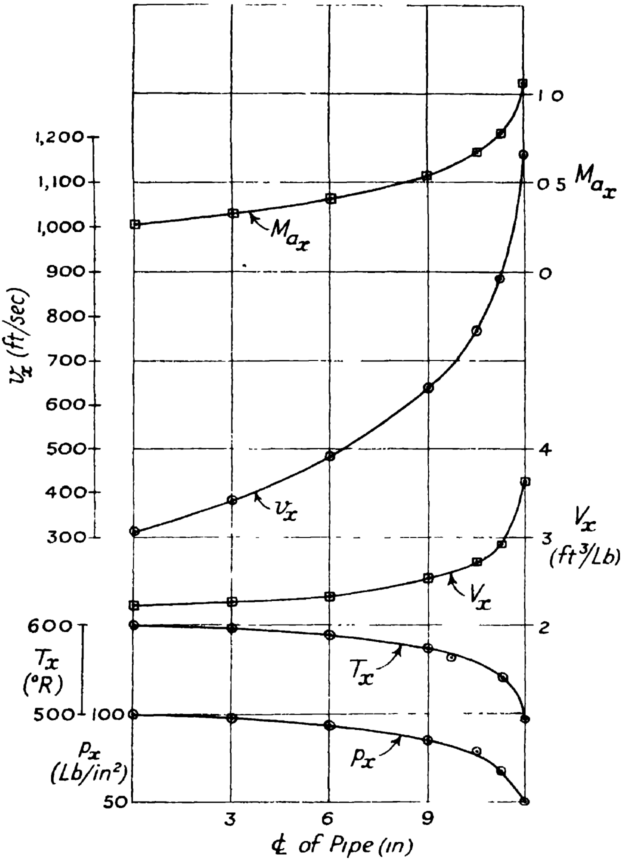


FIG. 254

Then $$v_2 = rv_1$$

$$= 3{\cdot}7 \times 316 = 1{,}170 \text{ ft/sec}$$

As $p_1 V_1 = R w_1 T_1$,

$$w_1 = \frac{144 \times 100 \times 1}{53{\cdot}3 \times 600} = 0{\cdot}45 \text{ Lb/ft}^3$$

$$\text{Weight of air flowing} = w_1 A_1 v_1$$

$$= 0{\cdot}45 \times 0{\cdot}01223 \times 316$$

$$= 1{\cdot}748 \text{ Lb/sec}$$

The variation of T, p, V, v and M_a are shown plotted, on a base representing the longitudinal axis of the pipe, in Fig. 254. These values were obtained by applying eq. (22) to different sections along the pipe, the initial velocity v_1 being a known amount of 316 ft/sec in each application. The discharge W is also known and equal to 1·748 Lb/sec. It will be noticed from the curves of Fig. 254 that in the last 3 in. length of the pipe there is a rapid increase in the values of V, v and M_a and a rapid reduction in p and T. The value of M_a reaches unity at about $\frac{1}{8}$ in. from the discharge end.

The value of M_a at the pipe exit cannot exceed unity in a converging pipe, as it would cause a decrease in entropy (§ 18.4). A subsonic flow through a converging pipe behaves in the same manner as a pipe of uniform cross-section and attains its maximum velocity when a Mach number of unity is reached.

Thus, using the suffix x for any section at x ft from the inlet end and applying eq. (22),

$$12 \log r + 56{\cdot}2 \log \left(\frac{2{,}660 + 0{\cdot}00037v_1^2 - 0{\cdot}000215v_1^2r^2}{2{,}660 + 0{\cdot}000155v_1^2}\right) = \frac{\log \frac{d_x}{d_1}}{2 \tan \theta}$$

Substituting $v_1 = 316$ ft/sec and the correct value of d_x, the value of r can be obtained for any assumed value of x. Then

$$v_x = rv_1$$

$$V_x = \frac{A_x v_x}{W}$$

From energy equation,

$$T_x = T_1 + \frac{v_1^2 - v_x^2}{2gJc_p}$$

As sonic velocity $v_s = 49\sqrt{T_x}$ for Fahrenheit units (§ 17.2),

$$M_{a_x} = \frac{v_x}{49\sqrt{T_x}}$$

The curves of Fig. 254 were plotted from these equations.

19.6. Effect of Angle of Slope of Tapering Pipe. The effect of the angle of slope of the sides of a tapering pipe can be seen by applying the solutions given in § 19.3 and § 19.4. In the following investigation a number of 12 in. long, tapering pipes have been considered, as shown in Fig. 247, each having the same inlet diameter of 1 in., but the outlet diameters, d_2, varied from 1·05 in. to 3·5 in., thus varying the angle of side slope θ. In each case the initial air supply at the inlet end is the same for each pipe, the velocity of the air supplied being at its sonic speed.

Then, referring to Fig. 247,

p_1 : 52·8 Lb/in.² $\quad v_1 = 1{,}097$ (= sonic speed) ft/sec

T_1 : 500°R $\quad W = 1{\cdot}7$ Lb/sec

d_1 : 1 in. $\quad l = 12$ in.

f : 0·0025

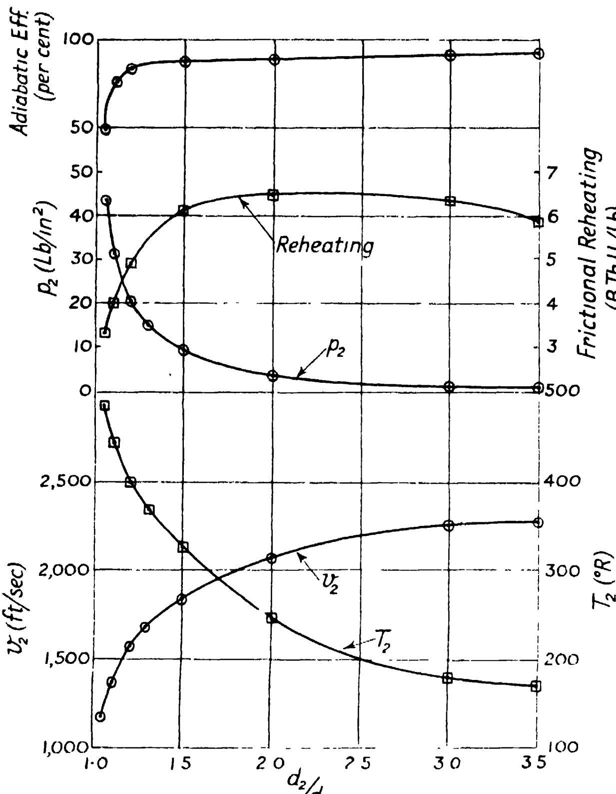


FIG. 255

The values of d_2 chosen are 1·05, 1·1, 1·2, 1·3, 1·5, 2, 3 and 3·5 in. The outlet pressure p_2 and the outlet temperature T_2 will vary with each value of d_2 chosen and must be calculated by the method demonstrated in § 19.3 and § 19.4.

First, the condition of supersonic flow will be assumed. This is attained by choosing values for the velocity ratio r to be greater than unity, as was done in § 19.3. Next, the calculations will be repeated for subsonic flow; in this case the values of r chosen will be less than unity, as in § 19.4.

1. WHEN FLOW IS SUPERSONIC THROUGHOUT. An example of this condition is the flow through the diverging cone of a nozzle exhausting at an extremely low back pressure. The method given in § 19.3 is applied to each pipe in turn, the values of r used in the solution being greater than unity. From the solution the values of p_2, v_2 and T_2 can be calculated for each assumed value of d_2. The values obtained are given in the following table, and are shown plotted in Fig. 255 on a base representing the ratio d_2/d_1. It will be noticed from the v_2 curve that there is no appreciable gain in the outlet velocity after the ratio $d_2/d_1 = 3$, and that most of the velocity gain occurs in the pipes having a d_2/d_1 ratio of less than 2. The bulk of the pressure drop also occurs within this range.

d_2/d_1	1·05	1·1	1·2	1·3	1·5	2·0	3·0	3·5
θ	0·112°	0·225°	0·45°	0·71°	1·2°	2·4°	4·75°	6·0°
r	1·06	1·245	1·418	1·525	1·66	1·88	2·05	2·07
v_2 (ft/sec)	1,163	1,365	1,555	1,674	1,820	2,062	2,250	2,272
T_2 (°R)	487	445	399	368	322	246	180	172
p_2 (Lb/in.^{2})	44·0	31·1	20·55	15·0	9·12	3·4	1·025	0·714

The condition of the air in the pipe is also shown plotted on the temperature-entropy diagram of Fig. 256 for values of d_2/d_1 of 1·05, 1·1, 1·2, 1·5 and 3·5. The curve AB represents the supersonic flow through the pipe in each case, all the curves being drawn to the same scale. It will be noticed from the curves the greatly increased temperature and pressure ranges due to the increased values of d_2, although the length of each pipe is the same. The effect of the frictional reheating is shown by the horizontal deviation of the expansion curves AB from the vertical constant-entropy line. When d_2 is 1·05 the expansion curve has almost reached the state of a horizontal line, which denotes an isothermal process. This means that at B nearly the whole of the energy expended is utilized in frictional reheating. It will be seen that the effect of frictional reheating is less pronounced as d_2 increases, and it has its least effect when $d_2 = 3{\cdot}5$ in.

The amount of frictional reheating, in British thermal units per pound of air, is given by the area of the T–S diagram under the curve AB. This has been calculated for each case and is shown plotted in Fig. 255. It will be noticed from the curve that the frictional reheating increases rapidly with the ratio d_2/d_1 and reaches its maximum value when $d_2/d_1 = 2$, after which it commences to fall gradually. This is because the velocity increase after this ratio is small, whilst, from the Darcy equation, the frictional resistance varies inversely with the diameter.

The adiabatic efficiency of the expansion is defined as the actual heat drop divided by the adiabatic heat drop. Or,

$$\text{adiabatic efficiency} = \frac{\text{actual heat drop}}{\text{actual heat drop} + \text{frictional reheat}}$$

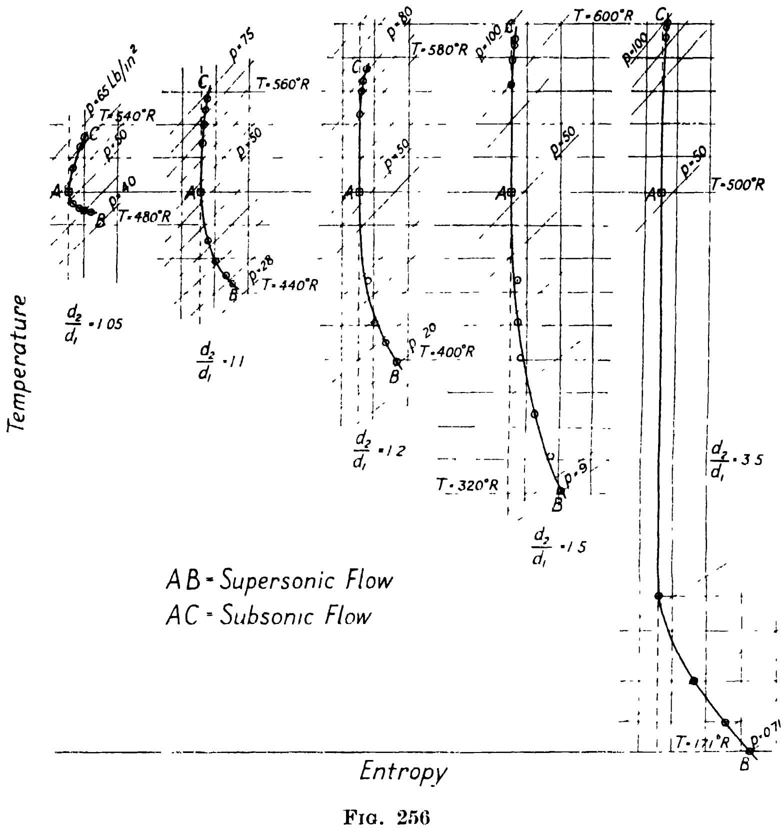


FIG. 256

Or, referring to the expansion shown in Fig. 164 (§ 14.4),

$$\text{adiabatic efficiency} = \frac{AC}{AB}$$

The adiabatic efficiency has been calculated for each expansion and is shown plotted in Fig. 255. It will be noticed from the curve that the efficiency is only 50 per cent when $d_2/d_1 = 1{\cdot}05$, but increases rapidly to 84 per cent for $d_2/d_1 = 1{\cdot}3$, after which there is a gradual increase to 94 per cent for $d_2/d_1 = 3{\cdot}5$.

It will be noticed from the table that extremely low temperatures were obtained by the exhausting air, especially for large ratios of d_2/d_1. This would cause inconvenience at the outlet end of the pipe owing to the freezing of the moisture contained in the exhausting air.

When $d_2/d_1 = 3{\cdot}5$ the flow at the outlet end of the pipe had a Mach number of 3·54 and a Reynolds number of 1,535,000.

2. WHEN THE FLOW IS SUBSONIC THROUGHOUT. An example of this condition is the flow through a diffuser, which converts the kinetic energy of a fluid stream into pressure head. The method demonstrated in § 19.4 can be applied to the tapering pipe for each value of d_2 chosen. The inlet condition of the air is the same as in Case (1) and the same range of values for d_2 will be used. As the flow is subsonic the assumed values of r must be less than unity. The values of p_2, T_2 and v_2 for each value of d_2 are calculated in the same way as in § 19.4; the results so obtained are given in the following table and are also shown plotted in Fig. 257.

d_2/d_1	1·05	1·1	1·2	1·3	1·5	2·0	3·0	3·5
θ	0·112°	0·225°	0·45°	0·71°	1·2°	2·4°	4·75°	6·0°
r	0·829	0·66	0·52	0·41	0·287	0·16	0·072	0·051
v_2 (ft/sec)	910	724	571	450	315	175·6	79	56
T_2 (°R)	531·4	556·5	573·0	583·2	591·7	597·5	599·5	599·7
p_2 (Lb/in.²)	61·3	73·2	80·5	88·6	96·3	98·1	99·2	100

It will be noticed from the curve for p_2 that the amount of pressure increase after the ratio $d_2/d_1 = 1{\cdot}5$ is almost negligible; hence, this amount of taper is sufficient for a diffuser with the given air supply condition.

The compression processes are shown plotted on the temperature-entropy diagram of Fig. 256, the curve AC representing the compression in each case. It will be noticed that the effect of frictional reheating is most pronounced when the ratio d_2/d_1 is smallest. For the case when $d_2/d_1 = 3{\cdot}5$ the compression is almost isentropic. This is to be expected on account of the small average velocity in this pipe.

The frictional reheating, in British thermal units per pound of air, has been calculated for each compression curve from the area of the T–S diagram under the curve AC, and is shown plotted in Fig. 257. It will be noticed that the frictional reheating gets less as the ratio d_2/d_1 increases.

This compression process corresponds to that of an air compressor, and the adiabatic efficiency of the compression is defined in a similar manner. Then,

$$\text{adiabatic efficiency} = \frac{\text{actual total heat increase}}{\text{actual total heat increase} + \text{frictional reheat}}$$

The adiabatic efficiencies have been calculated for all d_2/d_1 ratios and are shown plotted in Fig. 257. It will be noticed from the curve

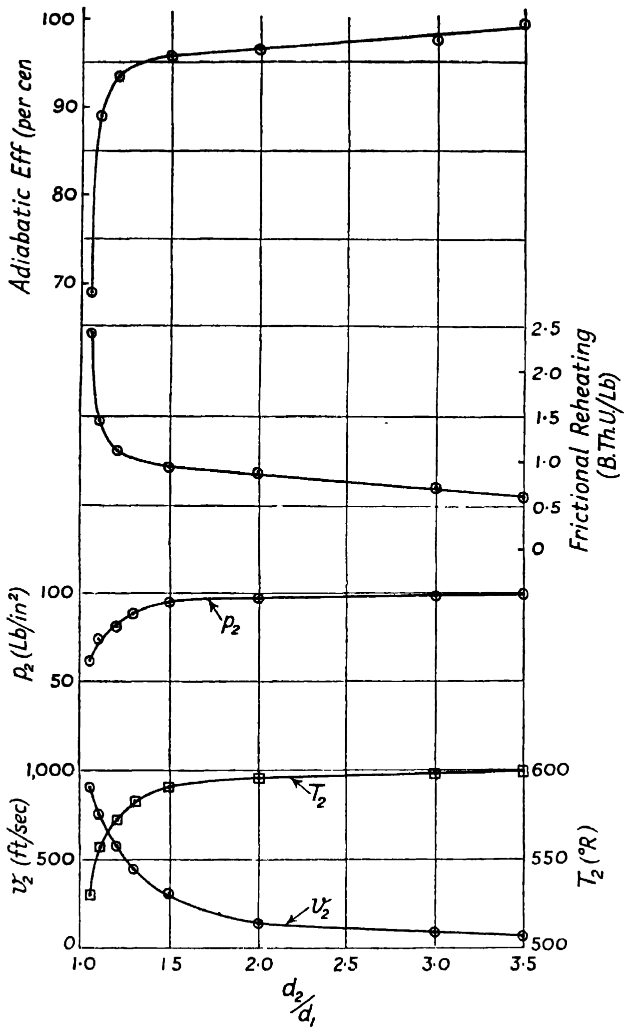


FIG. 257

that the gain in adiabatic efficiency is small beyond the value of the ratio d_2/d_1 — 1·5.

In the above problem it has been assumed that no pressure jumps (§ 17.5) have occurred in the pipe during the flow. A jump could be caused by the jet of air at inlet separating from the sides

of the passage and proceeding at a supersonic speed. In which case a pressure jump to subsonic velocity would occur when the conditions were suitable (§ 19.8). A pressure jump would cause more frictional reheating, owing to the shock-wave losses at the jump, and thus reduce the efficiency. The diffuser should be designed so that this does not occur, by adjusting the ratio d_2/d_1 to suit the back pressure p_2.

19.7. Application to a Converging-diverging Nozzle. The equations of § 19.1 can also be applied to the problem of the flow of gases through a converging-diverging nozzle. It is found that the

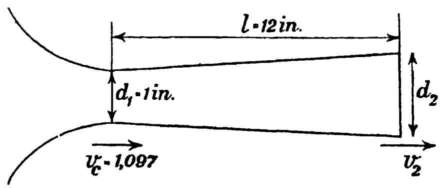


FIG. 258

frictional loss in the converging cone is extremely small and may be neglected in the calculations. This is due to the short length of this cone and because the flow through it is subsonic.

Let air at 100 Lb/in.^{2} pressure and at a temperature of 600°R be expanded through the converging-diverging nozzle shown in Fig. 258. The diverging cone has the same dimensions as the tapering pipe dealt with in § 19.6 [Case (1)]. The throat diameter is 1 in. and the critical velocity at the throat is at the sonic speed as in the tapering pipe of § 19.6.

Let p = initial air pressure at entrance to nozzle
= 100 Lb/in.^{2},
T = initial temperature of the air
= 600°R,
p_c = critical pressure at throat
= $0{\cdot}528p$
= 52·8 Lb/in.^{2},
T_c = critical temperature at throat.

The expansion in the converging cone is assumed to be isentropic and can be represented on the heat-entropy chart. From chart,

$$T_c = 500°\text{R}$$

$$\text{Adiabatic heat drop} = H_d = 24 \text{ B.Th.U.}$$

Applying eq. (14), Chapter 14,

$$v_c = 224\sqrt{24}$$
$$= 1{,}097 \text{ ft/sec}$$

This is the sonic speed of the air at temperature T_c.

The air thus enters the diverging cone in the same state as in the tapering pipe of § 19.6; hence, the results obtained for the supersonic flow through the tapering pipe can be used.

In order to investigate the effect of the side slope of the diverging cone on the nozzle efficiency, the value of the outlet diameter d_2 is varied from $d_2 = 1{\cdot}05$ in. to $d_2 = 3{\cdot}5$ in., as was done in § 19.6. The velocity of discharge v_2 for each assumed value of d_2 can be obtained from the table of § 19.6 [Case (1)].

Adiabatic efficiency of nozzle

$$= \frac{\text{total adiabatic heat drop} - \text{frictional reheating}}{\text{total adiabatic heat drop}}$$

The frictional reheating was calculated from the areas under the expansion curves on the T–S diagram (§ 19.6). The values of the adiabatic heat drop, the frictional reheating and the adiabatic efficiency have been calculated for all assumed values of d_2 and are given in the following table. It will be noticed that the efficiency varies between 89 and 94·8 per cent over the whole range of values of d_2. The discharge velocities v_2 are also entered in the table.

It is interesting to compare these results with those obtained from the usual accepted method of calculating the discharge velocities of this type of nozzle, obtained from the method given in § 14.3. Thus, applying eq. (14), Chapter 14,

$$v_2 = 224\sqrt{kH_d}$$

This equation was applied to the nozzle for each of the assumed values of d_2, the value of H_d used being the total adiabatic heat drop from p to p_2. The frictional loss was assumed to be 10 per cent of the total adiabatic heat drop in each case; hence, $k = 0{\cdot}9$. The values of v_2 were then calculated from the equation

$$v_2 = 224\sqrt{0{\cdot}9H_d} \quad . \qquad . \; (25)$$

and were also entered in the table.

d_2/d_1	1·05	1·1	1·2	1 5	2·0	3·0	3·5
Frictional Reheat (B.Th.U./Lb)	3 38	4 03	4 93	6 16		6 33	5 88
Actual Adiabatic Heat Drop		37 2	48·8	67·2	83 2	101·5	104·8
Total Adiabatic Heat Drop	30 63	41·23	53·73	73 36	89·72	107·83	110 68
Adiabatic Eff.	0 89	0 902	0·908	0·916	0·927	0·924	0·948
v_2 from eq. (25)	1,175	1,365	1,557	1,821	2,012	2,206	2,236
v_2 from § 19.6	1,163	1,365	1,555	1,820	2,062	2,250	2,272

By comparing the values of v_2 obtained by each method, it will be seen that they agree very closely in the range $d_2/d_1 = 1{\cdot}1$ to $1{\cdot}5$. The maximum variation for the other values does not exceed 0·25 per cent. From these results it appears that the simple method of solution given in § 14.4 is of sufficient accuracy for practical purposes for nozzles of these dimensions and for the initial air condition chosen, providing a ten per cent frictional loss is assumed.

19.8. Pressure Jump in Diverging Cone. A gas flowing at a supersonic speed through a diverging pipe is liable to choke at the outlet end if the back pressure p_2 is above the pressure given by the adiabatic law. This choking condition causes a sudden jump in pressure at a section of the pipe at which the conditions are unstable (§ 14.8). The pressure jump at a given section of the passage forms a stationary transverse shock wave which can be photographed by the Schlieren method; a reproduction of such a photograph is shown in Fig. 223. After the jump occurs the gas flow becomes subsonic, but the jump is accompanied by an increase of pressure and temperature. A considerable disturbance is caused during the jump and there is considerable frictional reheating of the gas producing a corresponding increase in entropy. These pressure jumps occur in the diverging cone of nozzles, in diffusers, and may occur in a uniform pipe if the Mach number at the outlet end has reached unity.

The values of the final temperature, pressure and velocity at a section where a jump has occurred can be calculated from the following four equations.

Let p_0, T_0, V_0 and v_0 apply to the gas condition at the supersonic, or upstream, side of the shock wave;

p_1, T_1, V_1 and v_1 apply to the gas condition at the subsonic, or downstream, side of the shock wave;

A — area of cross section (square feet) at which the jump occurs;

W = weight of gas flowing per second.

From equation of continuity of flow,

$$W = \frac{Av_0}{V_0} = \frac{Av_1}{V_1} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

From gas equation,

$$144p_1V_1 = RT_1 \quad . \quad . \quad . \quad . \quad . \quad (27)$$

From energy equation,

$$c_p(T' - T_1) = \frac{v_1^2}{2gJ} \quad . \quad . \quad . \quad . \quad . \quad (28)$$

where T' = stagnation temperature of gas (§ 18.13)

$$= T_0 + \frac{v_0^2}{2gJc_p}$$

From momentum equation, applied to both sides of shock wave,

$$144(p_1 - p_0)A = \frac{W}{g}(v_0 - v_1) \qquad . \quad . \quad . \ (29)$$

From eq. (28)
$$T' = T_1 + \frac{v_1^2}{b}$$

where $b = 2gJc_p$. Hence

$$v_1 = \sqrt{b}(T' - T_1)^{1/2} \qquad . \quad . \ (30)$$

From eq. (29) $\frac{144Ag}{W}(p_1 - p_0) = (v_0 - v_1)$

Let $c = 144Ag/W$. Then

$$c(p_1 - p_0) = v_0 - v_1$$

Hence
$$p_1 = \frac{cp_0 + v_0}{c} - \frac{v_1}{c}$$

$$= k - \frac{v_1}{c} \qquad . \quad . \quad . \quad . \ (31)$$

where $k = (cp_0 + v_0)/c$.

Substituting for v_1 from eq. (30),

$$p_1 = k - \frac{\sqrt{b}}{c}(T' - T_1)^{1/2}. \qquad . \ (32)$$

From eq. (27)
$$V_1 = \frac{RT_1}{144p_1} \qquad . \quad . \quad . \quad . \ (33)$$

From eq. (26)
$$\frac{W}{A} = \frac{v_1}{V_1}$$

Substituting for V_1 and v_1 from eqs. (33) and (30),

$$\frac{W}{A} = \frac{\sqrt{b}(T' - T_1)^{1/2} \times 144p_1}{RT_1}$$

Substituting for p_1 from eq. (32),

$$\frac{W}{A} = \frac{144\sqrt{b}(T' - T_1)^{1/2}\left[k - \frac{\sqrt{b}}{c}(T' - T_1)^{1/2}\right]}{RT_1} \qquad . \ (34)$$

From eq. (34) the value of T_1 can be calculated by trial or by plotting, if the gas conditions before the jump are known. The method is demonstrated in the following worked-out example.

Sometimes the total pressure jump is divided into two or more stages of smaller jumps, thus causing a series of two or more shock waves to form in the diverging cone. This can be seen in the four lower pressure curves of Stodola's experiments (Fig. 169).

EXAMPLE 1

Air is flowing with a supersonic velocity through the diverging cone of a converging-diverging nozzle, shown in Fig. 259. Curve *acdb* represents the adiabatic flow through the nozzle and includes the effect of frictional reheating; the point *c* represents the critical pressure. Assuming a pressure jump occurs at section *x*, calculate the values of the temperature, pressure and velocity at the completion of the jump. At the commencement of the jump the temperature, pressure and velocity are 367°R, 16·2 Lb/in.^{2} and 1,672 ft/sec respectively. The cross-sectional area at *x* is 0·00853 ft^2 and the weight of air flowing is 1·7 Lb/sec.

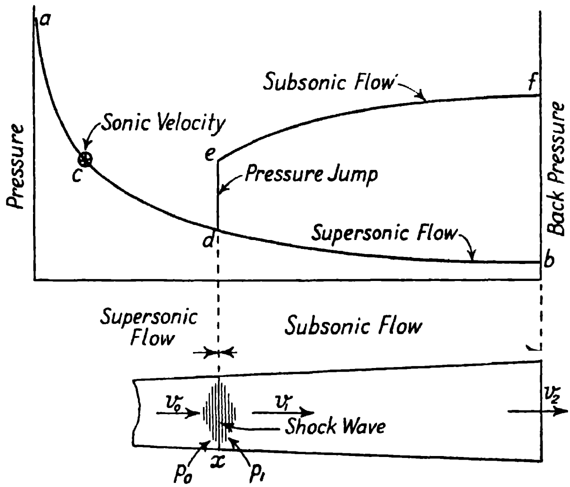


FIG. 259

First calculate the constants, b, c and k.

$$b = 2gJc_p$$

$$= 64{\cdot}4 \times 778 \times 0{\cdot}24 = 12{,}020$$

Hence $$\sqrt{b} = 109{\cdot}6$$

$$c = \frac{144Ag}{W}$$

$$\frac{144 \times 0{\cdot}00853 \times 32{\cdot}2}{1{\cdot}7} = 23{\cdot}28$$

$$k = p_0 + \frac{v_0}{c}$$

$$= 16{\cdot}2 + \frac{1{,}672}{23{\cdot}28} = 88{\cdot}1$$

$$T' = T_0 + \frac{v_0^2}{b}$$

$$= 367 + \frac{(1{,}672)^2}{12{,}020} = 600°\text{R}$$

From eq. (34)

$$144\sqrt{b}(T' - T_1)^{1/2}\left[k - \frac{\sqrt{b}}{c}(T' - T_1)^{1/2}\right] - \frac{WRT_1}{A} = 0$$

Inserting the values of b, c, k, T' and W/A,

$$144 \times 109\cdot6(600 - T_1)^{1/2}\left[88\cdot1 - \frac{109\cdot6}{23\cdot28}(600 - T_1)^{1/2}\right]$$

$$\frac{1\cdot7 \times 53\cdot3T_1}{0\cdot00853} = 0$$

From this equation the value of T_1 can be obtained by plotting. To do this, assume several values of T_1, and calculate the value of the equation for each assumed value of T_1. Then plot the values obtained for the equation on a base representing T_1. The correct value of T_1 is that given where the curve intersects the zero base line.

By plotting, $\quad T_1 = 558°\text{R}$

Applying eq. (30),

$$v_1 = \sqrt{b}(T' - T_1)^{1/2}$$

$$= 109\cdot6(600 - 558)^{1/2} = 714 \text{ ft/sec}$$

Applying eq. (31),

$$p_1 = k - \frac{v_1}{c}$$

$$= 88\cdot1 - \frac{714}{23\cdot28} = 57\cdot5 \text{ Lb/in.}^2$$

The pressure jump *de* is shown, drawn to scale, in Fig. 259. The flow beyond the jump is now subsonic and is represented by the compression curve *ef*. The curve can be plotted by the method demonstrated in § 19.4 for the subsonic flow in a diverging pipe; the solution gives a final back pressure of 70·2 Lb/in.^{2} at the outlet end *f*. Hence, the diverging cone to the right of the jump has acted as a diffuser, thus increasing the pressure from 57·5 Lb/in.^{2} to 70·2 Lb/in.^{2}

19.9. Entropy Increase during Pressure Jump. The pressure jump dealt with in § 19.8 is accompanied by a considerable disturbance caused by shock; this, in turn, is absorbed by frictional reheating of the gas which causes an increase in its entropy.

Let S_0 = entropy per pound of gas before the pressure jump,

S_1 = entropy per pound of gas after the pressure jump.

Then, from equation for change of entropy,

$$S_1 - S_0 = c_p \log_e \frac{T_1}{T_0} + \frac{R}{J} \log_e \frac{p_0}{p_1}$$

Substituting for p_1 from eq. (32),

$$S_1 - S_0 = c_p \log_e \frac{T_1}{T_0} + \frac{R}{J} \log_e \frac{p_0}{\left[k - \frac{\sqrt{b}}{c}(T' - T_1)^{1/2}\right]}$$

Expanding,

$$S_1 - S_0 = c_p \log_e T_1 - c_p \log_e T_0 + \frac{R}{J} \log_e p_0$$

$$- \frac{R}{J} \log_e \left[k - \frac{\sqrt{b}}{c}(T' - T_1)^{1/2}\right] \quad . \quad . \quad (35)$$

All frictional reheating ceases when $dS/dT_1 = 0$; that is, when the line of condition on the T–S diagram, representing the jump, becomes vertical. This condition denotes the limit of the jump. Hence, differentiating eq. (35) and treating T_1 as the variable,

$$\frac{dS}{dT_1} = \frac{c_p}{T_1} - \frac{\frac{R}{J} \times \frac{\sqrt{b}}{2c}}{k(T' - T_1)^{1/2} - \frac{\sqrt{b}}{c}(T' - T_1)} = 0 \quad . \quad (36)$$

The value of T_1 obtained from this equation is the actual value on the completion of the jump; it is found that this value is the same as that obtained from eq. (34).

It can be proved that eq. (36) reduces to the same equation as eq. (34) by substituting the following values in eq. (36)—

$$c_p = \frac{b}{2gJ} \text{ (from § 19.8)}$$

$$y = \frac{cW}{144A} \text{ (from § 19.8)}$$

This proves that the line of condition of the jump is vertical at the end of the jump, as shown in the T–S diagram of Fig. 260. In this figure the whole process of the expansion of 1 Lb of air in a converging-diverging nozzle is shown, drawn to scale. This expansion corresponds to the process shown by the pressure diagram in Fig.

259 of § 19.8, the same letters representing the corresponding processes. In Fig. 260—

ab = adiabatic flow with supersonic speed at discharge, and allowing for frictional resistance, when no jump occurs.

de = pressure jump commencing at point *d*. The actual shape of this line is unknown, but it is vertical at *e*.

ef = subsonic flow in diverging cone beyond jump. The pressure at *f* is the back pressure at the mouth of the nozzle.

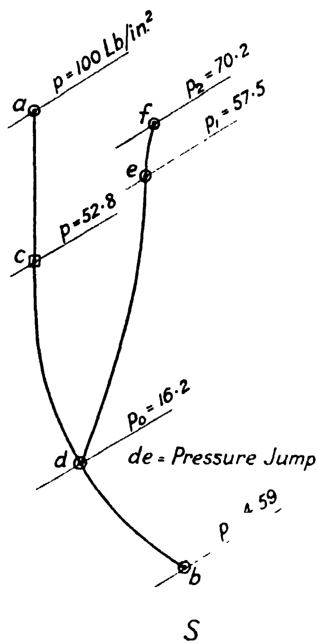


FIG. 260

The area of the *T–S* diagram under the curve *de*, down to absolute zero temperature, represents the frictional reheating during the jump. This is found to be approximately 6 B.Th.U./Lb of air, and represents a considerable amount when compared with the total heat drop in the nozzle.

19.10. Relation between Pressure Jumps and Back Pressure in Nozzle. The method of calculating the changes of pressure, temperature and velocity during a pressure jump at a chosen section in the diverging cone of a nozzle, given in § 19.9, has been applied to several sections of the converging-diverging nozzle shown in Fig. 261. The method was applied to the diverging cone, and the

following particulars apply to the nozzle and to the condition of air flow considered—

diameter of throat = 1 in.,
length of diverging cone = 21 in.,
slope of sides of diverging cone = θ = 1·2°,
initial air pressure at inlet to nozzle = p = 100 Lb/in.²,
initial air temperature at inlet to nozzle = T' = 600°R,

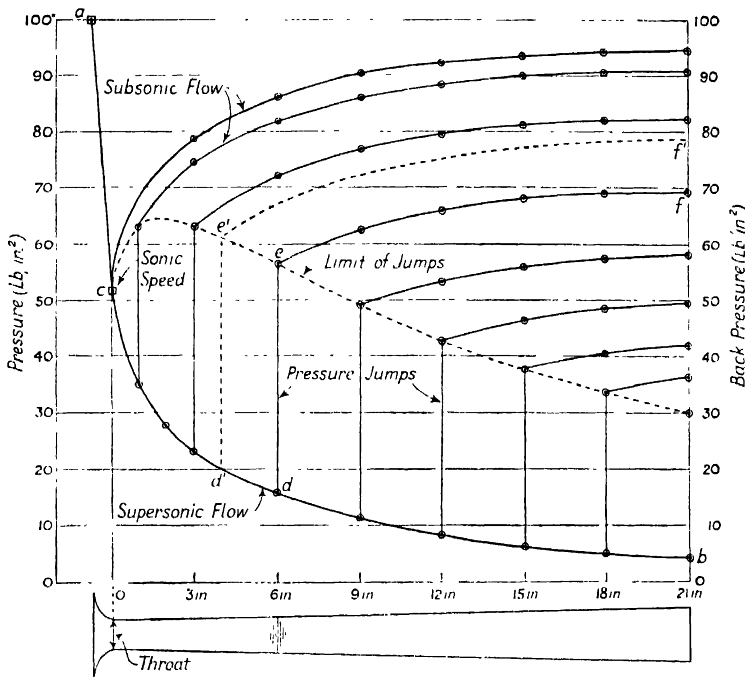


FIG. 261

weight of air flowing per second = W = 1·7 Lb,
critical pressure at throat = $0{\cdot}528p$ = 52·8 Lb/in.²,
sonic velocity at throat = 1,097 ft/sec.

Let x = distance of section considered from throat,
suffix 0 apply to condition of air before jump occurs,
suffix 1 apply to condition of air after jump is completed,
suffix 2 apply to condition of air at nozzle mouth.

The method developed in § 19.9 was applied to the sections at x = 1, 3, 6, 9, 12, 15, 18 and 21 in., and the pressure curve of each jump has been plotted, as was done in Fig. 259 of § 19.8. These are shown in Fig. 261, plotted to scale on a base representing the centre-line of the nozzle. The final back pressure, p_2, calculated after each

jump, is also shown in the figure. A curve, shown dotted, has been drawn through the corresponding points e, representing the completion of the jumps. This has converted the pressure diagrams into

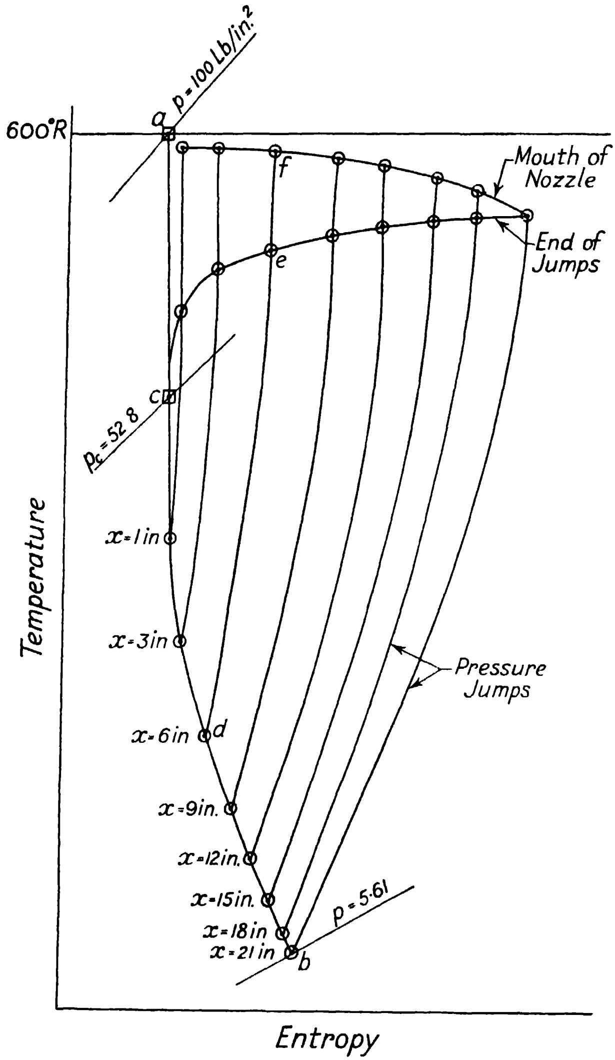


FIG. 262

a chart from which, for any given back pressure, the pressure diagram can be drawn parallel to the existing curves so that the section at which the jump occurs can be read off the chart. Thus,

if the back pressure for this particular nozzle problem is 79 Lb/in.^{2}, this is represented by the point f'. The pressure diagram for this back pressure can now be drawn parallel to the corresponding lines on the chart. The dotted line $f'e'd'$ represents this pressure diagram, the point d' giving the position of the jump. From the diagram of

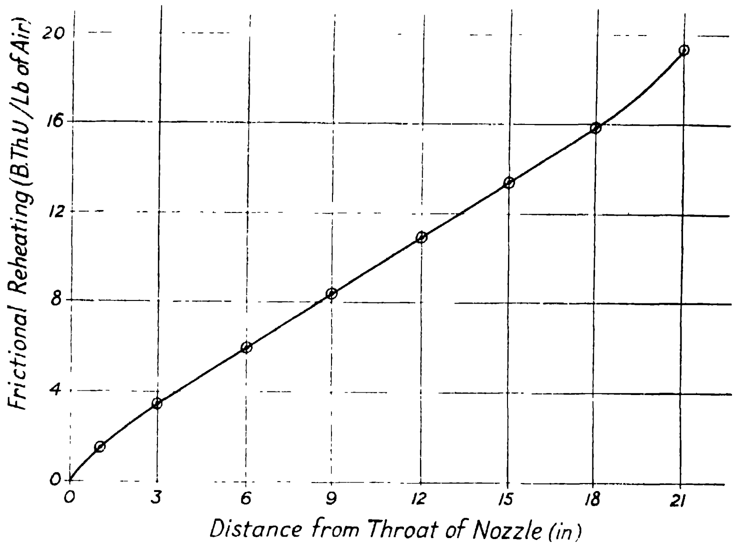


FIG. 263

Fig. 261 it is found that the jump occurs at 4 in. from the throat for a back pressure of 79 Lb/in.^{2}

It will be noticed from the diagram that the pressure jump at the mouth of the nozzle corresponds to a back pressure of 30 Lb/in.^{2} If the back pressure had a value between 30 Lb/in.^{2} and that of the point b, oblique shock waves would occur in the free jet of air after it had left the nozzle mouth. The oblique shock waves can be photographed by the Schlieren method. Such a condition is shown in the photographs of Figs. 233 (b) and 233 (c); the oblique shock waves, and the reflected shock waves they cause, can be seen in the photographs.

The pressure jumps for each of the chosen sections are also shown plotted to scale on the temperature-entropy diagram of Fig. 262; each of these diagrams correspond to the T–S diagram of Fig. 260, and are shown plotted on the supersonic flow line *acb*. In this diagram *de* represents the pressure jump and *ef* the adiabatic compression, with friction, which occurs during the remainder of the flow through the diverging nozzle. The line of condition *de* is vertical at e (§ 19.9) and the area under *de*, down to absolute zero temperature, gives the frictional reheating during the jump.

in Fig. 261, after the jump has occurred, have been calculated and are entered in Table II; they are also shown plotted in Fig. 265.

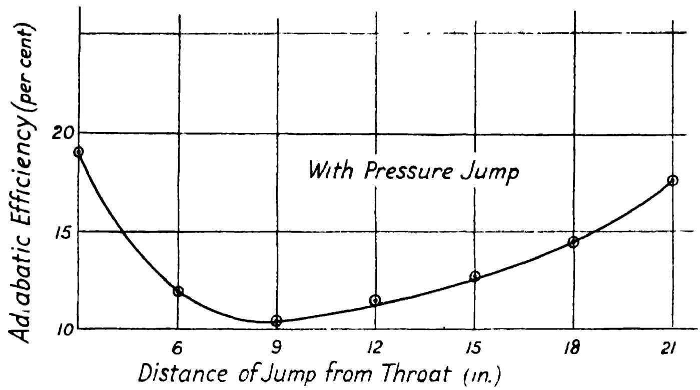


FIG. 265

TABLE II

(Includes effect of pressure jump)

Back Pressure (Lb/in.^{2})	Distance of Jump from Throat (in.)	T_2 (°R)	v_2 (ft/sec)	Adiabatic Heat Drop H_d (B.Th.U.)	Actual Heat Drop (B.Th.U.)	Adiabatic Efficiency (%)
83·4	3	595·5	233	6·8	1·3	19·1
70·2	6	593·6	277	13·5	1·6	11·85
59·0	9	591·1	327	20·2	2·1	10·4
50·3	12	588·0	381	26·1	3·0	11·5
42·6	15	583·6	449	31·6	4·0	12·65
37·0	18	578·3	511	36·3	5·2	14·45
30·5	21	569·2	608	42·9	7·6	17·7

It will be seen from Table II that the values of the adiabatic efficiency are between 10·4 per cent and 19·1 per cent only. These low results show that great care should be taken when designing nozzles to avoid the occurrence of a pressure jump in the diverging cone.

The above results also apply in the design of diffusers; no pressure jump should be allowed to occur.

19.12. Comparison of Calculated Pressures with Experimental Results. The curves of Fig. 266 show a comparison between the calculated pressures in a converging-diverging nozzle with those obtained from actual measurement. Air at an initial pressure of

19·76 Lb/in.^{2} was expanded through the nozzle shown in the figure. Eight small holes, drilled in the side of the diverging cone throughout its length, were fitted with pressure gauges of the U-tube type and the pressures measured during the flow. This was first done during a frictional adiabatic flow having a low back pressure. The results obtained are represented by the full-line curve *ACB*, the flow being supersonic after passing the throat. The back pressure was then

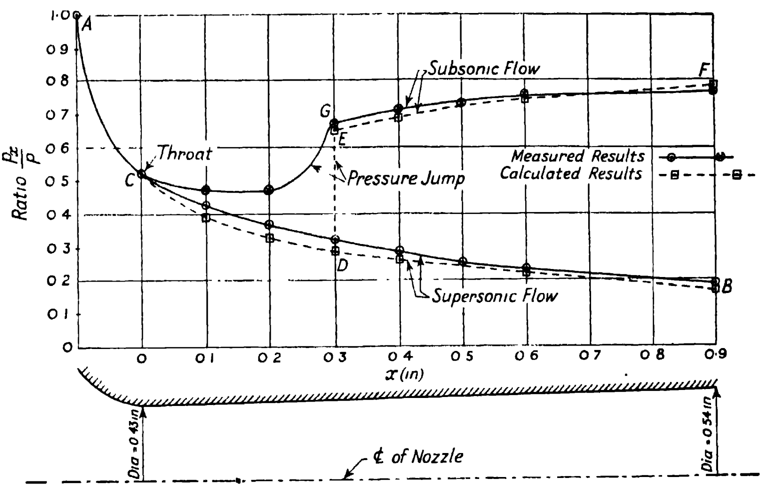


FIG. 266

increased so that a pressure jump occurred in the diverging cone and the pressures were again measured. These results are also shown plotted in the figure, and are represented by the full-line curve *ACGF*. A pressure jump immediately before the point *G* can be seen in the figure.

The pressure values were then calculated for both conditions of flow by the methods given in § 19.3–§ 19.8 and are shown plotted to the same scale in Fig. 266. The frictional adiabatic flow is now represented by the broken line *ACDB*; the choked flow causing the pressure jump *DE* is represented by the broken line *ACDEF*. These calculated results were obtained by using a value of Darcy's f of 0·005 for the frictional coefficient.

In the plotting, the vertical ordinate represents the pressure ratio p_x/p, where

p = initial static pressure of the air = 19·76 Lb/in.^{2},

p_x = pressure at a point x in. from throat.

The base of the curves represents the centre-line of the nozzle.

It will be noticed from the results that the calculated values agree closely with the measured values except with the length CG preceding the jump. The jump in this case appears to cause a slight pressure increase between the throat and the jump. The flow condition in the vicinity of the jump is unstable and the jump is not so abrupt as shown by the calculated jump DE.

19.13. Equation for Frictionless Flow in Terms of Mach Number. The problem of fluid flow through a tapering pipe is simplified considerably if friction is neglected. The general momentum equation given as eq. (18) can be modified for frictionless flow by inserting $f = 0$ in the equation for the constant $\bar{B}$. Then, by substituting Mach numbers for all velocities, the equation is reduced to a very simple form.

The method of substitution is shown in the following—

$$R = \left(\frac{\gamma - 1}{\gamma}\right) c_p J \qquad (37)$$

$$v_s = \sqrt{\gamma g R T} \qquad (38)$$

Then

$$v = M_a \sqrt{\gamma g R T} \qquad (39)$$

Stagnation or total temperature

$$= T' = T + \frac{v^2}{2gJc_p} \qquad (40)$$

Substituting for T from eq. (38), and for Jc_p from eq. (37),

$$T' = \frac{v_s^2}{\gamma g R} + \frac{\gamma - 1}{\gamma}\frac{v^2}{2gR}$$

$$= \frac{v_s^2}{\gamma g R}\left(1 + \frac{\gamma - 1}{2} M_a^2\right) \qquad (41)$$

Applying eq. (41) to the two ends of the tapering pipe, as T' is constant,

$$\frac{v_{s_1}^2}{\gamma g R}\left(1 + \frac{\gamma - 1}{2} M_{a_1}^2\right) = \frac{v_{s_2}^2}{\gamma g R}\left(1 + \frac{\gamma - 1}{2} M_{a_2}^2\right)$$

Hence

$$\frac{v_{s_1}^2}{v_{s_2}^2} = \frac{1 + \frac{\gamma - 1}{2} M_{a_2}^2}{1 + \frac{\gamma - 1}{2} M_{a_1}^2} \qquad (42)$$

From eq. (38)

$$\frac{T_1}{T_2} = \left(\frac{v_{s_1}}{v_{s_2}}\right)^2$$

Hence, substituting in eq. (42),

$$\frac{T_1}{T_2} = \frac{1 + \frac{\gamma - 1}{2} M_{a_2}{}^2}{1 + \frac{\gamma - 1}{2} M_{a_1}{}^2} \quad . \quad . \quad . \ (43)$$

Using the momentum equation for a tapering pipe [eq. (18)], and inserting $f = 0$ in the constant $\bar{B}$, as friction is neglected,

$$\frac{\bar{C}}{\bar{A}} \log \frac{v_2}{v_1} + \left(\frac{\bar{D}}{2\bar{B}} - \frac{\bar{C}}{2\bar{A}}\right) \log \left[\frac{\bar{A} + \bar{B}v_2{}^2}{\bar{A} + \bar{B}v_1{}^2}\right] = -\frac{\log \left(\frac{A_2}{A_1}\right)^{1/2}}{2 \tan \theta} \quad . \ (44)$$

where $\bar{A} = -4 \tan \theta RT'$.

Substituting for T' from eq. (40), and for R from eq. (37) in the second term only,

$$\left.\begin{aligned} \bar{A} &= -4 \tan \theta \left(RT_1 + \frac{\gamma - 1}{\gamma} \frac{v_1{}^2}{2g}\right) \\ \text{or} \quad \bar{A} &= -4 \tan \theta \left(RT_2 + \frac{\gamma - 1}{\gamma} \frac{v_2{}^2}{2g}\right) \end{aligned}\right\} \text{as } T' \text{ is the same at each end of the pipe}$$

$$\bar{B} = \frac{4 \tan \theta}{2g} \left(\frac{\gamma - 1}{\gamma}\right), \ (\text{as } f = 0)$$

$$\bar{C} = RT_1 + \frac{\gamma - 1}{\gamma} \frac{v_1{}^2}{2g}$$

$$\bar{D} = -\left(\frac{\gamma + 1}{2g\gamma}\right)$$

Hence $\dfrac{\bar{C}}{\bar{A}} = -\dfrac{1}{4 \tan \theta}$

and $\dfrac{\bar{D}}{2\bar{B}} = -\left(\dfrac{\gamma + 1}{\gamma - 1}\right) \dfrac{1}{8 \tan \theta}$

Substituting the above constants in eq. (44),

$$-\frac{1}{4 \tan \theta} \log \frac{v_2}{v_1} + \left\{-\left(\frac{\gamma + 1}{\gamma - 1}\right) \frac{1}{8 \tan \theta} + \frac{1}{8 \tan \theta}\right\} \times$$

$$\log \left[\frac{-4 \tan \theta \left(RT_2 + \frac{\gamma - 1}{\gamma} \frac{v_2{}^2}{2g}\right) + \frac{4 \tan \theta}{2g} \frac{\gamma - 1}{\gamma} v_2{}^2}{-4 \tan \theta \left(RT_1 + \frac{\gamma - 1}{\gamma} \frac{v_1{}^2}{2g}\right) + \frac{4 \tan \theta}{2g} \frac{\gamma - 1}{\gamma} v_1{}^2}\right]$$

$$= -\frac{\log \left(\frac{A_2}{A_1}\right)^{1/2}}{2 \tan \theta}$$

which reduces to

$$-\log\frac{v_2}{v_1}-\frac{1}{\gamma-1}\log\frac{T_2}{T_1}=\log\frac{A_2}{A_1}$$

Hence
$$\frac{v_1}{v_2}\times\left(\frac{T_1}{T_2}\right)^{\frac{1}{\gamma-1}}=\frac{A_2}{A_1} \tag{45}$$

Substituting for v_1 and v_2 from eq. (39),

$$\frac{M_{a_1}\sqrt{T_1}}{M_{a_2}\sqrt{T_2}}\times\left(\frac{T_1}{T_2}\right)^{\frac{1}{\gamma-1}}=\frac{A_2}{A_1}$$

Hence
$$\frac{A_2}{A_1}=\frac{M_{a_1}}{M_{a_2}}\left(\frac{T_1}{T_2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

Substituting for T_1/T_2 from eq. (43),

$$\frac{A_2}{A_1}=\frac{M_{a_1}}{M_{a_2}}\left(\frac{1+\frac{\gamma-1}{2}M_{a_2}^2}{1+\frac{\gamma-1}{2}M_{a_1}^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{46}$$

It will be noticed that eq. (46) provides a simpler method of solution but can be used only when the effect of friction is negligible.

EXAMPLE 2

The static pressure at the entry to a diverging duct discharging air to atmosphere is 13·0 Lb/in.²; the velocity 600 ft/sec and the total temperature 15°C. Calculate the exit velocity and the ratio of exit and entry areas, neglecting frictional resistance. Assume $\gamma = 1{\cdot}4$ and $c_p = 0{\cdot}24$. (*Lond. Univ.*)

Let suffix 1 apply to entrance and suffix 2 to the discharge end. Applying eq. (32), Chapter 18, to the inlet end,

total, or stagnation, temperature

$$= 273 + 15 = 288^\circ\text{K}$$

$$= T_1 + \frac{v_1^2}{2gJc_p}$$

Then
$$T_1 = 288 - \frac{(600)^2}{64{\cdot}4\times 1{,}400\times 0{\cdot}24}$$

$$= 288 - 16{\cdot}6 = 271{\cdot}4^\circ\text{K}$$

For the adiabatic compression in the duct,

$$\frac{T_1}{T_2}=\left(\frac{p_1}{p_2}\right)^{\frac{\gamma-1}{\gamma}}$$

Hence $$\frac{271{\cdot}4}{T_2} = \left(\frac{13}{14{\cdot}7}\right)^{\frac{0{\cdot}4}{1{\cdot}4}}$$

$$= 0{\cdot}9656$$

from which $$T_2 = 281°\text{K}$$

Applying eq. (32), Chapter 18, to the outlet end,

$$T' = T_2 + \frac{v_2^2}{2gJc_p}$$

Hence $$288 - 281 = \frac{v_2^2}{64{\cdot}4 \times 1{,}400 \times 0{\cdot}24}$$

from which $$v_2 = \sqrt{(21{,}600 \times 7)} = 388 \text{ ft/sec}$$

Applying eq. (45),

$$\frac{A_2}{A_1} = \frac{v_1}{v_2} \times \left(\frac{T_1}{T_2}\right)^{\frac{1}{\gamma - 1}}$$

$$= \frac{600}{388} \times \left(\frac{271{\cdot}4}{281}\right)^{2{\cdot}5}$$

$$= 1{\cdot}417$$

EXERCISES 19

1. A uniformly tapering pipe has a diameter of 1 in. at its narrow end, 3 in. at its broad end, and is 24 in. long. Air having a velocity of 1,097 ft/sec is supplied to the pipe at its narrow end; the pressure and temperature of the air being 52·8 Lb/in.² and 500°F abs. The air is discharged from the broad end of the pipe into a chamber of very low pressure so that the flow throughout the whole length of the pipe is supersonic. Darcy's $f = 0{\cdot}0025$.

Calculate the velocity, the absolute temperature and the pressure of the air discharging from the broad end of the pipe.

Ans. $v_2 = 2{,}000$ ft/sec; $T'_2 = 267°$R; $p_2 = 3{\cdot}85$ Lb/in.²

2. A short uniformly tapering pipe is 18 in. long, 1 in. diameter at its narrow end and 1¾ in. diameter at its broad end. A supersonic stream of air at a pressure of 100 Lb/in.² and a temperature of 600°R is supplied to the narrow end of the pipe and discharges from the broad end at a pressure of 25 Lb/in.² Darcy's $f = 0{\cdot}0025$. Calculate the temperature and velocity of the discharging air at the broad end, and the weight of air flowing through the pipe. *Ans.* $T_2 = 491°$R; $v_2 = 3{,}110$ ft/sec; $W = 7{\cdot}15$ Lb/sec.

3. A high-speed air stream at 100 Lb/in.² pressure, and 600°R temperature, is supplied to the narrow end of the same pipe as Question No. 2, but the back pressure at the broad end is increased to 150 Lb/in.² so that the flow is now subsonic and the pipe is acting as a diffuser.

Calculate the temperature and velocity of the discharge at the broad end of the pipe and the weight per second of the air flow through the pipe. Darcy's $f = 0{\cdot}0025$. *Ans.* 679°R; 247 ft/sec; 2·46 Lb/sec.

4. Air at an absolute pressure of 18·3 Lb/in.^{2} and at a temperature of 15°C is expanded through a converging-diverging nozzle. The back pressure at the mouth of the nozzle is extremely low so that the flow through the diverging cone remains supersonic throughout and no pressure jump occurs. Assuming the throat velocity to be sonic, calculate the velocity, temperature and pressure at the exit end of the nozzle and the weight of air discharged. Throat dia. = 0·43 in.; exit dia. = 0·54 in.; length of diverging cone = 0·9 in.; $\gamma = 1{\cdot}4$; $c_p = 0{\cdot}24$; $R = 96$ ft-Lb Centigrade units. Assume a frictionless adiabatic flow up to the throat and that Darcy's $f = 0{\cdot}005$ in the diverging cone. The air velocity entering the inlet end of the nozzle may be neglected as small.

Ans. $v_2 = 1{,}560$ ft/sec; $T_2 = 175{\cdot}8$°K; $p_2 = 3{\cdot}18$ Lb/in.^{2}
$W = 0{\cdot}0674$ Lb/sec.

5. If the back pressure of the nozzle air-flow problem of Question No. 4 be raised sufficiently to cause a pressure jump to occur at a section in the diverging cone situated at 0·3 in. from the throat, calculate the velocity, temperature and pressure after the jump occurred. The velocity, temperature and pressure at this section before the jump are 1,354 ft/sec, 203°K, and 5·08 Lb/in.^{2} respectively. Sketch the path of the jump on the $T - S$ diagram for air and, hence, find approximately the amount of frictional reheating per pound of air during the jump.

Ans. 782 ft/sec; 259·8°K; 12·68 Lb/in.^{2}; 2·2 C.H.U.

6. Calculate the velocity, temperature and pressure of the air at the mouth of the nozzle of Question No. 4 after the occurrence of the pressure jump of Question No. 5. Darcy's $f = 0{\cdot}005$.

Ans. 504 ft/sec; 277·2°K; 15·55 Lb/in.^{2}

CHAPTER 20

HYDRAULIC MACHINES, METERS AND VALVES

20.1. The Hydraulic Accumulator. The hydraulic accumulator is a cylinder used for temporarily storing the energy of water.

Hydraulic machines such as lifts or cranes are required to do a large amount of work during a small interval of time, which is followed by an idle period. For example, a lift or crane requires

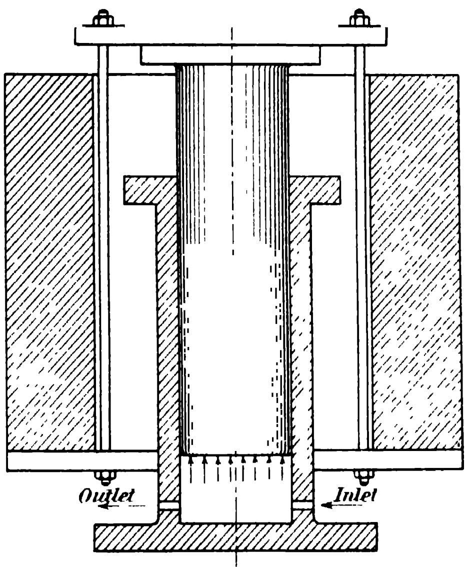


FIG. 267

the energy to be supplied during the upward motion of the load only, practically no energy being used during the downward motion. But as the pumps are supplying the energy continuously, it may be stored in an accumulator during the idle periods of the machine and given out at an increased rate during the working periods. The uniform supply of energy from the pumps need not, therefore, be as large as that required by the machine when doing its maximum rate of work, as the machine will then draw from the accumulator.

The accumulator consists of a vertical cylinder containing a sliding ram (Fig. 267). A container, fixed to the ram, is filled with heavy material such as slag, or the ram is loaded with weights. Water is delivered by the pumps into the cylinder when not required by the machine it is working. The pressure of the water lifts up the

heavy ram until the cylinder is full. The accumulator has then stored its maximum amount of energy. During its period of maximum work, the machine will draw from the accumulator and the ram will descend.

The maximum amount of energy the accumulator can store is known as the capacity of the accumulator.

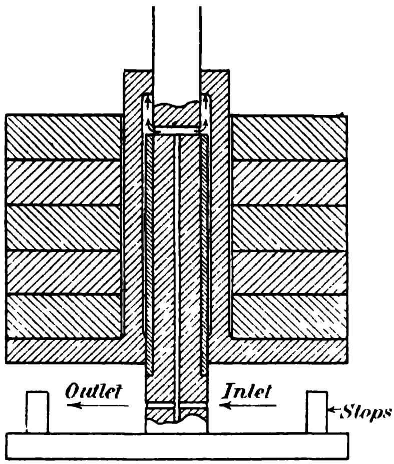


FIG. 268

Let A = area of base of ram in square feet,

and H = lift of ram in feet.

Then volume of accumulator $= AH$ ft^3

Let p = intensity of pressure of water supplied in pounds per square feet.

Then

$$\text{weight of ram} = pA \text{ Lb}$$

$$\text{Work done in lifting ram} = pAH \text{ ft-Lb}$$

This equals the energy stored, which is the capacity of the accumulator.

Therefore,

$$\text{capacity of accumulator} = pAH$$
$$= p \times \text{volume}$$

Another form of accumulator, known as Tweddell's differential accumulator, is shown in Fig. 268. The advantage of this accumulator is that the water can be stored at a high pressure by a comparatively small load on the ram. It consists of a fixed ram of which the lower portion is made larger than the upper portion by surrounding it with a brass bush. Sliding on the fixed ram is a

loaded cylinder, which is forced upwards by the pressure of the water from the main supply. The water enters and leaves the cylinder by a hole through the centre of the fixed ram.

Let a = sectional area of brass bush in square feet
= effective area of ram.

Then,

$$\text{load on cylinder} = pa$$

Therefore, by making the area of the bush small, it is possible to store at a high pressure with a small load.

Capacity of accumulator

$$= paH$$
$$= p \times \text{volume}$$

A sectional view of an actual accumulator is shown in Fig. 269.

If the pipes leading to an accumulator are very long, great trouble is experienced owing to surging, which is caused by the inertia effect of the water column. This can be overcome by fitting some form of relief valve (§ 20.9) as close to the accumulator as possible

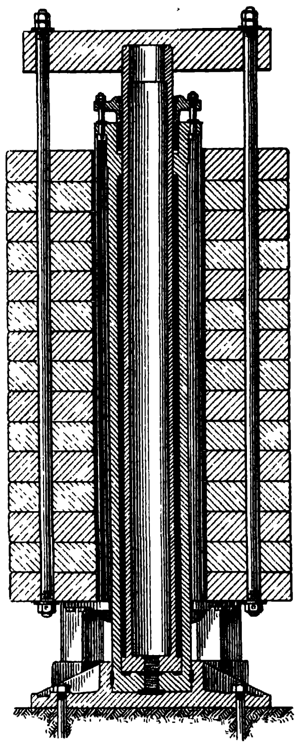


FIG. 269. SECTION OF HYDRAULIC ACCUMULATOR (CAST-IRON WEIGHT TYPE)

(Courtesy of The Hydraulic Engineering Co., Ltd.)

EXAMPLE 1

An accumulator has a ram of 6 in. diameter and a lift of 18 ft. Water is supplied at a pressure of 800 Lb/in.² Find the necessary load on the ram and the capacity in horse-power hours.

$$\text{Load on ram} = p \times a$$
$$= 800 \times \frac{\pi}{4} \times (6)^2$$
$$= 22{,}600 \text{ Lb}$$
$$\text{Capacity} = paH$$
$$= 22{,}600 \times 18 = 407{,}000 \text{ ft-Lb}$$
$$\frac{407{,}000}{33{,}000 \times 60} = 0{\cdot}206 \text{ h.p.-hours}$$

EXAMPLE 2

An accumulator has a 12-in. ram and 15 ft lift, and is loaded with 80 tons total weight. If packing friction is equivalent to 5 per cent of the load on the ram, determine the horse-power being delivered to the mains if the ram falls steadily through its full range in 1·5 min, and if at the same time the pumps are delivering 1 ft³/sec through the accumulator. (*A.M.I.C.E.*)

First find the pressure of water produced by the falling ram of the accumulator: this will be the pressure of water supplied to mains.

Intensity of pressure when ram is falling

$$\frac{\text{weight} \times \dfrac{95}{100}}{\text{area}}$$

$$\frac{80 \times 2{,}240 \times 0{\cdot}95}{\dfrac{\pi}{4}} \text{ Lb/ft}^2$$

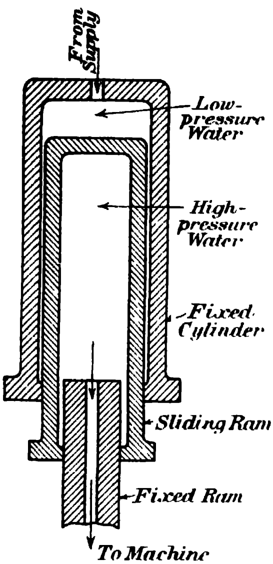


FIG. 270

Head of water due to this pressure

$$= \frac{80 \times 2{,}240 \times 0{\cdot}95}{\dfrac{\pi}{4} \times 62{\cdot}4}$$

$$= 3{,}475 \text{ ft of water}$$

Work supplied by pumps per minute

$$= WH$$

$$= (62{\cdot}4 \times 60) \times 3{,}475$$

$$= 13{,}000{,}000 \text{ ft-Lb}$$

Work done by accumulator per minute

$$= \text{weight} \times \text{distance moved}$$

$$= (80 \times 2{,}240 \times 0{\cdot}95) \times 10$$

$$= 1{,}703{,}000 \text{ ft-Lb}$$

$$\text{Horse-power delivered} \quad \frac{13{,}000{,}000 + 1{,}703{,}000}{33{,}000}$$

$$445$$

20.2. The Hydraulic Intensifier. The hydraulic intensifier is used for increasing the intensity of pressure of water by means of the energy of a larger quantity of water at low pressure. This is necessary when the pressure of the water supplied to a machine is not of sufficient intensity.

An intensifier consists of a fixed ram (Fig. 270) through which the high-pressure water flows to the machine. Mounted externally on the fixed ram is a hollow sliding ram containing the high-pressure water. The sliding ram is encased by a fixed cylinder which contains the low-pressure water from the main supply. The low-pressure water presses on the end of the sliding ram, forcing it downwards on to the fixed ram; this increases the pressure of the water in the sliding ram.

Let A = external area of end of sliding ram,

a = area of end of fixed ram,

P = intensity of pressure of low-pressure water in fixed cylinder,

p = intensity of pressure of high-pressure water in sliding ram.

As total upward force = total downward force

$$pa = PA$$

from which

$$p = \frac{PA}{a}$$

When the sliding ram is at the bottom of its stroke the valve admitting the high-pressure water to the machine is closed. Low-pressure water from the main is then admitted to the inside of sliding ram and the fixed cylinder is open to exhaust; this causes the sliding ram to rise. When it reaches the top of its stroke the valve admitting high-pressure water to the machine is opened and the valve admitting low-pressure water to the inside of the sliding cylinder is closed. At the same time the fixed cylinder valve closes to exhaust and opens to the main. Low-pressure water then flows into the fixed cylinder and forces the sliding cylinder downwards; this produces the high-pressure water in the sliding cylinder which is forced into the machine. The intensifier is thus single-acting, supplying high-pressure water during the downward stroke only. Double-acting intensifiers are made which give a continuous supply of high-pressure water.

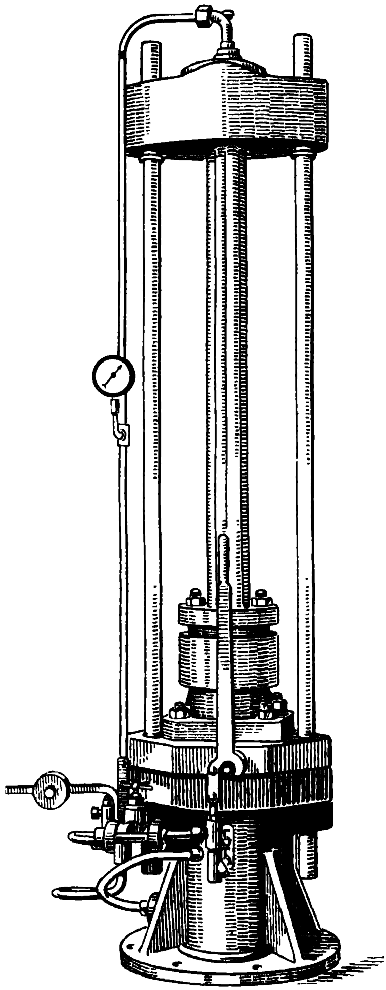


FIG. 271. HYDRAULIC INTENSIFIER (*Courtesy of The Hydraulic Engineering Co., Ltd.*)

It is possible to raise the pressure of water to 10 tons/in.^{2} by means of an intensifier.

The view of an actual intensifier is shown in Fig. 271.

EXAMPLE 3

Water is supplied to an hydraulic intensifier at a pressure of 24 Lb/in.² The diameters of the sliding and fixed rams of the intensifier are 2 in. and 5 in. respectively. Find the pressure of the water leaving the intensifier.

$$p = \frac{PA}{a} = 24 \times \frac{5^2}{2^2}$$

$$= 150 \text{ Lb/in.}^2$$

20.3. Water Meters. 1. THE KENT VENTURI METER. This consists of the ordinary Venturi meter, which has already been dealt with in § 3.6, on to which is attached a special apparatus for indicating the flow of water. The quantity of water flowing through the meter is proportional to the square root of the difference of pressure heads at the entrance and throat. The flow is plotted by a pencil on to a drum which is revolved by clockwork, whilst the total flow through the meter is recorded by the small dials shown in Fig. 272.

The instrument, shown in Fig. 272, consists of two cast-iron cylinders, or chambers, containing mercury in each of which rests a float. These cast-iron cylinders form the two limbs of a U-tube and are connected at their bases by a tube. The water pressure at the entrance and throat of the Venturi meter is transmitted to the cylinders through pipes. The floats are connected to vertical toothed racks each of which turns a pinion as the floats rise or fall with the pressure difference in the Venturi meter. These pinions transmit the motion to two other racks which are outside of the cylinders. The left-hand rack operates the pen carriage which plots the pressure difference on the squared paper surrounding the drum; the latter is rotated by clockwork. The squared paper used on the drum is so divided that the flow may be read direct. As the vertical displacement of the pencil is proportional to the pressure difference, the paper must be divided so as to read the square root of the pressure difference. It should be noted that the left-hand float will rise the same amount as the right-hand float falls, both displacements being in proportion to the pressure difference of the Venturi meter.

The right-hand external rack operates the integrator roller carriage to register the flow. The vertical displacement of the rack is proportional to the pressure difference; this must be reduced to the square root of the pressure difference in order to register the flow. Inside of the clockwork drum, and rotating with it, is an integrating drum, the development of which is shown in Fig. 273. The drum is divided by a parabolic curve ABC. The shaded surface above the curve is raised above the surface below. The right-hand rack is connected to a carriage which is in contact with the drum and which gears with the recording dials. When the raised surface

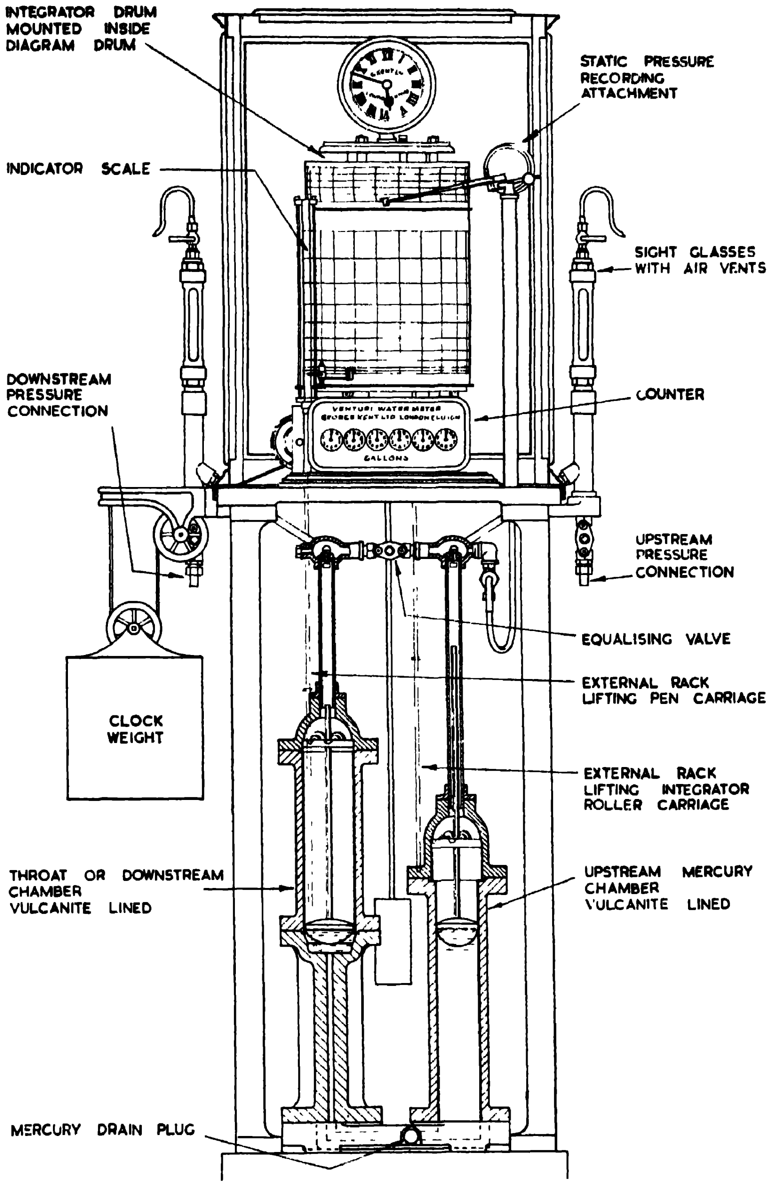


FIG. 272. KENT VENTURI METER
Part sectional front elevation of the "A" type (normal) recorder

(Courtesy of George Kent, Ltd.)

of the drum comes in contact with the carriage the latter is put out of gear with the recorder and no flow is registered. Thus, if the carriage is at a height D, the flow will only be registered over the portion of a revolution represented by DB; it will be out of gear during the portion BE. When the float is at the top of the cylinder there will be no flow taking place; the carriage will then be on the raised surface above A (Fig. 273) during the revolution of the drum, and no flow will be registered by the recorder.

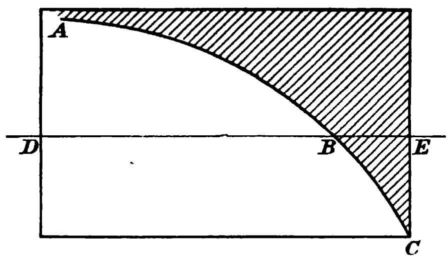


FIG. 273

2. THE DEACON METER. This meter is shown in Fig. 274. It consists of a cast-iron body C, into which is fitted a hollow cone A. The water flows into the meter through E, passes through the cone and leaves the meter through F. A disc D, having a diameter equal to the smallest diameter of the cone, is fixed to the rod G, which slides up and down in the boss B. A balance weight Q is attached to a wire W fixed to the top of the rod G, and keeps the disc D at the top of the cone when no water is flowing. When the water flows into the meter it forces down the disc D into a wider part of the cone and passes through the space between D and the cone sides This space increases as D descends; the vertical drop of D will, therefore, be in proportion to the quantity of water flowing.

The flow through the meter is recorded by means of a pencil connected to the wire from the rod G. The pencil is in contact with the surface of the drum R, which is revolved by clockwork. As the vertical motion of the pencil is proportional to the movement of the disc D, a curve giving the quantity of flow through the meter at any instant will be automatically drawn on suitably graduated squared paper placed around the revolving drum.

This meter is chiefly used for measuring the waste-water flow in water mains.

3. THE KENNEDY METER. This is a positive type of meter, the volume of water flowing being actually measured by continually filling a cylinder of known volume.

The meter consists of a cylinder (Fig. 275), in which slides a piston. The piston rod is connected to a rack, which slides up and

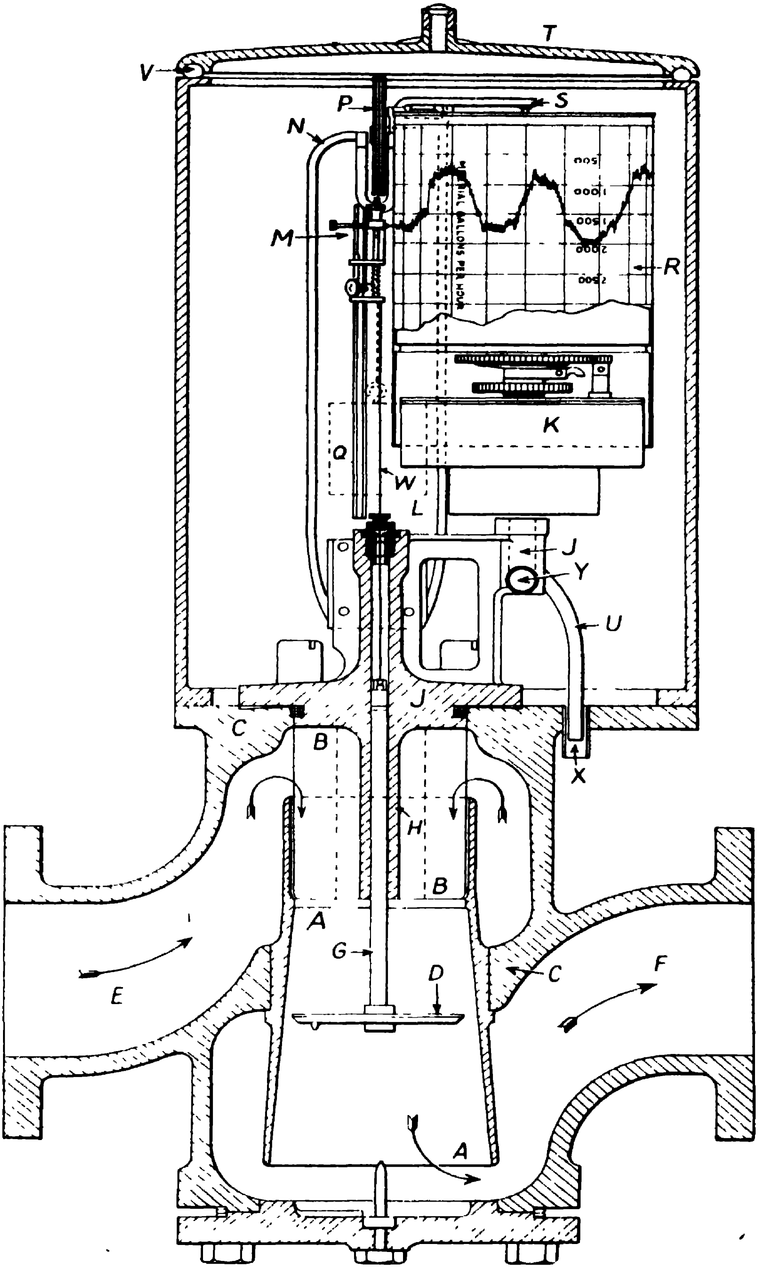


FIG. 274. DEACON'S RECORDING WASTE-WATER METER

(Courtesy of The Palatine Engineering Co., Ltd.)

down with the piston. The rack gears with a pinion, which operates a four-way cock. A diagrammatic view of the passages is shown in Fig. 276. The water from the supply pipe flows through *A* into the

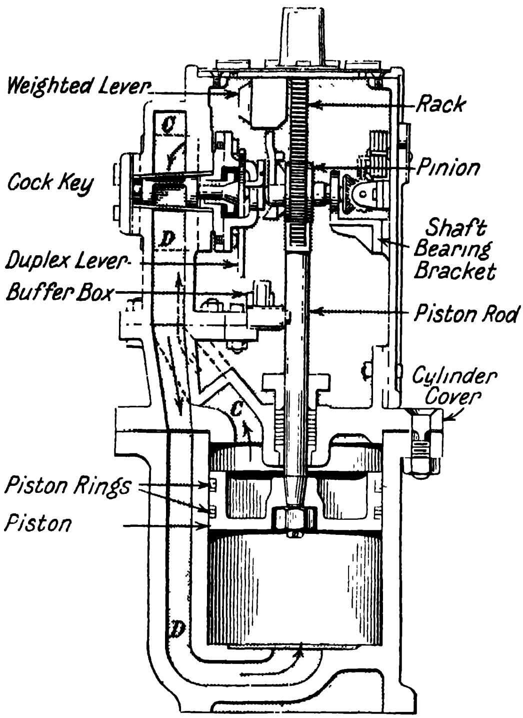


Fig. 275. The Kennedy Meter

pipe *D*, through which it enters the lower end of the cylinder and forces up the piston. As the piston rises, the rack turns the pinion. A weight is fixed to the end of a lever which is rotated upwards by a pin fixed to the pinion. When the piston reaches the top of its stroke, the weight is rotated to just beyond the vertical position; it then falls over suddenly and, by striking a lever, operates the cock into its reverse position. This new position of the cock cuts off the water supply from the lower end of the cylinder and admits it to the upper end. At the same time, the lower end is open to the outlet pipe *B* of the meter. The piston now moves downwards under the pressure of the incoming water, and forces the water in the lower end of the cylinder up the pipe *D* into the outlet pipe. In moving downwards, the rack operates the pinion, which causes the weight to be again raised. When the piston reaches the bottom of its stroke,

the weight falls over and turns the cock back to its former position. The upper part of the cylinder is now open to the outlet pipe and the lower part to the supply pipe. The piston will now be forced

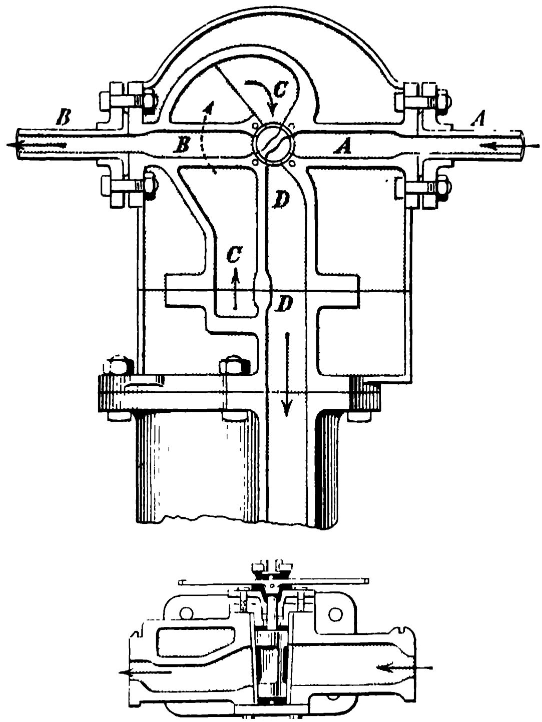


FIG. 276. KENNEDY METER: SECTION THROUGH PORTS

upwards, driving the water above it through the pipe C and into the outlet pipe. The cycle is then repeated.

For each stroke of the piston, a volume of water equal to the volume of the cylinder passes through the meter. The strokes are registered by means of a counter, operated by the pinion, which records the quantity of flow.

20.4. The Hydraulic Ram. The hydraulic ram is an automatic pump by means of which a large quantity of water falling through a small height is utilized in lifting a small quantity of water to a greater height.

A diagrammatic view of a hydraulic ram is shown in Fig. 277. Water from the natural supply A has an available head of H_1; by

means of the ram a small quantity of this water is raised through the height H_2 into the service tank E.

Let W = weight of water flowing per second from A,
w = weight of water raised per second to E.

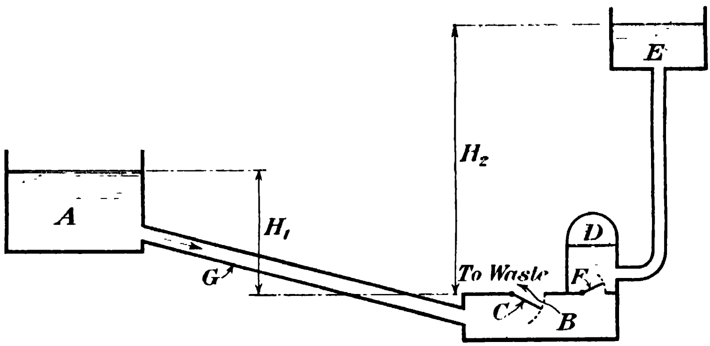


FIG. 277

Then, as energy supplied by A is theoretically equal to energy supplied to E,

$$WH_1 = wH_2$$

or

$$w = \frac{WH_1}{H_2}$$

If losses are taken into account,

$$\text{efficiency of ram} \quad \frac{wH_2}{WH_1}$$

The automatic action of the ram is due to the inertia forces of the water in the pipe G. The water commences to flow down the pipe G into the chamber B. The waste-water valve C is open and the water flows through it to waste. As the speed of the water in G increases, the dynamic pressure on the valve C increases, until it will ultimately be greater than the weight of the valve lid; the valve will then suddenly close. The closing of the valve C brings the water in G suddenly to rest, causing an increase of pressure in B. This increase of pressure lifts the valve F and some of the water will flow into the air vessel D, compressing the air in the vessel. This increased air pressure forces the water into the tank E. When the momentum of the water in B is destroyed, the valve F closes and the valve C opens, causing the flow from A to recommence; the cycle is then repeated. The automatic valves C and F may act by their weight or by a spring.

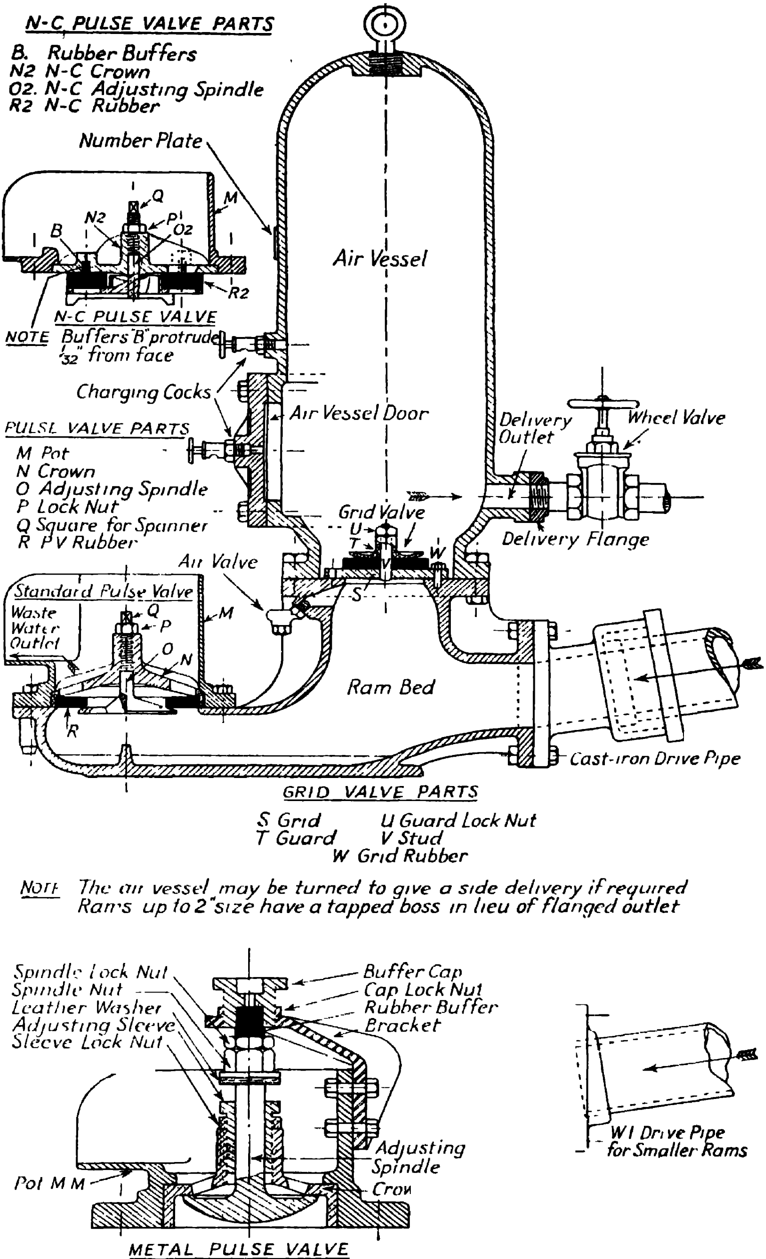


FIG. 278. VULCAN HYDRAULIC RAM

(*Courtesy of Green and Carter, Ltd.*)

Hydraulic rams are chiefly used on country estates and farms at which a large quantity of water under a low head is available.

The cross-sectional view of an actual hydraulic ram is shown in Fig. 278 and a plan and elevation of the complete installation is shown in Fig. 279.

The overall efficiency of the hydraulic ram is as large as 80 per cent, and water can be lifted to a height of fifty times the height

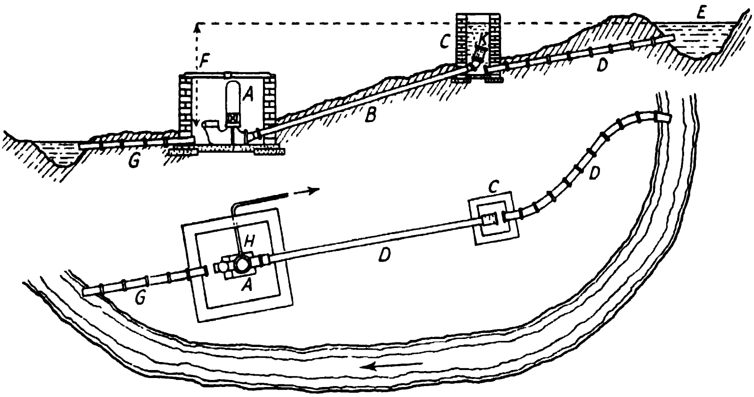


Fig. 279

of the working fall. Compound hydraulic rams are made which will raise water to any height to which it could be forced by an ordinary pump.

20.5. The Hydraulic Press. Hydraulic presses are used in most branches of industry; in principle they are the same as the simple press which was dealt with in § 1.4. They vary greatly in type according to the nature of the work required, but all consist of a ram sliding in a cylinder into which high-pressure water is forced. In some large forging presses water at a pressure of 5 tons/in.^{2} is used in the cylinder and produces a total force of 5,000 tons.

In all heavy presses some means must be adopted to bring about the return stroke of the ram. To do this, small return rams are fitted, their function being to bring the main ram back to the beginning of its stroke. The size of the return ram must be such that the area multiplied by water pressure is sufficient to lift the main ram. In designing the main ram, the area multiplied by water pressure should be large enough to do the work of the press and to overcome the resistance of the return rams.

For balancing purposes it is necessary to have two return rams to one main ram. An alternative method is to have one return ram in tandem with the main ram.

A view of a plastic moulding press is shown in Fig. 280; in this view the two return rams can be seen at the sides of the main ram.

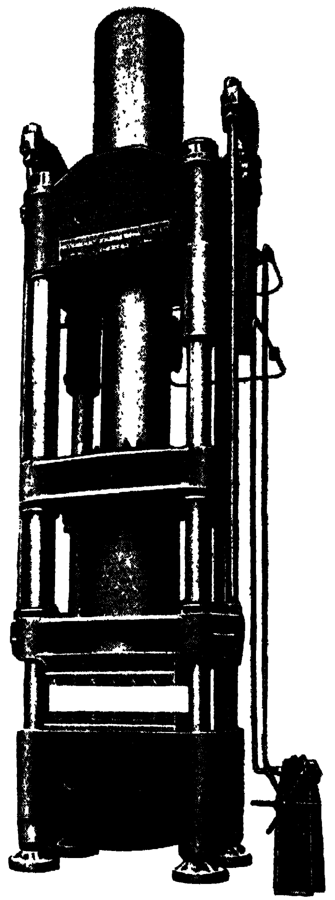


FIG. 280. PLASTIC MOULDING PRESS

Power, 850 tons. Daylight, 60 in. Stroke, 55 in. Platens, 52 in. × 48 in., steam heated. Prefilling for approach stroke. Cylinder, steel forging, moving on columns: ram fixed, giving very accurate guiding. A very useful press for deep mouldings.

(Courtesy of The Hydraulic Engineering Co., Ltd.)

EXAMPLE 4

The ram of a hydraulic press is 8 in. in diameter, and is worked from an intensifier of the piston and ram type which receives its low-pressure supply of water from a tank whose surface level is 50 ft above the level of the intensifier piston, through a pipe 2 in. in diameter and 400 ft long. The intensifier ram is 3 in. in diameter and the piston 36 in. in diameter. The friction of each of the three packings may be taken to be 3 per cent of the total pressure on the appropriate piston or ram. The frictional coefficient for the low pressure supply pipe is 0·005. Calculate the speed of advance of the press ram in inches per minute when exerting a force of 50 tons. Neglect all other losses. (*Lond. Univ.*)

Water pressure on ram of press

$$= 50 \times \frac{100}{97} = 51{\cdot}5 \text{ tons.}$$

Intensity of pressure on ram of press

$$= \frac{51{\cdot}5}{\frac{\pi}{4} 8^2} = 1{\cdot}025 \text{ tons/in.}^2$$

As this is the same pressure transmitted by the ram of the intensifier, intensity of pressure on intensifier ram

$$= 1{\cdot}025 \times \frac{100}{97} \text{ tons/in.}^2$$

As load on intensifier ram equals load on intensifiér sliding cylinder,

$$1{\cdot}025 \times \frac{100}{97} \times \frac{\pi}{4} \times 3^2 = p \times \frac{\pi}{4} \times 36^2 \times \frac{97}{100}$$

where p = pressure of low-pressure water supply:

Hence $$p = 1{\cdot}025 \times \left(\frac{100}{97}\right)^2 \times \left(\frac{3}{36}\right)^2 \times 2{,}240$$

$$= 16{\cdot}96 \text{ Lb/in.}^2$$

Hence, pressure head of low-pressure water

$$= \frac{16{\cdot}96 \times 144}{62{\cdot}4} = 39{\cdot}2 \text{ ft}$$

Therefore, head lost in friction in 2-in. pipe

$$= 50 - 39{\cdot}2 = 10{\cdot}8 \text{ ft}$$

Let v = velocity of water in 2-in. pipe in feet per second,
V = velocity of ram in feet per second.

Head lost in friction in 2-in. pipe

$$= 10{\cdot}8 = \frac{4flv^2}{2gd}$$

$$= \frac{4 \times 0{\cdot}005 \times 400v^2 \times 12}{2 \times 32{\cdot}2 \times 2}$$

from which $$v = 3{\cdot}8 \text{ ft/sec}$$

As quantity of water flowing along 2-in. pipe per second is less than the quantity flowing in the press cylinder per second by the ratio of the intensifier piston areas,

$$v \times \left(\frac{\pi}{4} \times \frac{2^2}{144}\right) = V \times \left(\frac{\pi}{4} \times \frac{8^2}{144}\right) \times \left(\frac{36}{3}\right)^2$$

from which $$V = 3{\cdot}8 \times \left(\frac{2}{8}\right)^2 \times \left(\frac{3}{36}\right)^2 \text{ ft/sec}$$

$$= 1{\cdot}2 \text{ in./min}$$

20.6. The Hydraulic Crane. The hydraulic crane is usually found at docks, sidings and warehouses, and is used for lifting loads up to 250 tons. It consists of a central pedestal supporting a mast from which is suspended a jib, or arm; the latter can be raised or lowered in order to reduce or increase the radius of action. The mast revolves about a vertical axis, the jib swinging with it; thus, by revolving the pedestal and lowering the jib, the suspended load may be moved to any place within the crane's area of action. The principle of the suspended jib enables the load to be lifted over obstacles on the ground.

The load is suspended by a wire rope which passes over pulleys to a hydraulic ram; this ram has an arrangement of pulleys for increasing the velocity ratio and is known as a jigger. The jigger is attached to the mast and consists of a sliding ram and cylinder

at the ends of which are pulleys (Fig. 281); it increases the velocity ratio of the ram and cable by acting on the principle of the multi-sheaved pulley blocks. One set of pulleys is fixed to the ram whilst the other set is fixed to the cylinder, the cable being wound over both sets of pulleys. High-pressure water is admitted into the cylinder, forcing out the ram; this increases the distance between the two sets of pulleys, thus winding in the cable. A six-sheaf pulley block system will give a velocity ratio of six to one · this means that the suspended load will move at six times the speed of the ram. A modern hydraulic crane may have a lifting speed of 250 ft/min.

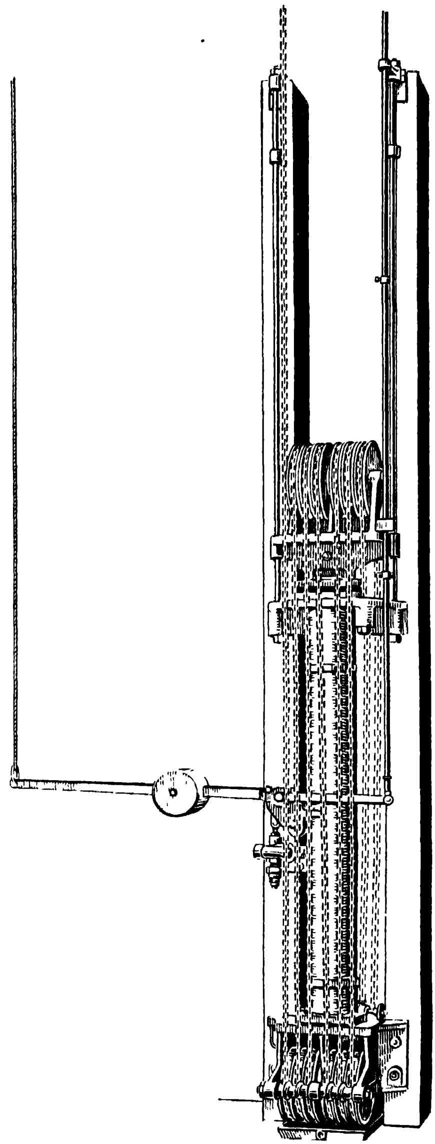


FIG. 281. SINGLE POWER JIGGER

(Courtesy of The Hydraulic Engineering Co., Ltd.)

EXAMPLE 5

The following particulars refer to a hydraulic crane –

Diameter of ram, 12 in.
Velocity ratio of crane hook to ram, 5 : 1.
Length of supply pipe from accumulator, 500 ft.
Diameter of supply pipe, 2 in.
Pressure at accumulator, 750 Lb/in.^{2}
Mechanical friction of ram, pulleys, etc., equivalent to a pressure of 50 Lb/in.^{2} on the ram.
Coefficient of friction for the pipe, 0·010.

Plot a curve showing the relation between the load lifted and the speed of lifting. (*Lond. Univ.*)

Let W = load lifted in pounds.

Then load on ram $= 5W$

Let v = velocity of water in 2-in. pipe,
V = velocity of lifting in feet per second.

Then velocity of ram $= \dfrac{V}{5}$

$$\text{Intensity of pressure on ram} = \frac{5W}{\frac{\pi}{4} \times 1} + (50 \times 144) \text{ Lb/ft}^2$$

$$= 6{\cdot}36W + 7{,}200 \text{ Lb/ft}^2$$

$$\text{Head of water on ram} = \frac{p}{w} = \frac{6{\cdot}36W + 7{,}200}{62{\cdot}4} \text{ ft of water}$$

$$= 0{\cdot}1W + 115 \text{ ft of water}$$

$$\text{Head of water in accumulator} = \frac{750 \times 144}{62{\cdot}4} = 1{,}730 \text{ ft of water}$$

Head lost in friction in pipe = head in accumulator—head on ram
$= 1{,}730 - (0{\cdot}1W + 115)$
$= 1{,}615 - 0{\cdot}1W$ ft of water

Hence, $$1{,}615 - 0{\cdot}1W = \frac{4flv^2}{2gd}$$

$$= \frac{4 \times 0{\cdot}01 \times 500v^2 \times 12}{62{\cdot}4 \times 2}$$

from which $$v = \sqrt{840 - 0{\cdot}052W} \text{ ft/sec}$$

As quantity of flow along pipe per second equals flow per second in ram cylinder,

$$v \times \frac{\pi}{4}\left(\tfrac{1}{6}\right)^2 = \frac{V}{5}\frac{\pi}{4}(1)^2$$

from which $$V = \frac{5}{36}v$$

$$= \frac{5}{36}\sqrt{840 - 0{\cdot}052W}$$

$$= \sqrt{16{\cdot}2 - 0{\cdot}001003W}$$

By substituting various values of W in this equation the corresponding values of V are obtained.

W	0	2,000	4,000	6,000	8,000	10,000	12,000	14,000
V	4·0	3·76	3·48	3·18	2·86	2·48	2·02	1·47

A curve may now be plotted with these results.

20.7. The Hydraulic Lift. The hydraulic lift obtains its motion from a jigger, in the same way as the crane (§ 20.6). The jigger should be fixed with the ram working downwards, so that its weight will be supported by the cables; this prevents any tendency of the ram to move independently of the lift cage. The lift cage runs between guides of hard wood or round steel, and is usually suspended by four lifting ropes, each one being of sufficient strength to support the load. Sliding balance weights are provided to balance the weight of the cage.

Views of hydraulic suspended lifts are shown in Fig. 282.

Modern hydraulic lifts now have a lifting speed of 350 ft/min in this country; in the United States lifting speeds of 400 ft/min are in use.

The earlier form of hydraulic lift consisted of a sliding ram and cylinder; the platform or cage was supported on the end of the ram and pushed up by it. Hence, the stroke of the ram was the same as the lift of the platform. This type of lift is known as a direct-acting lift.

EXAMPLE 6

A hydraulic direct-acting lift has a ram 6 in. in diameter. The pipe connecting the valve box to the cylinder is short and is $\frac{3}{4}$ in. in diameter. The pressure in the valve box is 800 Lb/in.² Neglecting frictional losses and assuming the valve fully open, find the maximum load that can be lifted steadily at a velocity of 2 ft/sec. Find also the maximum velocity with which the lift with this load could descend steadily with an open exhaust. (*A.M.I. Mech.E.*)

Let v = velocity of water in $\frac{3}{4}$ in. pipe.

Then, as quantity per second flowing through pipe equals quantity per second flowing in cylinder,

$$v \times \frac{\pi}{4}\left(\tfrac{3}{4}\right)^2 = 2 \times \frac{\pi}{4}(6)^2$$

from which $$v = 128 \text{ ft/sec}$$

$$\text{Velocity head of water in pipe} = \frac{v^2}{2g} = \frac{128^2}{64\cdot 4} = 255 \text{ ft of water}$$

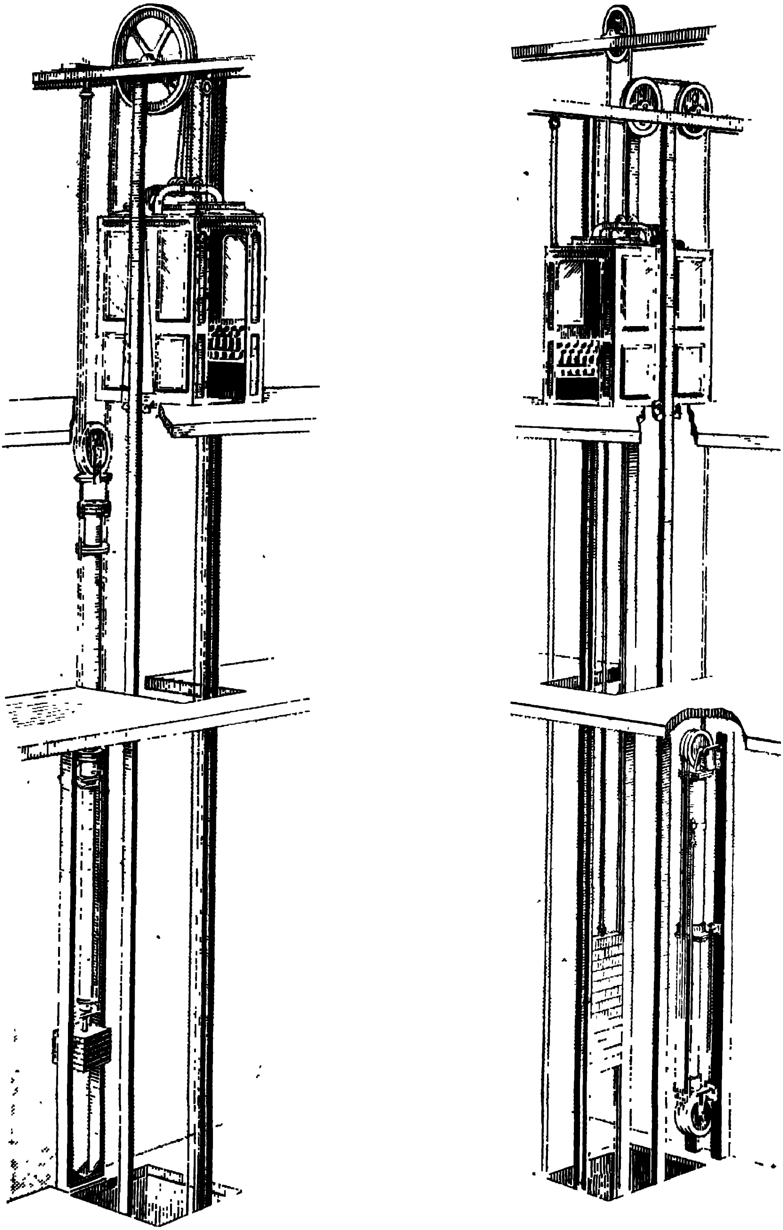


FIG. 282. HYDRAULIC SUSPENDED LIFTS

(*Courtesy of The Hydraulic Engineering Co., Ltd.*)

Total intensity of pressure on ram = pressure in valve box + pressure due to velocity in pipe

$$= 800 + \frac{62 \cdot 4 \times 255}{144}$$

$$= 910 \cdot 5 \text{ Lb/in.}^2$$

$$\text{Load on ram} = 910 \cdot 5 \times \frac{\pi}{4}(6)^2$$

$$= 25{,}700 \text{ Lb}$$

Let V = velocity of descent in feet per second.

Then, velocity in $\frac{3}{4}$ in. pipe $= \left(\frac{6}{\frac{3}{4}}\right)^2 V = 64V$

In descending, the ram will give a velocity head to water in the $\frac{3}{4}$ in. pipe; this will be the only resistance. Hence,

pressure head due to ram = velocity head in $\frac{3}{4}$-in. pipe

that is $$\frac{910 \cdot 5 \times 144}{62 \cdot 4} = \frac{(64V)^2}{2g}$$

from which $$V = 5 \cdot 75 \text{ ft/sec}$$

20.8. The Hydraulic Capstan. Hydraulic capstans are used for winding a haulage rope and are found in railway goods yards and at docks. They consist of a vertical drum operated by a hydraulic engine. A cable is attached to the wagon or ship which is to be moved, the free end being wrapped round the capstan's drum; the capstan's engine is then started by pressing a lever with the foot; this causes the drum to rotate and wind up the haulage cable.

The hydraulic engine used for capstans is usually of the "Brotherton" type. This engine consists of three fixed radial cylinders at 120°, each containing a piston, with piston rods fixed to the same crank pin. The engine contains one working valve, with three parts, each connected to one of the cylinders. High-pressure water is admitted to the head of the cylinder, forcing the piston along the cylinder for the working stroke. During the return stroke the exhaust port is opened and the used water flows out to waste.

The hauling drum is keyed on to the crank shaft, and is the only part of the machine above the ground. The engine is started by a foot treadle, thus leaving the hands free to manipulate the rope. The foot treadle operates a balanced mitre valve which admits water from the mains to the engine; when the foot is removed from the treadle the valve automatically closes.

20.9. Hydraulic Valves. 1. SLIDE VALVES. These valves are for operating hydraulic machinery and consist of the "*D*" slide valve and the piston valve (Fig. 283); they are similar in action to the

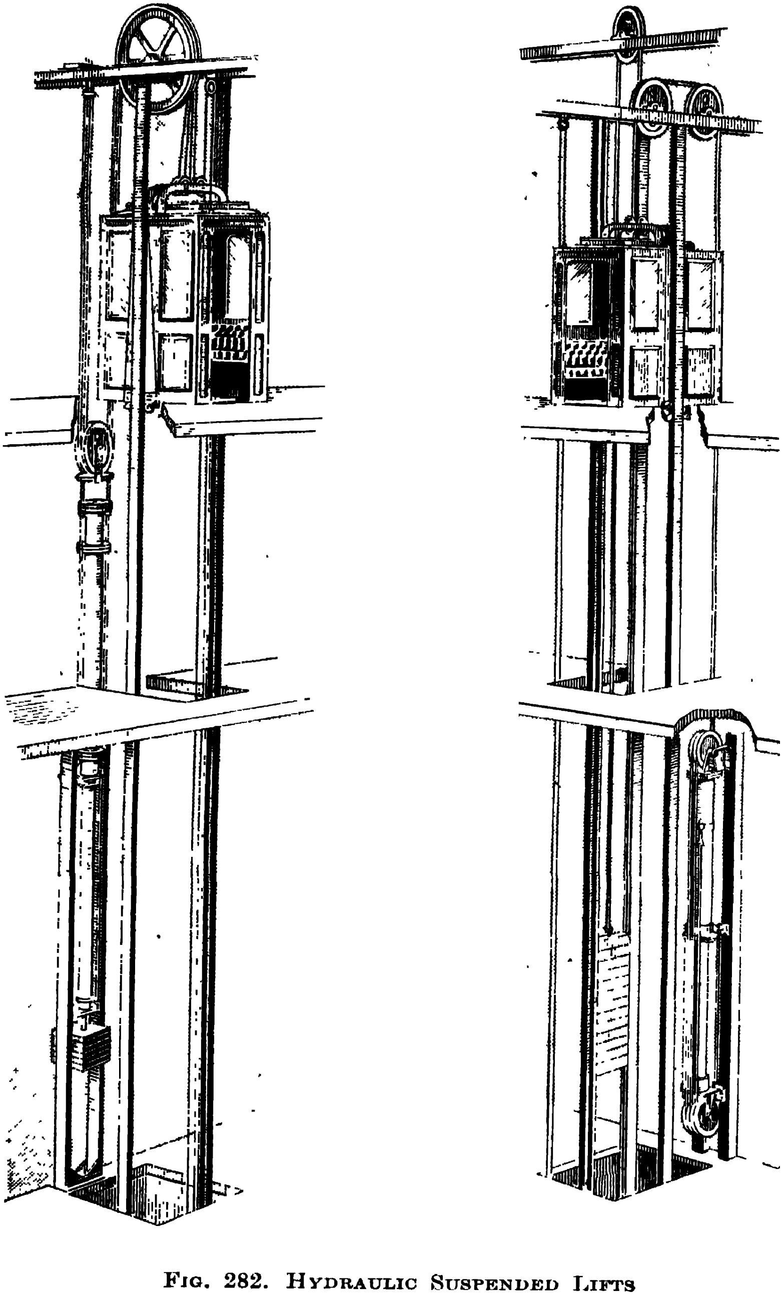


Fig. 282. Hydraulic Suspended Lifts

(Courtesy of The Hydraulic Engineering Co., Ltd.)

Total intensity of pressure on ram = pressure in valve box + pressure due to velocity in pipe

$$= 800 + \frac{62 \cdot 4 \times 255}{144}$$

$$= 910 \cdot 5 \text{ Lb/in.}^2$$

$$\text{Load on ram} = 910 \cdot 5 \times \frac{\pi}{4}(6)^2$$

$$= 25{,}700 \text{ Lb}$$

Let V = velocity of descent in feet per second.

Then, velocity in $\frac{3}{4}$ in. pipe $= \left(\frac{6}{\frac{3}{4}}\right)^2 V = 64V$

In descending, the ram will give a velocity head to water in the $\frac{3}{4}$ in. pipe; this will be the only resistance. Hence,

pressure head due to ram = velocity head in $\frac{3}{4}$-in. pipe

that is $$\frac{910 \cdot 5 \times 144}{62 \cdot 4} = \frac{(64V)^2}{2g}$$

from which $$V = 5 \cdot 75 \text{ ft/sec}$$

20.8. The Hydraulic Capstan. Hydraulic capstans are used for winding a haulage rope and are found in railway goods yards and at docks. They consist of a vertical drum operated by a hydraulic engine. A cable is attached to the wagon or ship which is to be moved, the free end being wrapped round the capstan's drum; the capstan's engine is then started by pressing a lever with the foot; this causes the drum to rotate and wind up the haulage cable.

The hydraulic engine used for capstans is usually of the "Brotherton" type. This engine consists of three fixed radial cylinders at 120°, each containing a piston, with piston rods fixed to the same crank pin. The engine contains one working valve, with three parts, each connected to one of the cylinders. High-pressure water is admitted to the head of the cylinder, forcing the piston along the cylinder for the working stroke. During the return stroke the exhaust port is opened and the used water flows out to waste.

The hauling drum is keyed on to the crank shaft, and is the only part of the machine above the ground. The engine is started by a foot treadle, thus leaving the hands free to manipulate the rope. The foot treadle operates a balanced mitre valve which admits water from the mains to the engine; when the foot is removed from the treadle the valve automatically closes.

20.9. Hydraulic Valves. 1. SLIDE VALVES. These valves are for operating hydraulic machinery and consist of the "*D*" slide valve and the piston valve (Fig. 283); they are similar in action to the

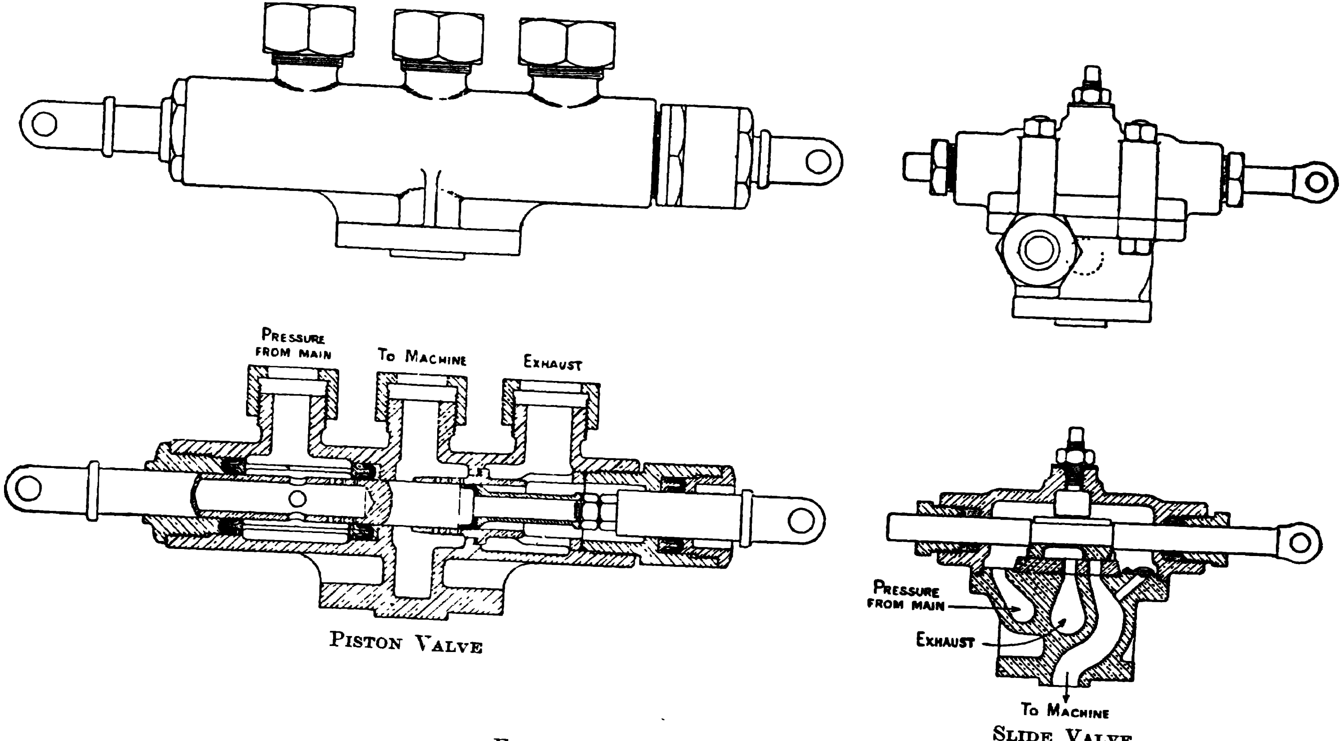


FIG. 283. WORKING VALVES
(Courtesy of The Hydraulic Engineering Co., Ltd.)

ordinary steam-engine valves. For water pressures up to 1,000 Lb/in.^{2} the "*D*" slide valve may be used; but for very high pressures, as in lifts and cranes, the piston valve must be used.

As the valves slide to and fro they uncover or cover the various ports, thus admitting or cutting off the water supply to the machine or to exhaust. The operation of the valve can be clearly seen from Fig. 283.

2. Mitre Valve. This valve is used on cranes that are required to lift and lower rapidly; it consists of vertical spindle valves with mitred ends working on seats to suit, and requires very little effort to operate. A view of this valve is shown in Fig. 284. The valve spindles are operated by levers.

3. Stop Valves. Stop valves are used for shutting off the main water supply. They are spindle valves, and are lowered on to the seat by revolving the spindle in a screw thread, a hand wheel being fitted for this purpose. For small valves, an unbalanced valve may be used; but for large valves working under high pressure it would require too large an effort to close the valve by hand. To overcome this, a balanced stop valve is used. The balanced stop valve has the water admitted to both sides of the valve when open, thus relieving the valve spindle of the water pressure.

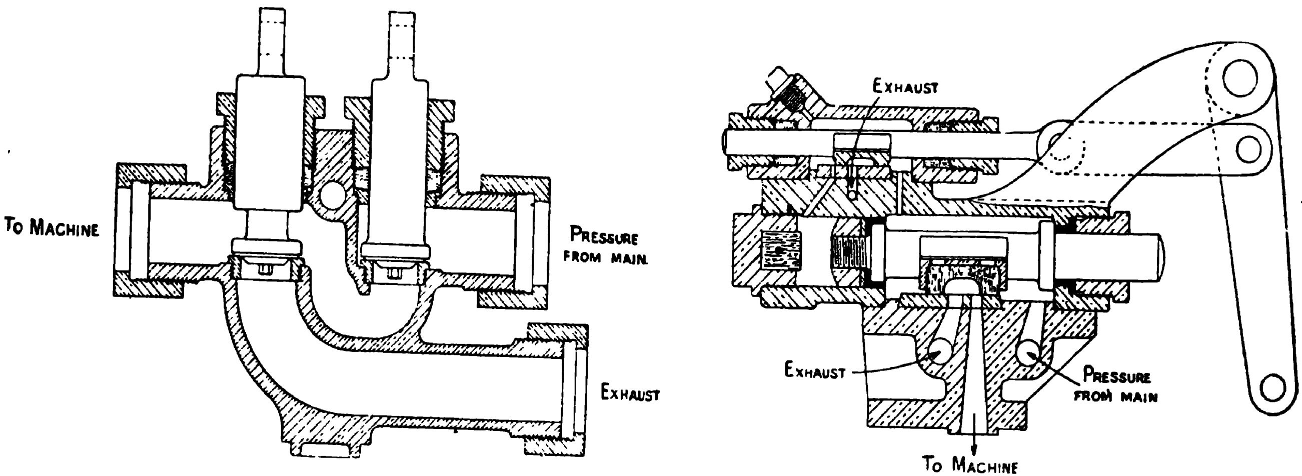
Views of an unbalanced and a balanced stop valve are shown in Fig. 285.

4. Relief Valves. One form of relief valve is the safety valve which is arranged to open and reduce the pressure after a certain maximum pressure has been reached. These are fitted to accumulators and to machines with a rising ram of a predetermined stroke. If the ram should rise beyond its proper limit, owing to accidental causes, the pressure of water would become excessive and dangerous; the relief valve will then open and reduce the pressure. Its action, therefore, is the same as the steam safety valve on a boiler. The form of relief valve for this purpose is a lever and weight-loaded valve; a spring-loaded valve may also be used.

Another use of relief valves is to check the rise in pressure in a long pipe due to the sudden stopping of the flow; such valves are known as momentum valves. They consist of pistons working in a chamber against a spring. These valves are also fitted on machines which receive heavy shocks such as shell forging presses; when used for this purpose they are known as shock absorbers.

20.10. Hydraulic Joints and Packing. Hydraulic pipes of less than 2 in. diameter are usually of wrought iron with screw joints. The ends of each length of pipe are tapped and screwed into a coupling, the thread being first covered with hemp and white lead in order to prevent leakage.

Hydraulic pipes of more than 2 in. diameter are of cast iron with oval or circular flanges cast on the ends. These flanges are bolted

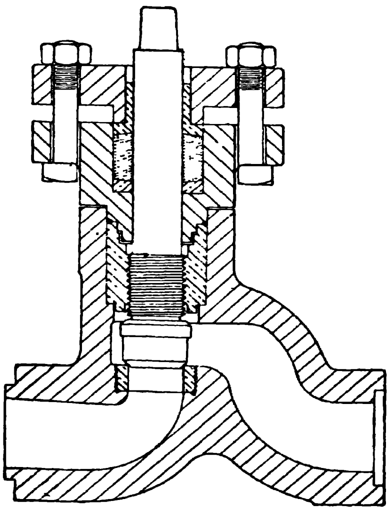


Mitre Valve

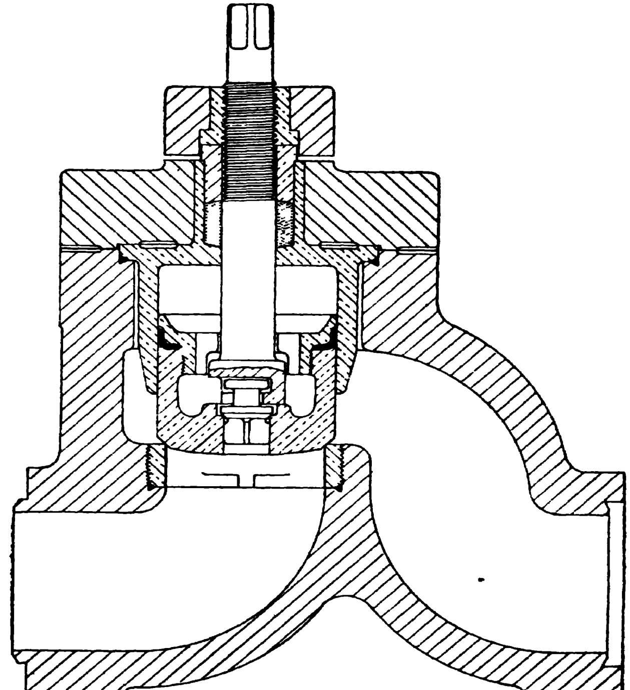
Water-shot Valve

Fig. 284. Working Valves

(Courtesy of The Hydraulic Engineering Co., Ltd.)



Unbalanced Stop Valve



Double Balanced Stop Valve

Fig. 285. Working Valves

(Courtesy of The Hydraulic Engineering Co., Ltd.)

together, a strip of packing being placed between them to prevent leakage. The packing consists of a thin sheet of rubber cut to the shape of the flange, or it may consist of one of the many patent hydraulic packing sheets which are on the market; copper rings are also used in place of sheet packing.

Hydraulic glands, pistons, etc. are packed with hemp or yarn soaked in tallow and well pressed into position. Also leather packing rings are used, the leather being first soaked in grease. These leather packing rings are named after the shape of their sections and are known as "U" leathers, "cup" leathers, and "hat" leathers; plain leather washers are also used.

The seams of wrought-iron tanks, ships, etc. are made watertight by "caulking"; the metal is caused to "flow" over the seam by blows from a caulking tool. The seams of wood vessels and boats are made watertight by placing a layer of white lead between the planks.

EXERCISES 20

1. A hydraulic accumulator has a ram of 9 in. diameter and a lift of 15 ft. Find the load on the ram and the capacity if supplied with water at 60 Lb/in.^{2} pressure. *Ans.* 3,810 Lb; 57,200 ft-Lb.

2. A hydraulic intensifier has ram diameters of 3 in. and 7 in. Find the pressure at which the water is raised when the pressure of the supply is 75 Lb/in.^{2} *Ans.* 408 Lb/in.^{2}

3. A hydraulic lift has a ram diameter of 6 in. and is supplied with water at a pressure of 400 Lb/in.^{2} Find the total load the lift will carry if the efficiency is 85 per cent.

If the lift has a velocity of 2 ft/sec, find the horse-power required when lifting. *Ans.* 9,600 Lb; 41·2 h.p.

4. 40 h.p. is to be transmitted from an accumulator through a 4 in. pipe, 5,000 ft long. If the loss is to be 2 per cent, find the diameter of the ram which is loaded with 120 tons. (Assume coefficient of friction in pipe to be 0·01.) (*Lond. Univ.*) *Ans.* 19·9 in.

5. An accumulator maintains a pressure of 1,200 Lb/in.^{2} in a 3 in. hydraulic main. A hydraulic lift is supplied with pressure water from this main, and the point at which the supply to the lift is drawn off is at a distance of 2,000 ft from the accumulator. The ram at the lift is 8 in. in diameter, and the load on it, inclusive of its own weight, is 12 tons. The friction of the ram, cage, etc. may be taken as equivalent to an addition of $6\frac{1}{2}$ per cent of the gross load on the ram. Determine the speed at which the lift will ascend, if the value of the coefficient of resistance, f, for the hydraulic main is 0·008. Neglect the loss due to shock at entrance to cylinder. (*Lond. Univ.*) *Ans.* 2·69 ft/sec.

6. Give a careful sketch showing the construction of a hydraulic ram, and explain its action fully by aid of reference letters. In what circumstances would you make use of such a machine and why? (*A.M.I.C.E.*)

7. Describe with sketches the hydraulic ram, and explain its action. (*Lond. Univ.*)

8. An accumulator has a 12 in. ram and 20 ft lift, and is loaded with 100 tons total weight. If packing friction accounts for 2 per cent of the total force on

the ram, determine the horse-power being delivered to the mains if the ram falls steadily through its full range in 2 min, and if at the same time the pumps are delivering 240 gal/min. (*A.M.I.Mech.E.*) *Ans.* 395 h.p.

9. A hydraulic lift raises a load of 8 tons through a height of 40 ft once every 2 min, the speed of lifting being 2 ft/sec. It is worked from an accumulator which is being continuously charged by a pump. The pressure of the water is 500 Lb/in.^{2}, the efficiency of the lift 75 per cent and the efficiency of the pump 85 per cent. Find the power required to drive the pump, and the minimum capacity of the accumulator. Frictional losses in the pipes may be neglected. (*Lond. Univ.*) *Ans.* h.p. = 17; volume = 11·08 ft^3.

CHAPTER 21

RECIPROCATING PUMPS

21.1. Types of Reciprocating Pump. There are two main types of pump, centrifugal and reciprocating; the latter type only will be dealt with in this chapter. A reciprocating pump is driven by power from an external source and consists of a cylinder in which a piston or plunger is moved backwards and forwards. This movement of the plunger creates alternately a vacuum pressure and a positive pressure in the cylinder by means of which the water is raised. If a plunger is used, or if the water acts on one side of the piston only, the pump is single-acting. In this case it sucks the water into the cylinder on the outward stroke and forces it out during the inward stroke. If the water acts on both sides of the piston it will suck and deliver during one stroke; such a pump is said to be double-acting.

Pumps which raise the water by suction only are known as suction pumps. Such pumps are only suitable for low lifts, as the maximum height through which water could be lifted by this type of pump is theoretically equal to the barometer reading and actually to about 25 ft. Pumps which lift water by means of pressure are known as force pumps.

The reciprocating pump is always used for producing very high pressures. For very large quantities at low or medium pressures the centrifugal pump is used.

21.2. Force Pump. A diagrammatic view of a force pump is shown in Fig. 286. The rotation of the crank causes the plunger P to move backwards and forwards in the cylinder C. During the suction stroke the plunger moves to the right, which causes a vacuum in the cylinder. The atmospheric pressure on the water surface forces the water up the suction pipe S; this forces open the suction valve a, and the water enters the cylinder. On the return stroke of the plunger the water pressure closes the suction valve and opens the delivery valve b; the water is then forced up the delivery pipe D and so raised to the required height or pressure.

The theoretical volume of water raised per revolution is equal to the stroke volume of the cylinder if the pump is single-acting, and to twice this volume if double-acting.

Actually, the amount lifted is less than this volume, owing to losses.

A sectional view of an actual reciprocating pump is shown in Fig. 287. This pump is double-acting and is driven direct by the

steam piston and cylinder, shown on the left, which is connected to the plunger by a common piston rod.

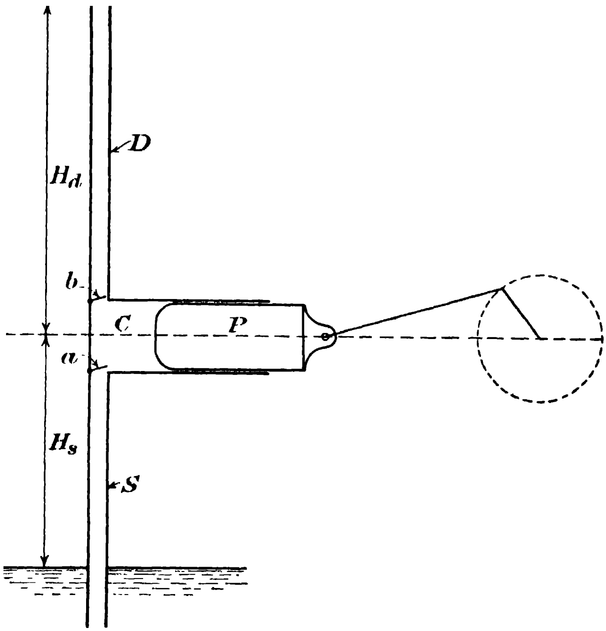


FIG. 286

21.3. Work Done by Pump. Referring to Fig. 286, let r be the radius of crank and L be the length of stroke, in feet. Then

$$L = 2r$$

Let A be the cross-sectional area of piston in square feet. Then,

theoretical volume of water pumped per stroke $= AL$

and theoretical weight of water per stroke $= wAL$

Let H_s = height of centre of cylinder above water surface

and H_d = height to which water is raised above centre of cylinder.

Then, total height lifted $= H_s + H_d$

Let v_d = velocity of water in delivery pipe.

$$\text{Velocity head of water leaving delivery pipe} = \frac{{v_d}^2}{2g}$$

As v_d is usually small and varies during the stroke, it may be neglected unless the total lift is very small.

Let W be the weight of water per second actually lifted.

Then, work done $= W(H_s + H_d)$ ft-Lb/sec

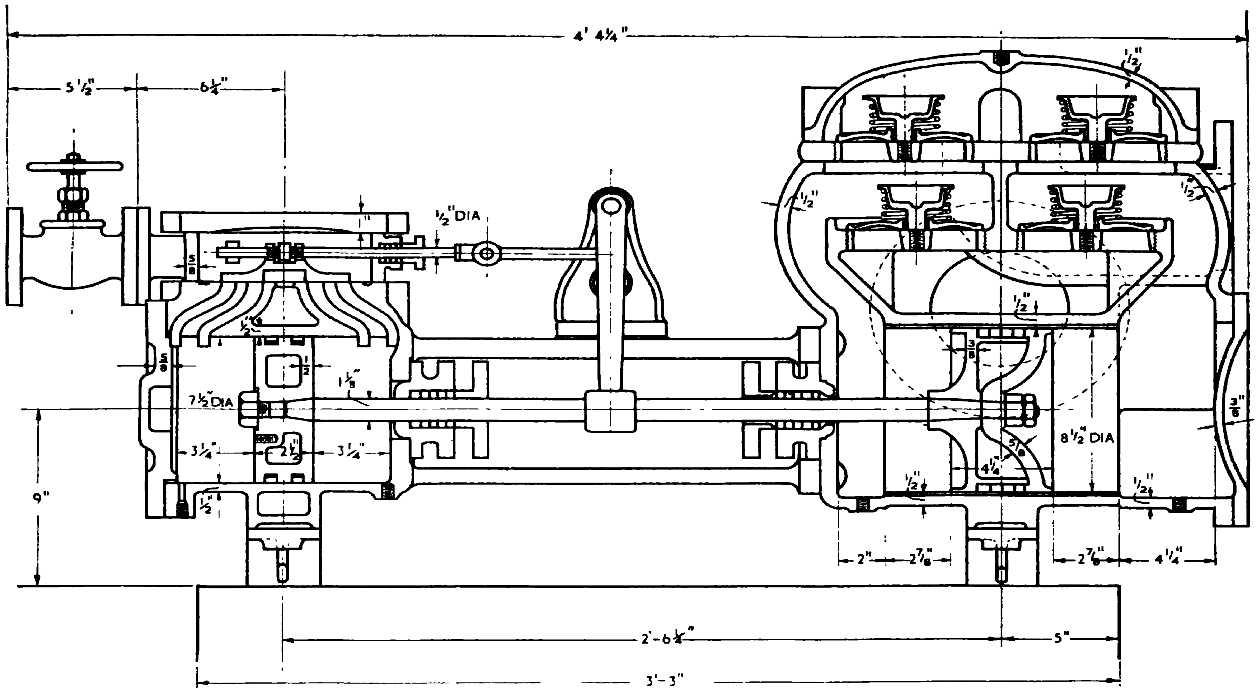


FIG. 287. SECTION OF HORIZONTAL DUPLEX STEAM PUMP

(Courtesy of Worthington-Simpson, Ltd.)

$$\text{Theoretical horse-power required} = \frac{W(H_s + H_d)}{550}$$

The actual horse-power required would be greater than this on account of frictional resistance of water and mechanical parts, and of leakage.

The ratio of the actual volume of water discharged to the volume swept through by the plunger is called the coefficient of discharge. Or,

$$\text{coefficient of discharge} = \frac{W}{62 \cdot 4ALn}$$

where n = number of suction strokes per second.

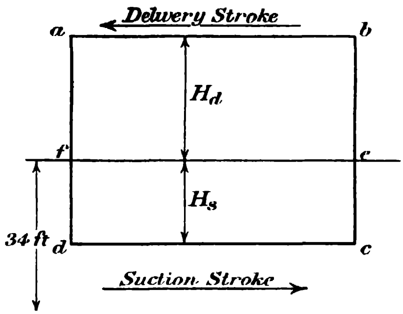


FIG. 288

The difference between the volume swept through by the plunger and the actual discharge is called the "slip."

In the case of pumps with a long suction pipe and a low delivery head, the pressure due to the inertia of the column of water in the suction pipe will be large compared with the pressure on the outside of the delivery valve, especially if the speed is great. This may cause the delivery valve to open before the end of the suction stroke, and a greater volume of water will be delivered than that swept through by the plunger. This makes the theoretical discharge less than the actual; the slip will then be negative and the coefficient of discharge will be greater than unity.

A diagram showing the work done by the pump during a complete cycle is shown in Fig. 288. The diagram shows the pressure on the plunger, or on one side of the piston if double-acting, plotted as the vertical ordinate, whilst the length of the stroke is represented by the horizontal ordinate. The horizontal line *fe* represents atmospheric pressure. The line *dc* is the pressure in the cylinder during the suction stroke, it being below the atmospheric line by the amount H_s. The line *ab* represents the pressure in the cylinder during the delivery stroke, and is above the atmospheric line by the

amount H_d. The area *dcef* is the work done by the plunger on the suction stroke, and *abef* is that done on the delivery stroke. Then, total work done per revolution is given by the area *abcd*. If the pump is double-acting, the work done is twice this amount.

Such a diagram may be obtained automatically by means of an indicator placed on the cylinder, and is consequently called an indicator diagram.

An actual diagram taken by an indicator would be similar to Fig. 288 if the pump were running at a low speed.

EXAMPLE 1

A single-acting reciprocating pump has a piston area of 1·5 ft² and a stroke of 12 in. The cross-sectional area of the delivery pipe is 0·3 ft² and the water is lifted through a total height of 40 ft. If the speed of the pump is 60 r.p.m., and the actual quantity of water lifted 550 gal/min, find the slip, the coefficient of discharge, and the theoretical horse-power required to drive the pump.

$$\text{Volume swept through by piston} = 1{\cdot}5 \times 1{\cdot}0$$
$$= 1{\cdot}5 \text{ ft}^3$$
$$\text{Theoretical volume pumped per second} = 1{\cdot}5 \times \frac{60}{60}$$
$$= 1{\cdot}5 \text{ ft}^3$$
$$\text{Actual volume pumped per second} = \frac{550}{60 \times 6{\cdot}24}$$
$$= 1{\cdot}47 \text{ ft}^3$$
$$\text{Slip} = 1{\cdot}5 - 1{\cdot}47$$
$$= 0{\cdot}03 \text{ ft}^3/\text{sec}$$
$$\frac{0{\cdot}03}{1{\cdot}5} \times 100$$
$$= 2 \text{ per cent}$$
$$\text{Coefficient of discharge} = \frac{1{\cdot}47}{1{\cdot}5} = 98 \text{ per cent}$$
$$\text{Total pressure head} = H_s + H_d$$
$$= 40 \text{ ft of water}$$
$$\text{Theoretical horse-power} = \frac{550 \times 10}{60} \times \frac{40}{550}$$
$$= 6{\cdot}67$$

21.4. Variation of Pressure due to Acceleration of Piston. Owing to the reciprocating motion of the plunger or piston, it will have an acceleration at the beginning and a retardation at the end of each stroke. This will transmit a corresponding acceleration and retardation to the water in the suction and delivery pipes, the inertia of which will cause a variation of the pressure in the cylinder.

In order to simplify the problem it is usual to assume that the piston moves with simple harmonic motion. This would be the case if the connecting rod were very long compared with the length of the crank.

Consider the diagrammatic view of the crank and connecting rod of Fig. 289. Let the crank be rotating with an angular velocity ω.

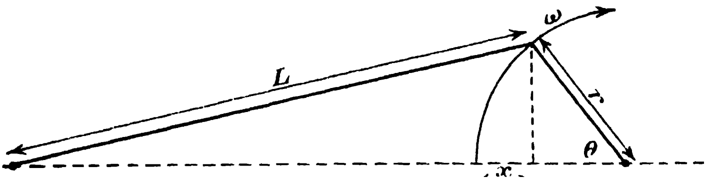


FIG. 289

Let it turn through an angle θ in the time t sec, and assuming simple harmonic motion,

$$\theta = \omega t$$

Displacement of piston from end of stroke

$$= x = r - r\cos\omega t$$

$$\text{Velocity of piston} = v = \frac{dx}{dt} = \omega r \sin\omega t \qquad . \quad . \quad (1)$$

$$\text{Acceleration of piston} = f = \frac{dv}{dt} = \omega^2 r \cos\omega t \qquad . \quad . \quad (2)$$

Let A be the area of piston and a be the area of pipe. Then, as volume of water flowing from pipe per second equals volume of water flowing into cylinder per second,

$$\text{velocity of water in pipe} = \text{velocity of piston} \times \frac{A}{a}$$

$$= \frac{A}{a}\omega r \sin\omega t$$

$$= \frac{A}{a}\omega r \sin\theta \qquad (3)$$

$$\text{Acceleration of water in pipe} = f \times \frac{A}{a}$$

$$= \frac{A}{a}\omega^2 r \cos\omega t$$

$$= \frac{A}{a}\omega^2 r \cos\theta \qquad (4)$$

Let l = length of pipe through which water is flowing. Then

weight of water in pipe = wal

Let p_a = intensity of pressure due to acceleration of water in pipe. From Newton's laws of motion,

accelerating force = mass × acceleration

That is,

$$p_a a = \frac{wal}{g} \times f\frac{A}{a}$$

or

$$p_a = \frac{wl}{g} \times f\frac{A}{a}$$

Let H_a = acceleration pressure in feet of water

$$= \frac{p_a}{w}$$

Then,

$$H_a = \frac{p_a}{w} = \frac{l}{g} \times f\frac{A}{a}$$

Substituting for f from eq. (2),

$$H_a = \frac{l}{g} \times \frac{A}{a}\,\omega^2 r \cos\theta \quad . \quad . \quad . \quad (5)$$

The pressure head due to acceleration acting on the piston will, therefore, vary with the angle θ.

At the beginning of the stroke when $\theta = 0$, $\cos\theta = 1$; then

$$H_a = \frac{l}{g}\frac{A}{a}\,\omega^2 r$$

At the middle of the stroke when $\theta = 90$, $\cos\theta = 0$; then

$$H_a = 0$$

At the end of the stroke when $\theta = 180$, $\cos\theta = -1$; then

$$H_a = -\frac{l}{g}\frac{A}{a}\,\omega^2 r$$

If simple harmonic motion is not assumed, the acceleration of piston at dead centres = $\omega^2 r\,[1 \pm (r/L)]$ where L is the length of the connecting rod.* Then, at beginning of stroke,

$$H_a = \frac{l}{g}\frac{A}{a}\,\omega^2 r\left(1 + \frac{r}{L}\right)$$

At end of stroke,

$$H_a = \frac{l}{g}\frac{A}{a}\,\omega^2 r\left(1 - \frac{r}{L}\right)$$

21.5. The Effect of Acceleration in Suction Pipe. Consider the suction stroke of the pump of Fig. 286. As the piston moves along the cylinder it must produce a vacuum sufficient to lift the water up the height H_s, and also to accelerate the water. The vacuum pressure in the cylinder must, therefore, equal $H_s + H_a$. If this vacuum pressure reaches 26 ft of water, that is 8 ft absolute, the

* See text-books on Theory of Machines.

water commences to vaporize and cavities of dissolved gases and vapour are formed. This will cause the water in the pipe to separate and flow in sections; the flow is then no longer continuous and vibrations and "knocking" will occur. This phenomenon is known as *separation* or *cavitation* and must be prevented.

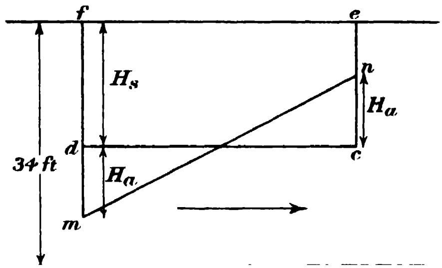


FIG. 290

The suction stroke of the indicator diagram of Fig. 288 must now be modified to take into account the acceleration head.

Let l_s = length of suction pipe
and a_s = cross-sectional area of suction pipe.

Then
$$H_a = \frac{l_s}{g} \times \frac{A}{a_s} \omega^2 r \cos \theta$$

At the beginning of the suction stroke this must be added to the suction head as the piston is accelerating the water. The equation gives a straight sloping line, the accelerating head being zero at the centre of the stroke. At the end of the stroke the water causes a positive pressure on the piston in retarding, which reduces the vacuum pressure in the cylinder.

The new indicator diagram for the suction stroke is shown in Fig. 290. The acceleration head H_a is added to the vacuum head at the beginning of the stroke and subtracted at the end. The work done is now represented by the area *fmne*; but as this equals the area *fdce*, the net work done remains as before. Thus, the inertia of the water does not affect the net work done, but only causes a variation of the pressure in the cylinder. The piston does work on the water in accelerating it during the first half of the stroke, but receives it back in retarding it during the latter half.

If simple harmonic motion had not been assumed, the straight sloping line *mn* would have been slightly curved.

In designing pumps, the point *m* (Fig. 290) must not fall below the cavitation pressure of the water. Or

$$H_s + H_a \text{ must not be greater than 26 ft of water}$$

This may be arranged by varying H_s, l_s, the ratio A/a_s, or the speed of the pump.

EXAMPLE 2

A single-acting pump has a plunger diameter of 5 in. and a stroke of 1 ft. The length of the suction pipe is 30 ft and the diameter 3 in. Find the acceleration head at the beginning of stroke when the pump is running at 30 r.p.m. If the height of pump's centre is 10 ft above the water level in the sump, find the pressure head in the cylinder at beginning of stroke.

At beginning of stroke, $\cos\theta = 1$. Then

$$H_a = \frac{l_s}{g} \times \frac{A}{a}\,\omega^2 r$$

$$\frac{30}{32{\cdot}2} \times \left(\frac{5}{3}\right)^2 \left(\frac{2\pi 30}{60}\right)^2 \times 0{\cdot}5$$

$$= 12{\cdot}75 \text{ ft of water}$$

$$\text{Pressure head in cylinder} = 34 - H_s - H_a$$

$$= 34 - 10 - 12{\cdot}75$$

$$= 11{\cdot}25 \text{ ft of water}$$

21.6. The Effect of Acceleration in the Delivery Pipe. The column of water in the delivery pipe will be accelerated at the beginning of the delivery stroke and retarded at the end, in the same way as that in the suction pipe. But, as the delivery pipe may be much longer than the suction pipe, the lift of the latter being limited to 26 ft, the accelerating head in this case may be very large.

Let l_d = length of delivery pipe
and a_d = cross-sectional area of delivery pipe.

Then,
$$H_a = \frac{l_d}{g} \times \frac{A}{a_d}\,\omega^2 r \cos\theta$$

In Fig. 291 the indicator diagram of the delivery stroke of Fig. 288 is shown with the acceleration head added. The work done is the area *efqp*, and is not affected by the acceleration of the water. The minimum pressure head in the cylinder is represented by the point q and equals

$$H_d - H_a$$

above atmosphere. In absolute units this becomes

$$34 + H_d - H_a$$

This amount must not be less than 8 ft of water, otherwise cavitation will take place at the end of the stroke. The limiting condition is, therefore, when

$$34 + H_d - H_a = 8$$

or, when

$$H_a = 26 + H_d$$

If the delivery pipe of the pump is vertical, both sides of this equation will increase with the length of the pipe; in which case it is highly improbable that H_a would be greater than H_d.

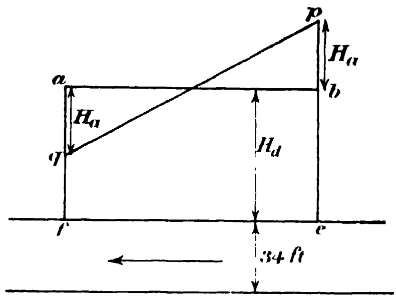


FIG. 291

Consider the delivery pipe of a pump to be bent either to the form shown in Fig. 292 or to that of Fig. 293. Let the length of pipe and height lifted be the same in both cases. The conditions at points a in both figures will be the same, there being no difference in the values of H_a and H_d in either case. If separation takes place

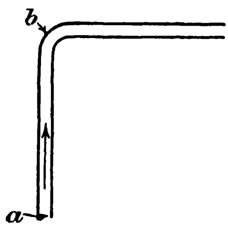


FIG. 292

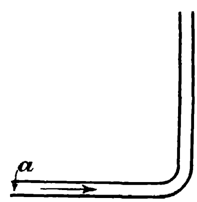


FIG. 293

it would do so at the point b of Fig. 292; for at this point H_d is zero and there is still a considerable length of pipe beyond b which affects H_a.

EXAMPLE 3

A single-acting pump has a piston diameter of 6 in. and a crank radius of 1 ft. The delivery pipe is 3 in. in diameter and 100 ft long. The water is lifted 100 ft above the centre of the pump. Find the maximum speed at which the pump may be run so that no cavitation takes place during the delivery stroke. Neglect the velocity head in the delivery pipe and assume separation occurs at an absolute pressure of 8 ft of water.

Referring to Fig. 291, separation takes place when

$$H_d + 34 - H_a = 8$$

or
$$H_a = H_d + 26$$
$$= 100 + 26 = 126 \text{ ft of water}$$

But, at end of stroke,
$$H_a = \frac{l_d}{g} \times \frac{A}{a}\,\omega^2 r$$

Therefore
$$126 = \frac{100}{32\cdot 2} \times \left(\frac{6}{3}\right)^2 \omega^2 \times 1$$

from which
$$\omega = 3\cdot 22 \text{ radians per second}$$

Let n = number of revolutions per minute. Then
$$\omega = \frac{2\pi n}{60}$$

Hence
$$n = \frac{3\cdot 22 \times 60}{2\pi}$$
$$= 30\cdot 6 \text{ r.p.m.}$$

21.7. Work Done against Friction in Pipes. There will be a frictional resistance to the flow in the suction and delivery pipes which follows the ordinary friction laws dealt with in Chapter 7.

For any point of the stroke the velocity in the pipe is given by eq. (3)
$$v = \frac{A}{a}\,\omega r \sin \theta$$

$$\text{Head lost in friction} = h_f = \frac{4fl}{d} \times \frac{v^2}{2g}$$
$$\frac{4fl}{2dg}\left(\frac{A}{a}\,\omega r \sin \theta\right)^2$$

where f = coefficient of friction.

At the two ends of the stroke, $\sin \theta = 0$; therefore the velocity in the pipe is zero, and there will be no loss of head due to friction.

h_f has its maximum value when $\theta = 90°$, that is at the middle of the stroke, when
$$h_f = \frac{4fl}{d2g}\left(\frac{A}{a}\,\omega r\right)^2$$

The equation for h_f is a parabola. If the frictional head is added to the indicator diagrams of Figs. 290 and 291 the combined indicator diagram will be as shown in Fig. 294, the parabola *mrn* being the work done against friction in the suction pipe, and the parabola *qsp* being that of the delivery pipe.

Total work done during suction stroke
$$= \text{area } efmrn$$
$$= \text{area } efdc + \text{area } mrn$$

Total work done during delivery stroke

$$= \text{area } efqsp$$
$$= \text{area } abef + \text{area } qsp$$

As the mean ordinate of a parabola is equal to two-thirds of the maximum ordinate,

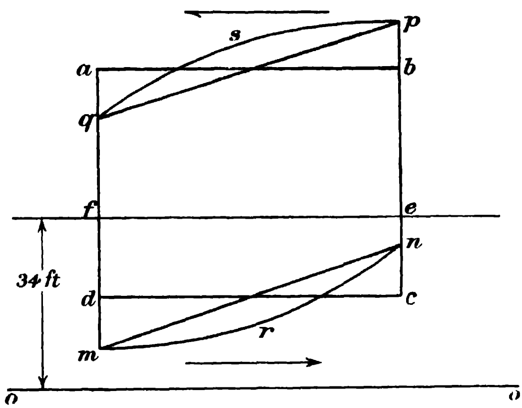


FIG. 294

mean ordinate of suction pipe parabola

$$= \frac{2}{3} h_{f_s}$$
$$= \frac{2}{3} \times \frac{4fl_s}{d_s 2g} \left(\frac{A}{a_s} \omega r\right)^2$$

where the suffix s applies to the suction pipe.

Work done against friction during suction stroke

$$= \text{area of parabola } mrn$$

Work done against friction per second

$$= \frac{2}{3} \times \frac{4fl_s}{d_s 2g} \left(\frac{A}{a_s} \omega r\right)^2 \times W$$

In the same way, work done against friction during delivery stroke per second

$$= \frac{2}{3} h_{f_d} \times W$$
$$= \frac{2}{3} \times \frac{4fl_d}{d_d 2g} \left(\frac{A}{a_d} \omega r\right)^2 \times W \text{ ft-Lb}$$

where the suffix d refers to the delivery pipe and W is weight of water pumped per second.

Total work done per second

$$= W\left(h_s + h_d + \frac{2}{3}h_{f_s} + \frac{2}{3}h_{f_d}\right)$$

The vacuum pressure on the piston during the suction stroke for any angle θ of the crank

$$= H_s + H_a + h_{f_s}$$

$$= H_s + \frac{l_s A\omega^2 r\cos\theta}{ga_s} + \frac{4fl_s}{d_s 2g}\left(\frac{A}{a_s}\omega r\sin\theta\right)^2 \text{ ft of water}$$

The pressure on the piston, above atmosphere, during the delivery stroke is equal to

$$H_d + \frac{l_d A\omega^2 r\cos\theta}{ga_d} + \frac{4fl_d}{d_d 2g}\left(\frac{A}{a_d}\omega r\sin\theta\right)^2 \text{ ft of water}$$

It will be noticed that in both these equations the acceleration head is a maximum at the ends of the stroke and zero at the centre, whilst the frictional head is zero at the ends and a maximum at the centre of the stroke.

Example 4

A single-acting pump has a stroke of 1 ft and a piston diameter of 6 in. The centre of the pump is 15 ft above level of water in sump and 100 ft below delivery water level. The lengths of the suction and delivery pipes are 20 ft and 120 ft respectively, and their diameters are 3 in. The coefficient of friction for these pipes is 0·01. If the pump is working at 30 r.p.m., find the pressure head on the piston at the beginning, middle, and end of both strokes, and find the horse-power required to drive the pump. (Ignore the velocity head of the discharge water.)

(1) Suction Stroke

At ends of stroke,
$$H_a = \frac{l_s}{g} \times \frac{A}{a_s}\omega^2 r$$
$$= \frac{20}{32{\cdot}2} \times \left(\frac{6}{3}\right)^2 \left(\frac{2\pi 30}{60}\right)^2 \times \tfrac{1}{2}$$
$$= 12{\cdot}3 \text{ ft of water}$$

At middle of stroke,
$$h_{f_s} = \frac{4fl_s}{d_s 2g}\left(\frac{A}{a_s}\omega r\right)^2$$
$$= \frac{4 \times 0{\cdot}01 \times 20}{\frac{1}{4} \times 64{\cdot}4}\left(4 \times \frac{2\pi 30}{60} \times \tfrac{1}{2}\right)^2$$
$$= 1{\cdot}96 \text{ ft of water}$$

Pressure at beginning of stroke
$$= H_s + H_a$$
$$= 15 + 12{\cdot}3$$
$$= 27{\cdot}3 \text{ ft of water (vacuum)}$$

$$\text{Pressure at end of stroke} = H_s - H_a$$
$$= 15 - 12{\cdot}3$$
$$= 2{\cdot}7 \text{ ft of water (vacuum)}$$

$$\text{Pressure at middle of stroke} = H_s + h_{f_s}$$
$$= 15 + 1{\cdot}96$$
$$= 16{\cdot}96 \text{ ft of water (vacuum)}$$

(2) DELIVERY STROKE

At ends of stroke,
$$H_a = \frac{l_d}{g} \times \frac{A}{a_d}\,\omega^2 r$$
$$= \frac{120}{32{\cdot}2} \times \left(\frac{6}{3}\right)^2 \left(\frac{2\pi 30}{60}\right)^2 \times \tfrac{1}{2}$$
$$= 73{\cdot}8 \text{ ft of water}$$

At middle of stroke,
$$h_{f_d} = \frac{4fl_d}{d_d 2g}\left(\frac{A}{a_d}\,\omega r\right)^2$$
$$= \frac{4 \times 0{\cdot}01 \times 120}{\frac{1}{4} \times 64{\cdot}4}\left(4 \times \frac{2\pi 30}{60} \times \tfrac{1}{2}\right)^2$$
$$= 11{\cdot}75 \text{ ft of water}$$

$$\text{Pressure at beginning of stroke} = H_d + H_a$$
$$= 100 + 73{\cdot}8$$
$$= 173{\cdot}8 \text{ ft of water (above atm.)}$$

$$\text{Pressure at end of stroke} = 100 - 73{\cdot}8$$
$$= 26{\cdot}2 \text{ ft of water (above atm.)}$$

$$\text{Pressure at middle of stroke} = H_d + h_{f_d}$$
$$= 100 + 11{\cdot}75$$
$$= 111{\cdot}75 \text{ ft of water (above atm.)}$$

$$\text{Work done per stroke} = p \times \text{area} \times \text{length}$$
$$= wH \times \text{volume of cylinder}$$
$$= \text{weight of water per stroke} \times H$$

$$\text{Weight of water per stroke} = W = 62{\cdot}4 \times \frac{\pi}{4}(0{\cdot}5)^2 \times 1$$
$$= 12{\cdot}25 \text{ Lb}$$

Work done during suction stroke
$$= W\left(H_s + \frac{2}{3}h_{f_s}\right)$$
$$= W\left\{15 + \left(\frac{2}{3} \times 1{\cdot}96\right)\right\}$$
$$= 16{\cdot}31\,W \text{ ft-Lb}$$

Work done during delivery stroke

$$= W\left(H_d + \frac{2}{3}h_{f_d}\right)$$

$$= W\left\{100 + \left(\frac{2}{3} \times 11{\cdot}75\right)\right\}$$

$$= 107{\cdot}8W \text{ ft-Lb}$$

Total work done per revolution $= W(107{\cdot}8 + 16{\cdot}31)$

$$= 12{\cdot}25 \times 124{\cdot}1 = 1{,}520 \text{ ft-Lb}$$

$$\text{Horse-power required} = \frac{1{,}520 \times 30}{33{,}000}$$

$$= 1{\cdot}38$$

21.8. Maximum Vacuum Pressure during Suction Stroke. It is not quite clear from Fig. 294 which part of the suction stroke will have the maximum vacuum pressure. A part of the curve mrn may fall below m; in which case cavitation may occur at some point other than the beginning of the stroke.

The velocity head of the water in the suction pipe is converted into pressure head on entering the cylinder, therefore the maximum vacuum pressure will occur just inside the suction pipe at the section where it enters the cylinder.

Let the total vacuum pressure in the pipe at this section $= H$, and v_s = velocity in suction pipe. Then,

$$H = H_s + H_a + \frac{v_s^2}{2g} + h_f$$

Substituting the values of H_a, v_s, and h_f in terms of θ,

$$H = H_s + \frac{l_s}{g}\frac{A}{a_s}\,\omega^2 r\cos\theta + \frac{v_s^2}{2g}\left(1 + \frac{4fl_s}{d_s}\right)$$

$$= H_s + \frac{l_s}{g}\frac{A}{a_s}\,\omega^2 r\cos\theta + \left(\frac{A}{a_s}\right)^2\frac{\omega^2 r^2\sin^2\theta}{2g}\left(1 + \frac{4fl_s}{d_s}\right)$$

Differentiating,

$$\frac{dH}{d\theta} = -\frac{l_s}{g}\frac{A}{a_s}\,\omega^2 r\sin\theta + \left(\frac{A}{a_s}\right)^2\frac{\omega^2 r^2\sin\theta\cos\theta}{g}\left(1 + \frac{4fl_s}{d_s}\right)$$

$$= 0 \text{ for maximum vacuum pressure} \quad . \quad . \quad . \quad . \quad (6)$$

But $$\sin\theta\cos\theta = \frac{\sin 2\theta}{2}$$

Hence

$$-l_s\sin\theta + \frac{A}{a_s}\,r\,\frac{\sin 2\theta}{2}\left(1 + \frac{4fl_s}{d_s}\right) = 0$$

Hence $\theta = 0$ for maximum vacuum pressure.

Another solution of eq. (6) is obtained by dividing through by $\sin\theta$; then,

$$\cos\theta = \frac{l_s a_s}{Ar\left(1 + \frac{4fl_s}{d_s}\right)}$$

As this equation, when applied to actual pumps, always gives a value greater than unity, its solution does not hold.

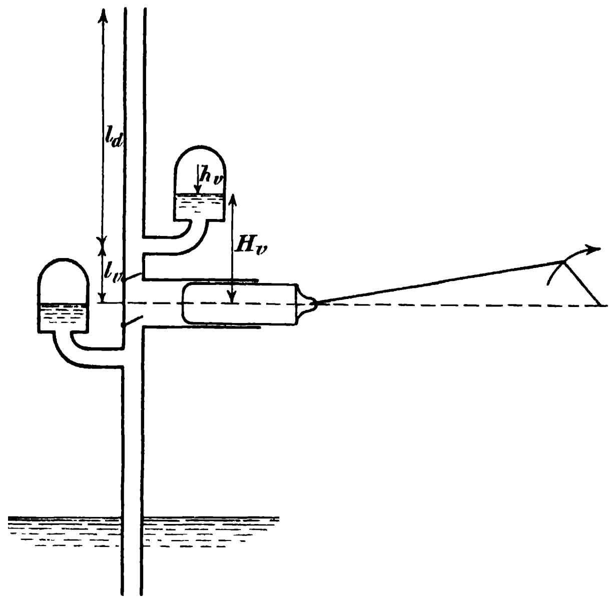


FIG. 295

The vacuum pressure is, therefore, a maximum at the beginning of the stroke.

Hence, if cavitation occurs during the suction stroke, it will do so at the beginning.

21.9. The Reduction of the Acceleration Head by Means of an Air Vessel. As the pressure of the pump must not fall below the cavitation pressure of the water, the maximum speed of a pump is limited by the acceleration head. The acceleration head depends on the length of the suction or delivery pipe, and it may be considerably reduced by fitting an air vessel on these pipes as near to the cylinder as possible. Suppose an air vessel be fitted on the delivery pipe of the pump in Fig. 295. The air vessel is a cast-iron chamber having an opening at the base, through which the water may flow. As the level of the water in the chamber rises, the air trapped in the upper

portion of the chamber is compressed, and will force the water out as soon as the pressure of the latter falls.

The water in the delivery pipe beyond the air vessel is assumed to flow with a uniform velocity throughout the cycle. During the middle portion of the delivery stroke, when the piston is forcing the water into the delivery pipe with a velocity greater than the mean, the additional water will flow into the air vessel. At the ends of the stroke, when the water is forced into the delivery pipe with a velocity less than the mean, the water will flow out of the air vessel and so make up the deficiency. The constant flow in the delivery pipe beyond the air vessel is thus maintained. The only volume of water which is now accelerated is that in the delivery pipe between the air vessel and cylinder; this is made small by fitting the air vessel as near the cylinder as possible.

The pressure of the air in the air vessel will vary as the water flows in and out; this variation is reduced by making the air vessel large compared with the area of the delivery pipe. In order to simplify the problem, it is assumed that the air vessel is so large that the change of water level in it may be neglected. This is the same as assuming the air pressure in the air vessel to be constant.

Let l_d = length of delivery pipe beyond air vessel,

l_v = length of delivery pipe between cylinder and air vessel,

v_d = constant velocity of water in delivery pipe beyond air vessel.

Then,
$$H_a = \frac{l_v}{g}\frac{A}{a_d}\omega^2 r \cos\theta$$

Head lost in friction in delivery pipe beyond air vessel

$$= \frac{4fl_d v_d^2}{d_d 2g}$$

Head lost in friction in delivery pipe between air vessel and cylinder

$$= \frac{4fl_v}{d_d 2g}\left(\frac{A}{a_d}\omega r \sin\theta\right)^2$$

Also
$$v_d = \frac{\text{volume of water per second}}{\text{area of delivery pipe}}$$

If pump is single-acting,

$$v_d = \frac{2rAn}{60a_d}$$

where n is the number of revolutions per minute.

If pump is double-acting,

$$v_d = \frac{4rAn}{60a_d}$$

Total pressure head at beginning of delivery stroke

$$= H_d + \frac{v_d^2}{2g} + \frac{4fl_dv_d^2}{d_d2g} \quad \frac{l_vA}{ga_d}\,\omega^2 r$$

Total pressure head at end of stroke

$$= H_d + \frac{v_d^2}{2g} + \frac{4fl_dv_d^2}{d_d2g} \quad \frac{l_vA}{ga_d}\,\omega^2 r$$

Total pressure head at middle of stroke,

$$= H_d + \frac{v_d^2}{2g} + \frac{4fl_dv_d^2}{d_d2g} + \frac{4fl_v}{d_d2g}\left(\frac{A}{a_d}\,\omega r\right)^2$$

The last term in each of these equations is small and may usually be neglected.

The same reasoning applies if an air vessel is fitted on the suction pipe, the water accelerated being reduced to the amount between the air vessel and cylinder. The above formula will hold for the suction pipe if the suffix s is substituted for the suffix d. In this case the pressure head will be below atmosphere.

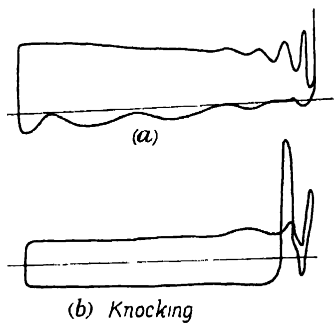


FIG. 296

Total work done per second

$$= W\left(H_s + H_d + \frac{4fl_dv_d^2}{d_d2g} + \frac{4fl_sv_s^2}{d_s2g}\right)$$

The total pressure head in the air vessel reckoned above the centre of the cylinder will be approximately equal to the total pressure head in the delivery pipe above the same datum.

Let h_v = pressure of air in air vessel in feet of water,

and H_v = height of water level in air vessel above centre of cylinder.

Then $$h_v \quad H_v = H_d + \frac{4fl_dv_d^2}{d_d2g} + 34$$

if all small quantities are neglected.

Actual indicator diagrams taken from a Tangye pump are shown in Fig. 296. In this pump an air vessel was fitted on the delivery pipe only. Fig. 296 (a) shows the indicator diagram taken when the pump was running normally; a diagram taken when the pump was knocking is shown in Fig. 296 (b), the sudden pressure rise caused by the knock is noticeable.

EXAMPLE 5

A reciprocating pump draws water from a sump through a suction pipe 6 in. in diameter and 40 ft long, the water level being 10 ft below the level of the cylinder. The cylinder diameter is 9 in., stroke 15 in., and the length of the connecting rod 5 ft. The driving crank rotates at 20 r.p.m. Determine the pressure in the cylinder at the beginning of the stroke (*a*) when no air vessel is fitted; (*b*) when an air vessel is fitted at the cylinder level and distance 5 ft from it. (*Lond. Univ.*)

(*a*) *No Air Vessel*

(1) Assuming simple harmonic motion,

$$H_a = \frac{Al_s}{a_s g}\,\omega^2 r$$

$$= \left(\frac{9}{6}\right)^2 \times \frac{40}{32{\cdot}2}\left(2\pi \times \frac{20}{60}\right)^2 \times \frac{7{\cdot}5}{12}$$

$$= 7{\cdot}67 \text{ ft of water}$$

Total pressure head in cylinder

$$= H_s + H_a$$

$$= 10 + 7{\cdot}67 = 17{\cdot}67 \text{ ft of water below atm}$$

(2) If harmonic motion is not assumed,

$$H_a = \frac{Al_s}{a_s g}\,\omega^2 r\left(1 + \frac{r}{L}\right)$$

$$= 7{\cdot}67 \times 1{\cdot}125 = 8{\cdot}67 \text{ ft of water}$$

Total pressure head in cylinder

$$= 10 + 8{\cdot}67 = 18{\cdot}67 \text{ ft of water below atm.}$$

(*b*) *When Air Vessel is Fitted*

Assume pump is single-acting.

Then $$v_s = \frac{A}{a_s} \times 2r\,\frac{n}{60}$$

$$= \left(\frac{9}{6}\right)^2 \times 2 \times \frac{7{\cdot}5}{12} \times \frac{20}{60} = 0{\cdot}937 \text{ ft/sec}$$

$$h_f = \frac{4fl_s v_s^2}{d_s 2g}$$

$$\frac{4 \times 0{\cdot}01 \times 35 \times 0{\cdot}937^2}{0{\cdot}5 \times 64{\cdot}4} = 0{\cdot}0382 \text{ ft of water}$$

(1) Assuming simple harmonic motion,

$$H_a = \frac{Al_v}{a_s g}\,\omega^2 r$$

$$= \left(\frac{9}{6}\right)^2 \times \frac{5}{32 \cdot 2} \times \left(2\pi \times \frac{20}{60}\right)^2 \times \frac{7 \cdot 5}{12}$$

$$= 0 \cdot 959 \text{ ft of water}$$

Total pressure head in cylinder

$$= H_s + H_a + h_f$$
$$= 10 + 0 \cdot 959 + 0 \cdot 0382$$
$$= 10 \cdot 9972 \text{ ft of water below atm.}$$

(2) If simple harmonic motion is not assumed,

$$H_a = \frac{Al_v}{a_s g}\, \omega^2 r \left(1 + \frac{r}{L}\right)$$
$$= 0 \cdot 959 \times 1 \cdot 125 = 1 \cdot 078 \text{ ft of water}$$

Total pressure head in cylinder

$$= 10 + 1 \cdot 078 + 0 \cdot 0382$$
$$= 11 \cdot 1162 \text{ ft of water below atm.}$$

21.10. Work Saved by Fitting Air Vessel. The following applies to either suction or delivery strokes. Consider in the first case the pump to be single-acting. If there is no air vessel on the pipe, the diagram representing the work lost in friction during the revolution is a parabola (§ 21.7), the area of which equals

$$W \times \frac{2}{3} \times \frac{4fl}{d2g}\left(\frac{A}{a}\,\omega r\right)^2 \quad . \quad . \quad . \quad (7)$$

where W = weight of water pumped per revolution.

Suppose an air vessel is now fitted just outside the cylinder. The velocity of flow in the pipe is now constant; the frictional loss will, therefore, also be constant and acts over both strokes. The diagram showing the work done against friction during the revolution will now be a rectangle of area

$$W \times \frac{4flv^2}{d2g}$$

where v = mean velocity of flow in pipe.

But $$v = \frac{A}{a}\,\frac{2r\omega}{2\pi} = \frac{A}{a}\,\frac{\omega r}{\pi},$$

Therefore, work done against friction

$$= W \times \frac{4fl}{d2g}\left(\frac{A}{a}\,\frac{\omega r}{\pi}\right)^2 \quad . \quad . \quad . \quad (8)$$

Subtracting eq. (8) from eq. (7), work saved by fitting air vessel

$$= W \times \frac{4fl}{d2g} \times \left(\frac{A}{a}\,\omega r\right)^2\left(\frac{2}{3} - \frac{1}{\pi^2}\right)$$

V_{f_1} = velocity of flow at outlet,
r = radius of wheel at inlet,
r_1 = radius of wheel at outlet,
α = angle entering water makes with wheel's tangent,
β = angle leaving water makes with wheel's tangent,
θ = angle of blade tip at inlet,
ϕ = angle of blade tip at outlet,
W = weight of water entering wheel in pounds per second,
H = total head of water supplied,
e = hydraulic efficiency of turbine,
n = number of revolutions per minute,
N = number of blades in wheel,
t = thickness of blades,
b = breadth of blades at inlet,
b_1 = breadth of blades at outlet.

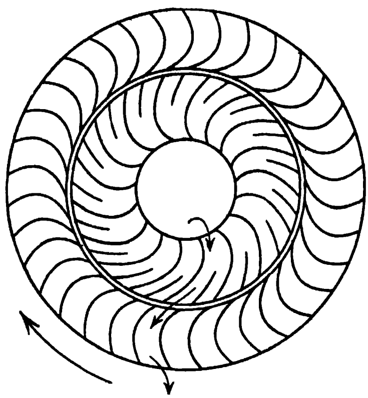


FIG. 297

22.3. Reaction Turbines. 1. OUTWARD-FLOW TURBINE. The outward radial-flow turbine consists of a wheel in the shape of a cylindrical disc mounted on a shaft and having blades around the perimeter (Fig. 297). The water flows into the wheel at the centre and passes through fixed radial guide blades into the moving blades. The object of the fixed guide blades is to guide the water into the moving blades at the correct angle α. The water passes through the moving blades, causing them to rotate, and is discharged at the outer edge. The wheel is surrounded by a water-tight casing and may run in a vertical or horizontal position. It may be submerged below the tail race or placed in a suction or draught tube (Fig. 321) above the foot of the fall. The latter position is the more convenient, as the wheel is then more accessible. Being a reaction turbine, the water in the wheel is under pressure; the wheel must, therefore, run full.

The flow of water through the wheel may be regulated by a cylindrical sluice gate situated between the moving blades and the guide blades. This is very unsatisfactory owing to the loss of head due to contraction when the gate is partly closed.

The revolving wheel causes a centrifugal head to be impressed on the water passing through it. This increases the relative velocity of the water in the outward-flow type and consequently tends to increase the quantity of water passing through the wheel. If there

is a slight increase in speed, the centrifugal head is increased and the wheel tends to race.

The efficiency is increased by discharging the water radially, in which case the velocity of whirl at outlet is zero.

The diagrams of velocity for inlet and outlet are shown in Fig. 298.

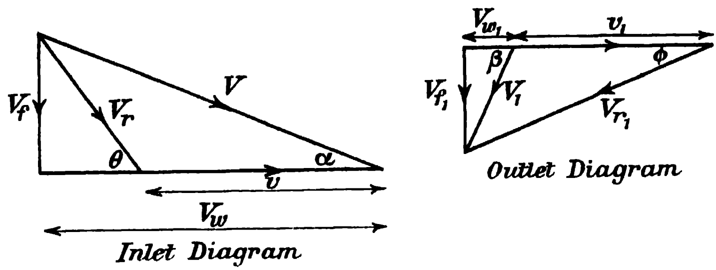


FIG. 298

It should be noted that—

$$V_w = V \cos \alpha$$
$$V_f = V \sin \alpha$$
$$V_r \sin \theta = V \sin \alpha$$
$$V_r \cos \theta = V_w - v$$
$$v = \omega r = \frac{2\pi n r}{60}$$

Also
$$\frac{v}{v_1} = \frac{r}{r_1}$$

From velocity diagram at outlet,

$$V_{r_1} \cos \phi = v_1 + V_{w_1}$$
$$V_{w_1} = V_1 \cos \beta$$
$$V_{f_1} = V_1 \sin \beta$$
$$V_1 \sin \beta = V_{r_1} \sin \phi$$

If discharge is radial $\beta = 90^\circ$; then $V_{w_1} = 0$ and $V_1 = V_{f_1}$.

From eq. (4), Chapter 6—

work done on wheel per pound of water

$$= \frac{V_w v}{g} - \frac{V_{w_1} v_1}{g}$$

energy lost per pound of water passing through wheel

$$= H - \frac{V_1^2}{2g}$$

Therefore
$$\frac{V_w v}{g} - \frac{V_{w_1} v_1}{g} = H - \frac{V_1^2}{2g} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It should be noted that in a reaction turbine H does not equal $V^2/2g$.

$$\text{Hydraulic efficiency} = \frac{H - \dfrac{V_1^2}{2g}}{H}$$

$$= \frac{V_w v - V_{w_1} v_1}{gH}$$

If the discharge is radial, eq. (1) becomes

$$\frac{V_w v}{g} = H - \frac{V_1^2}{2g}$$

Radial area of flow at inlet $= (2\pi r - Nt)b = k2\pi rb$

where k is a factor which allows for area of blades.
Volume of water flowing through wheel per second

$$= (2\pi r - Nt)bV_f$$

Radial area of flow at outlet $= (2\pi r_1 - Nt)b_1 = k_1 2\pi r_1 b_1$

k_1 being the blade factor at outlet.

As quantity of water flowing through wheel at inlet equals quantity flowing at outlet,

$$\frac{V_f}{V_{f_1}} = \frac{(2\pi r_1 - Nt)b_1}{(2\pi r - Nt)b} = \frac{k_1 2\pi r_1 b_1}{k 2\pi rb}$$

2. Inward-flow Turbine. The inward radial-flow reaction turbine is similar in principle to the outward-flow, except that the water enters the wheel at the outer periphery and flows radially towards the centre; it then leaves the wheel in a direction parallel to the axis. The fixed guide blades surround the revolving blades externally, and the whole is surrounded by an outer casing. The centrifugal head impressed on the water by the revolving wheel is now acting against the radial flow of the water, so that any increase in speed of the wheel will tend to reduce the quantity of flow through the wheel, and consequently reduce the power. This is an advantage, as it tends to prevent racing. The wheel may be placed below the level of the tail race or in a suction tube above the foot of the fall. The highest efficiency is obtained when the discharge is radial and when the velocity of the leaving water is as small as possible.

The method of solution and the equations for an inward-flow turbine are the same as given for the outward-flow turbine.

3. Axial-flow Turbine. In this type of reaction turbine the water enters the wheel at the side and flows parallel to the axis (Fig. 299). It is sometimes known as a parallel-flow turbine. The diagrams of velocity and equations for this type of turbine are the

same as for the radial-flow types, except that the radius of flow is now constant. Therefore, .

$$v = v_1 \text{ and } V_f = V_{f_1}$$

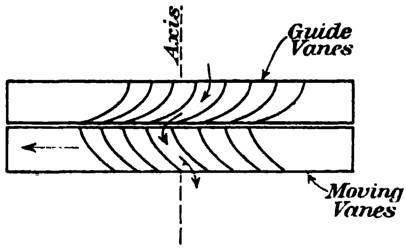


FIG. 299

Then

work done per pound of water

$$= \frac{V_w v}{g} - \frac{V_{w_1} v_1}{g}$$

$$= \frac{v(V_w - V_{w_1})}{g}$$

It is usual for the water to leave in a direction parallel to the axis.

EXAMPLE 1

Determine the hydraulic efficiency of a low-head inward-flow reaction turbine in which the guide blades make angles of 25° with the tangents to the blade circle and the receiving tips of the runners are inclined 105° to the tangents. The discharge is radial, the velocity of flow constant, and the water passes on to the moving blades without shock.

Calculate the velocity of flow if the supply head is 15 ft. (*Lond. Univ.*)

Referring to Fig. 298,

$$\alpha = 25°, \theta = 105°, \beta = 90°$$

$$V_f = V \sin 25° = 0{\cdot}4226 V$$

$$V_w = V \cos 25° = 0{\cdot}9063 V$$

$$v = 0{\cdot}9063 V - \frac{0{\cdot}4226 V}{\tan 105°} = 1{\cdot}0195 V$$

As $V_{w_1} = 0$,

$$\text{work done} = \frac{V_w v}{g} = \frac{0{\cdot}9063 V \times 1{\cdot}0195 V}{g}$$

$$= \frac{0{\cdot}924 V^2}{g}$$

$$\text{Energy rejected} = \frac{V_1^2}{2g} = \frac{V_f^2}{2g} = \frac{(0{\cdot}4226V)^2}{2g}$$

$$= \frac{0{\cdot}0895\,V^2}{g}$$

$$\text{Efficiency} = \frac{\dfrac{0{\cdot}924\,V^2}{g}}{\dfrac{0{\cdot}924\,V^2}{g} + \dfrac{0{\cdot}0895\,V^2}{g}}$$

$$= \frac{0{\cdot}924}{0{\cdot}924 + 0{\cdot}0895}$$

$$= \frac{0{\cdot}924}{1{\cdot}0135} = 91{\cdot}15 \text{ per cent}$$

$$H = \frac{V_w v}{g} + \frac{V_f^2}{2g}$$

or

$$15 = \frac{1{\cdot}0135\,V^2}{}$$

Then

$$= \sqrt{\frac{15 \times 32{\cdot}2}{1{\cdot}0135}}$$

$$= 21{\cdot}8 \text{ ft/sec}$$

and

$$V_f = 0{\cdot}4226\,V$$

$$= 0{\cdot}4226 \times 21{\cdot}8$$

$$= 9{\cdot}24 \text{ ft/sec}$$

Example 2

An inward-flow turbine works under a total head of 90 ft. The velocity of the wheel periphery at inlet is 50 ft/sec. The outlet pipe of the turbine is 1 ft in diameter, and the turbine is supplied with 50 gal of water per second. The radial velocity of flow through the wheel is the same as the velocity in the outlet pipe.

Neglecting friction, determine (*a*) the vane angle at inlet; (*b*) the guide blade angle; (*c*) the horse-power of the turbine. (*Lond. Univ.*)

As $V_1 = V_{f_1}$ the discharge is radial. Then

$$V_{w_1} = 0$$

Assume the turbine to be a reaction turbine.

$$V_f = V_1 = \frac{\text{discharge}}{\text{pipe area}} = \frac{50}{6{\cdot}24 \times 0{\cdot}785} = 10{\cdot}21 \text{ ft/sec}$$

$$\frac{V_w v}{g} = H - \frac{V_f^2}{2g}$$

Then $$\frac{V_w 50}{32 \cdot 2} = 90 - \frac{(10 \cdot 21)^2}{64 \cdot 4}$$

and $$V_w = 56 \cdot 9 \text{ ft/sec}$$

The diagram of velocity at inlet may now be drawn to scale and the values of θ and α measured. Or, they may be calculated from Fig. 298 as follows—

$$\tan \alpha = \frac{V_f}{V_w} = \frac{10 \cdot 21}{56 \cdot 9} = 0 \cdot 1795$$

Then $$\alpha = 10 \cdot 2^\circ$$

$$\tan \theta = \frac{V_f}{V_w - v} = \frac{10 \cdot 21}{6 \cdot 9} = 1 \cdot 47$$

Then $$\theta = 55 \cdot 8^\circ$$

Work done per pound of water

$$= \frac{V_w v}{g}$$

$$= \frac{56 \cdot 9 \times 50}{32 \cdot 2} = 88 \cdot 38 \text{ ft-Lb}$$

$$\text{h.p.} = \frac{W \times \text{work done}}{550}$$

$$= \frac{50 \times 10 \times 88 \cdot 38}{550} = 80 \cdot 2$$

EXAMPLE 3

An inward-flow reaction turbine is supplied with 21 ft³ of water per second under a head of 50 ft. It develops 100 h.p. at 375 r.p.m.; the inner and outer diameters of the wheel are 20 in. and 30 in. respectively. The velocity of the water at exit is 10 ft/sec, and it leaves the wheel radially. Determine the actual and theoretical hydraulic efficiencies of the wheel.

If the actual hydraulic efficiency of the wheel were 84 per cent, find the most suitable angles for the guide and wheel vanes at inlet. Assume the width of wheel constant. (*Lond. Univ.*)

$$v = \frac{30}{12} \times \pi \times \frac{375}{60} = 49 \cdot 1 \text{ ft/sec}$$

$$v_1 = 49 \cdot 1 \times \frac{20}{30} = 32 \cdot 7 \text{ ft/sec}$$

As width of wheel is constant,

$$\frac{V_f}{V_{f_1}} = \frac{r_1}{r}$$

Then $$V_f = 10 \times \frac{20}{30} = 6{\cdot}67 \text{ ft/sec}$$

As discharge is radial,

$$V_1 = V_{f_1}$$

Theoretical work done per pound of water

$$= H - \frac{V_1^2}{2g}$$

$$= 50 - \frac{(10)^2}{2g} = 48{\cdot}45 \text{ ft-Lb}$$

Theoretical hydraulic efficiency

$$= \frac{48{\cdot}45}{50} = 96{\cdot}9 \text{ per cent}$$

Actual work done per pound of water

$$= \frac{\text{h.p.} \times 550}{W}$$

$$= \frac{100 \times 550}{21 \times 62{\cdot}4} = 42 \text{ ft-Lb}$$

$$\text{Actual efficiency} = \frac{42}{50} = 84 \text{ per cent}$$

As discharge is radial,
theoretical work done per pound

$$= \frac{V_w v}{g} = 48{\cdot}45$$

Then $$V_w = \frac{48{\cdot}45 \times 32{\cdot}2}{49{\cdot}1}$$

$$= 31{\cdot}8 \text{ ft/sec}$$

The values of α and θ may be found by drawing the velocity diagram at inlet to scale; or,

$$\tan \alpha = \frac{V_f}{V_w} = \frac{6{\cdot}67}{31{\cdot}8} = 0{\cdot}2097$$

Hence $$\alpha = 11{\cdot}9^\circ$$

$$\tan (180 - \theta) = \frac{V_f}{v - V_w}$$

$$= \frac{6{\cdot}67}{(49{\cdot}1 - 31{\cdot}8)} = 0{\cdot}385$$

Hence $$\theta = 158{\cdot}9^\circ$$

EXAMPLE 4

An outward-flow reaction turbine has a speed of 200 r.p.m., and a constant breadth of 9 in. The diameters of the wheel at inlet and outlet are 5 ft and 6 ft respectively. The wheel works under a total head of 120 ft, and the quantity of water passing through the wheel is 200 ft³/sec. If the hydraulic efficiency is 90 per cent, find the angles of the blades and guide vanes.

Work done per pound of water

$$= H - \frac{V_1^2}{2g} = 0{\cdot}9H$$

from which
$$V_1 = \sqrt{0{\cdot}1 \times 64{\cdot}4 \times 120}$$
$$= 27{\cdot}8 \text{ ft/sec}$$
$$v = \pi d \frac{n}{60}$$
$$= \pi \times 5 \times \frac{200}{60} = 52{\cdot}4 \text{ ft/sec}$$
$$v_1 = 52{\cdot}4 \times \frac{6}{5} = 62{\cdot}8 \text{ ft/sec}$$
$$V_f = \frac{\text{quantity per second}}{\text{radial area of flow}} = \frac{200}{\pi d \times \frac{9}{12}}$$
$$= \frac{200}{\pi \times 5 \times 0{\cdot}75} = 17 \text{ ft/sec}$$
$$V_{f_1} = 17 \times \frac{5}{6} = 14{\cdot}2 \text{ ft/sec}$$

Referring to outlet diagram of Fig. 298,

$$\sin\beta = \frac{V_{f_1}}{V_1} = \frac{14{\cdot}2}{27{\cdot}8} = 0{\cdot}511$$

Hence
$$\beta = 30{\cdot}5^\circ$$
$$V_{w_1} = V_1 \cos\beta = 27{\cdot}8 \cos 30{\cdot}5^\circ$$
$$= 23{\cdot}9 \text{ ft/sec}$$
$$\tan\phi = \frac{V_{f_1}}{v_1 + V_{w_1}} = \frac{14{\cdot}2}{62{\cdot}8 + 23{\cdot}9} = 0{\cdot}164$$

Then
$$\phi = 9\tfrac{1}{2}^\circ$$
$$\frac{V_w v}{g} - \frac{V_{w_1} v_1}{g} = eH$$

Therefore, as V_{w_1} is negative.

$$\frac{V_w 52{\cdot}4}{32{\cdot}2} + \frac{(23{\cdot}9 \times 62{\cdot}8)}{32{\cdot}2} = 0{\cdot}9 \times 120$$

from which $V_w = 37{\cdot}4$ ft/sec

Referring to inlet diagram of Fig. 298,

$$\tan\alpha = \frac{V_f}{V_w} = \frac{17}{37{\cdot}4} = 0{\cdot}455$$

Then $\alpha = 24{\cdot}5°$

$$\tan\theta = \frac{V_f}{V_w - v} = \frac{17}{37{\cdot}4 - 52{\cdot}4} = -\,1{\cdot}132$$

Therefore $\theta = 131{\cdot}5°$

22.4. Impulse Turbines. The problems dealing with impulse turbines may be solved in a similar way to the reaction turbine problems, but the following points should be noted—

(*a*) The turbine must not run full; the pressure is atmospheric throughout.

(*b*) The total head is converted to velocity before entering the wheel. Then $V = \sqrt{(2gH)}$. This is sometimes stated as $V = k\sqrt{(2gH)}$, where k is a coefficient which takes into account losses in the nozzle or guide vanes.

(*c*) As $V = \sqrt{(2gH)}$, the hydraulic efficiency may be stated as

$$\frac{V^2 - V_1^2}{V^2}$$

(*d*) The velocities of flow V_f and V_{f_1} depend on the radial area and on the amount the wheel is full.

1. RADIAL-FLOW TURBINE. The flow may be inwards or outwards. The water enters the wheel through fixed guide blades as in the reaction turbine. As the water flows over the moving vanes, a centrifugal head is impressed on it by the revolving wheel, which is immediately converted to velocity head. This increases the relative velocity of the water in an outward flow and decreases it in an inward flow. The centrifugal head given to the water was proved in § 3.9 to be

$$\frac{v_2^2}{2g} - \frac{v_1^2}{2g}$$

Then
$$\frac{V_r^2}{2g} = \frac{V_{r_1}^2}{2g} + \left(\frac{v^2}{2g} - \frac{v_1^2}{2g}\right) \quad . \quad . \quad . \quad (2)$$

If the flow is inward, the relative velocity is thus reduced by the centrifugal force. This makes the speed of the inward-flow turbine easier to control than that of the outward flow. A small increase of

speed of the wheel, due to a temporary lightening of the load, increases the centrifugal force, which decreases the flow through the wheel and consequently decreases the power. The wheel thus tends to automatically adjust itself to the load. With the outward-flow turbine, the centrifugal force increases the flow and the wheel tends to race.

The radial-flow impulse turbine is not suitable for very low falls, as the wheel must be placed above the foot of the fall in order that

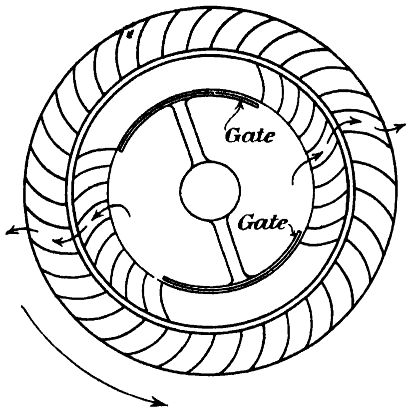


FIG. 300

it does not run full. A certain amount of the fall is thus lost. In a high fall this amount is not noticeable. The efficiency is greatest when V_1 is as small as possible.

In an impulse turbine it is possible to regulate the flow through the wheel without loss. The water need not be admitted over the whole circumference, as the pressure is atmospheric. The flow may therefore be regulated by means of a revolving gate (Fig. 300), which, when turned, completely shuts off the flow in the vanes covered by it, without interfering with the flow in the remaining vanes.

This method of regulating the flow is used in the Girard impulse turbine. In this type, the water is admitted to two opposite quadrants of the wheel when the revolving gate is fully open, the remaining two quadrants being covered by the gate.

2. AXIAL-FLOW TURBINE. The same conditions governing the radial-flow impulse turbine apply to the axial-flow impulse turbine. Except that in this type there is no centrifugal head impressed on the water as $v = v_1$; therefore the relative velocity is constant. Or,

$$V_r = V_{r_1}$$

The maximum efficiency occurs when V_1 is as small as possible.

The chief types of axial-flow impulse turbine are the Girard and the Pelton wheel. The latter type differs from the ordinary turbine and is dealt with separately in § 22.6.

EXAMPLE 5

The mean blade circle diameter of the runner of an axial-flow impulse turbine of the Girard type is 4½ ft. The guide blade angle is 24°, the receiving and discharging angles of the runner blades being 48° and 23° respectively. The breadth of the moving blades at inlet is 4 in.

Calculate the speed of the turbine so that the water may pass smoothly on to the blades when the turbine is working under a head of 280 ft, and find the horse-power developed if, with full circumferential admission, the passages are 85 per cent full at inlet. (*Lond. Univ.*)

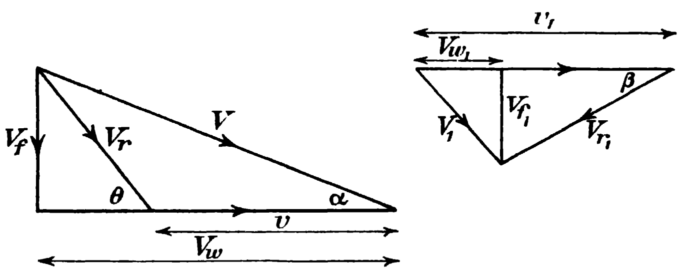


FIG. 301

The velocity diagrams are shown in Fig. 301.

$$V = \sqrt{2gH} = \sqrt{64{\cdot}4 \times 280} = 134 \text{ ft/sec}$$

$$V_f = V \sin 24° = 134 \times 0{\cdot}4067 = 54{\cdot}5 \text{ ft/sec}$$

$$V_w = V \cos 24° = 134 \times 0{\cdot}9135 = 122{\cdot}3 \text{ ft/sec}$$

$$v = V_u \quad \frac{V_f}{\tan 48°} = 122{\cdot}3 - \frac{54{\cdot}5}{1{\cdot}1106} = 73{\cdot}2 \text{ ft/sec}$$

$v_1 = v$, as the turbine is an axial flow

$$v = \pi d \frac{n}{60}$$

that is, $\quad 73{\cdot}2 = \pi \times 4{\cdot}5 \times \frac{}{60}$

from which $\quad n = 311$ r.p.m.

$$V_r = \frac{V_f}{\sin 48°} = \frac{54{\cdot}5}{0{\cdot}7431} = 73{\cdot}2 \text{ ft/sec}$$

$$V_{r_1} = V_r \quad 73{\cdot}2 \text{ ft/s}$$

as the turbine is an axial-flow impulse.

$$V_{w_1} = v_1 - V_{r_1} \cos 23^\circ$$
$$= 73{\cdot}2 - (73{\cdot}2 \times 0{\cdot}9205) = 5{\cdot}8 \text{ ft/sec}$$

Work done per pound

$$= \frac{V_w v}{g} - \frac{V_{w_1} v_1}{g}$$
$$= \frac{(122{\cdot}3 \times 73{\cdot}2)}{32{\cdot}2} - \frac{(5{\cdot}8 \times 73{\cdot}2)}{32{\cdot}2}$$
$$= 265 \text{ ft-Lb}$$

Quantity of water per second

$$= b \times \pi d \times V_f \times 0{\cdot}85$$
$$= \frac{4}{12} \times \pi \times 4{\cdot}5 \times 54{\cdot}5 \times 0{\cdot}85$$
$$= 218 \text{ ft}^3\text{/sec}$$

$$\text{Horse-power} = \frac{W \times \text{work done per pound}}{550}$$
$$= \frac{218 \times 62{\cdot}4 \times 265}{550}$$
$$= 6{,}550$$

EXAMPLE 6

In an outward-flow impulse turbine the available head is 81 ft. The rim speed of the wheel at inlet is $0{\cdot}4\sqrt{2gH}$, guide-vane angle and wheel-vane angle at outlet are 20°, inlet radius 0·75 ft, outlet radius 1 ft. The velocity of the water in the guides is 95 per cent of the theoretical velocity due to total head. The losses in the wheel to be taken as 6 per cent of total head. Find the hydraulic efficiency.

If the depth of the guides is 0·25 ft, what would be the horse-power of the turbine if used with admission over one-quarter of the circumference, allowing 10 per cent loss of area due to vanes? (*Lond. Univ.*)

Referring to velocity diagrams of Fig. 298,

$$v = 0{\cdot}4\sqrt{2gH} = 0{\cdot}4\sqrt{64{\cdot}4 \times 81} = 28{\cdot}95 \text{ ft/sec}$$
$$v_1 = \frac{vr_1}{r} = 28{\cdot}95 \times \frac{1}{0{\cdot}75} = 38{\cdot}6 \text{ ft/sec}$$
$$V = 0{\cdot}95\sqrt{2gH} = 0{\cdot}95\sqrt{64{\cdot}4 \times 81} = 68{\cdot}6 \text{ ft/sec}$$
$$V_w = V \cos 20^\circ = 68{\cdot}6 \times 0{\cdot}9397 = 64{\cdot}5 \text{ ft/sec}$$
$$V_f = V \sin 20^\circ = 68{\cdot}6 \times 0{\cdot}342 = 23{\cdot}45 \text{ ft/sec}$$
$$\tan \theta = \frac{V_f}{V_w - v} = \frac{23{\cdot}45}{64{\cdot}5 - 28{\cdot}95} = 0{\cdot}66$$

Then, $\theta = 33{\cdot}4°$

$$V_r = \frac{V_f}{\sin\theta} = \frac{23{\cdot}45}{0{\cdot}5505} = 42{\cdot}6 \text{ ft/sec}$$

Using eq. (2), and allowing for 6 per cent of total head loss in vanes,

$$\frac{V_{r_1}^{\ 2}}{2g} = \frac{V_r^2}{2g} - \left(\frac{v^2}{2g} - \frac{v_1^2}{2g}\right) - 0{\cdot}06H$$

that is $\quad V_{r_1}^{\ 2} = 42{\cdot}6^2 - (28{\cdot}95^2 - 38{\cdot}6^2) - (0{\cdot}06 \times 2g \times 81)$

from which $\quad V_{r_1} = 46{\cdot}5$ ft/sec

$$\begin{aligned} V_{w_1} &= V_{r_1}\cos 20° - v_1 \\ &= (46{\cdot}5 \times 0{\cdot}9397) - 38{\cdot}6 \\ &= 5 \text{ ft/sec} \end{aligned}$$

Work done per pound

$$\begin{aligned} &= \frac{V_w v}{g} - \frac{V_{w_1} v_1}{g} \\ &= \frac{(64{\cdot}5 \times 28{\cdot}95)}{32{\cdot}2} + \frac{(5 \times 38{\cdot}6)}{32{\cdot}2} \text{ (as } V_{w_1} \text{ is negative)} \\ &= 64 \text{ ft-Lb} \end{aligned}$$

$$\begin{aligned} e &= \frac{\text{work done per pound}}{\dfrac{V^2}{2g}} = \frac{64 \times 64{\cdot}4}{(68{\cdot}6)^2} \\ &= 87{\cdot}5 \text{ per cent} \end{aligned}$$

Radial area of flow at inlet

$$\begin{aligned} &= \pi db \times \frac{1}{4} \times \frac{90}{100} \\ &= \pi \times 1{\cdot}5 \times 0{\cdot}25 \times \frac{1}{4} \times \frac{90}{100} = 0{\cdot}265 \text{ ft}^2 \end{aligned}$$

Quantity of water per second

$$\begin{aligned} &= 0{\cdot}265\,V_f \\ &= 0{\cdot}265 \times 23{\cdot}45 = 6{\cdot}22 \text{ ft}^3 \end{aligned}$$

$$\begin{aligned} \text{Horse-power} &= \frac{W \times \text{work done per pound}}{550} \\ &= \frac{6{\cdot}22 \times 62{\cdot}4 \times 64}{550} \\ &= 45{\cdot}1 \end{aligned}$$

22.5. Summary of Equations for Turbine Problems. The following is a tabulated summary of equations and conditions governing all classes of turbines, which will be useful for reference when solving problems on turbines—

	Impulse	Reaction
Radial and axial flow	$V = \sqrt{2gH}$ Work done per pound $= \frac{V_w v}{g} - \frac{V_{w_1} v_1}{g}$ $= \frac{V^2}{2g} - \frac{V_1^2}{2g}$ Hydraulic eff. $= \frac{V^2 - V_1^2}{V^2}$ Wheel must not run full. V_f depends on area of flow and on amount full. Pressure is atmospheric.	Work done per pound $= \frac{V_w v}{g} - \frac{V_{w_1} v_1}{g}$ $= H - \frac{V_1^2}{2g}$ Hydraulic eff. $= \frac{H - \frac{V_1^2}{2g}}{H}$ Wheel must run full. V_f depends on area of flow. Pressure varies throughout.
Radial flow only	$\frac{v_1}{v} = \frac{r_1}{r}$ $\frac{V_{r1}^2}{2g} = \frac{V_r^2}{2g} - \left(\frac{v^2}{2g} - \frac{v_1^2}{2g}\right)$	$\frac{v_1}{v} = \frac{r_1}{r}$
Axial flow only	$v = v_1$ $V_{r_1} = V_r$	$v = v_1$

22.6. Early Types of Turbine. The following account of the earlier types of water turbine shows how they have developed from the original water wheels. As stated at the end of § 22.1, the Francis turbine, the Pelton wheel and the Kaplan turbine have mainly superseded all the earlier types.

1. BARKER'S MILL OR SCOTCH TURBINE. This is a simple type of reaction turbine which is now obsolete.* It consists of a revolving cylindrical tank having arms through which the water is discharged backwards, as shown in Fig. 302. Problems on this type of turbine may be solved from the ordinary methods applied to reaction turbines. The velocity diagrams for inlet and outlet may be drawn in the same manner as in Fig. 298. It should be noted that the arm corresponds to the moving vane. As the water enters the vane radially and at the centre,

$$\alpha \text{ and } \theta = 90^\circ$$

$$v = 0$$

$$V_f = V_r = V$$

* It has been revived as a lawn sprinkler.

The diagram at inlet thus becomes a radial line as shown in Fig. 303.

Consider the velocity diagram at outlet. As the water leaves tangentially,

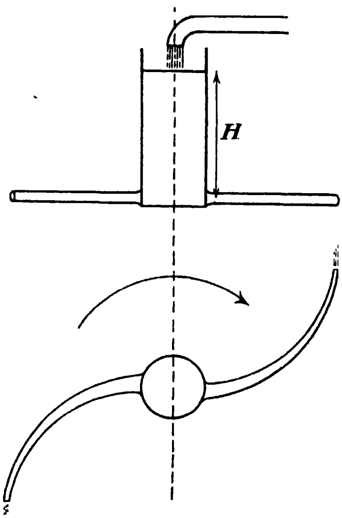


FIG. 302

$$\beta \text{ and } \phi = 0$$
$$V_{w_1} = V_1$$
$$V_{f_1} = 0$$

The velocity diagram at outlet is thus a horizontal line (Fig. 303), from which

$$V_{r_1} = v_1 + V_1. \qquad (3)$$

Work done per pound of water

$$= \frac{V_w v}{g} + \frac{V_{w_1} v_1}{g}$$
$$= 0 + \frac{(V_{r_1} - v_1)v_1}{g} \quad \text{(from eq. (3))}$$
$$= H - \frac{V_1^2}{2g}$$

where H = head of water in tank.

Total energy supplied per pound of water

$$= H = \frac{(V_{r_1} - v_1)v_1}{g} + \frac{V_1^2}{2g}$$

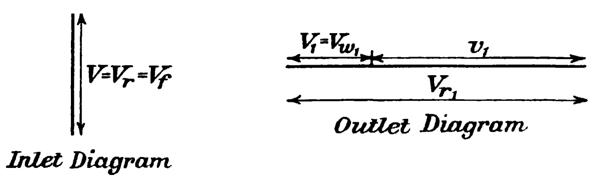


FIG. 303

Substituting for V_1 from eq. (3),

$$H = \frac{(V_{r_1} - v_1)v_1}{g} + \frac{(V_{r_1} - v_1)^2}{2g}$$
$$= \frac{V_{r_1}^2 - v_1^2}{2g}$$

$$\text{Efficiency} = \frac{2(V_{r_1} - v_1)v_1}{V_{r_1}^2 - v_1^2} = \frac{2v_1}{V_{r_1} + v_1}$$

It will be noticed that the vanes of the Barker's mill are an example of jet propulsion (§ 6.6).

2. Fourneyron Turbine. This is an outward radial-flow reaction type and was the first successful reaction turbine to be made. It has been used for heads of 1 ft to 360 ft and has an efficiency of about 75 per cent. It is governed by a cylindrical sluice gate which fits between the moving and fixed blade rings. As throttling the supply in this way causes a loss of head due to sudden contraction, transverse diaphragms are fitted through the blades which divide the wheel into four sections. The cylindrical gate may then close one or more sections completely without causing any loss of head.

3. Francis Turbine. The Francis turbine is an inward-flow radial reaction type and was the first type of inward flow to be constructed. This has the advantage of the centrifugal force acting against the flow, which reduces the tendency to race. In modern Francis turbines the flow is regulated by swivel guide blades (Figs. 322 and 324). The Francis turbine is the type now used for most water-power schemes of medium heads. Heads up to 1,000 ft are used and overall efficiencies of 90 per cent are attained; 50,000 horse-power can be obtained from a single runner.

4. Thomson Turbine. This is an inward-flow reaction turbine. The turbine wheel is surrounded by an eccentric chamber called a vortex chamber. The water enters the wheel at the largest part of the chamber and is guided to the moving blades by four pivoted guide blades. The flow may then be regulated by closing up the guide blades.

5. Jonval Turbine. The Jonval is an axial-flow impulse turbine. The simplest type consists of one horizontal ring of moving blades into which the water is directed by guide vanes placed above. The flow is regulated by a horizontal sluice which closes parts of the wheel.

A later type of Jonval wheel consisted of several concentric rings of moving blades. The power may then be regulated by closing one or more rings completely.

The invention of the suction tube is also due to Jonval.

6. Girard Turbine. There are two types of Girard turbine, an axial-flow and a radial-flow; both are impulse wheels. They may be used for heads up to 1,700 ft and have an overall efficiency of about 75 per cent. The guide passages do not extend over the whole circumference but over two opposite quadrants. The water supply is varied by a sliding circular sluice gate (Fig. 300) which completely shuts off the flow through the vanes it covers. By turning this sluice the flow may be stopped through as many vanes as required. This prevents any loss of head due to contraction when running at "part gate." The wheel of the axial flow is usually placed vertical; the radial flow may be horizontal or vertical.

7. PELTON WHEEL. The Pelton wheel is a special type of axial-flow impulse turbine and is used for very high heads. It is the most efficient type of impulse wheel, having an overall efficiency of 88 per

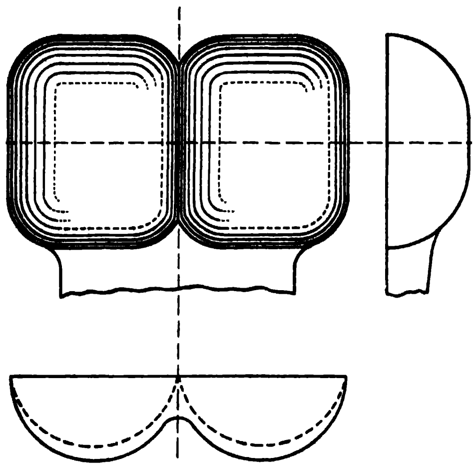


FIG. 304. PELTON WHEEL BUCKETS (EXTERNAL VIEWS)

cent. This type of wheel has been evolved from an earlier type of undershot water wheel (Fig. 79) used in the mines of California.

The jet impinges on the wheel from one or more nozzles and strikes the blade at the centre (Fig. 316), flowing axially in both

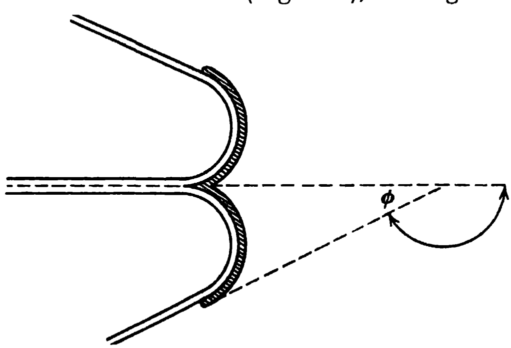


FIG. 305

directions. The blades are known as buckets and consist of a double hemispherical cup (Fig. 304). As the water flows axially in both directions, there is no axial thrust on the wheel. The flow of water through the wheel may be regulated by a throttle valve in the supply pipe or by a needle valve in the nozzle. The buckets are so shaped that the jet is discharged backwards. Usually, the total deflection of the bucket is 160° (Fig. 305). An arrangement of a

Pelton wheel, showing nozzle, made by Gilbert Gilkes and Gordon, Ltd., is shown in Fig. 315.

The work done and efficiency of the Pelton wheel may be obtained from the velocity diagrams as in the case of an ordinary axial-flow impulse turbine.

For a Pelton wheel,

$$\theta = 0$$

Also

$$\alpha = 0$$

Then, velocity diagram at inlet is a horizontal straight line, as shown in Fig. 306.

Hence

$$V_r = V - v$$

and

$$V_w = V = \sqrt{2gH}$$

$$V_f = 0$$

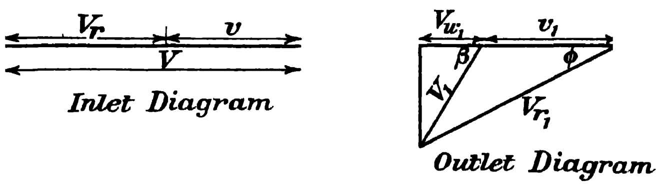


FIG. 306

From diagram at outlet (Fig. 306),

$$v_1 = v$$

$$V_{r_1} = V_r = V - v$$

$$V_{w_1} = V_{r_1} \cos\phi - v_1$$

$$= (V - v)\cos\phi - v$$

Work done per pound of water

$$= E = \frac{V_w v}{g} - \frac{V_{w_1} v_1}{g}$$

$$= \frac{Vv}{g} + \frac{[(V - v)\cos\phi - v]v}{g}$$

(as V_{w_1} is negative)

$$= \frac{1}{g}[Vv + v(V - v)\cos\phi - v^2] \qquad (4)$$

$$= eH$$

$$= H - \frac{V_1^2}{2g}$$

$$\text{Efficiency} = e = \frac{\frac{1}{g}[Vv + v(V - v)\cos\phi - v^2]}{\frac{V^2}{2g}}.$$

The speed of the wheel for maximum efficiency can be found by differentiating this equation in terms of v and equating to zero.

$$\frac{de}{dv} = V + (V - 2v)\cos\phi - 2v = 0$$

from which $$V(1 + \cos\phi) - 2v(1 + \cos\phi) = 0$$

Hence $$v = \frac{V}{2}$$

Therefore, the speed of the wheel for maximum efficiency will be equal to half the speed of the jet.

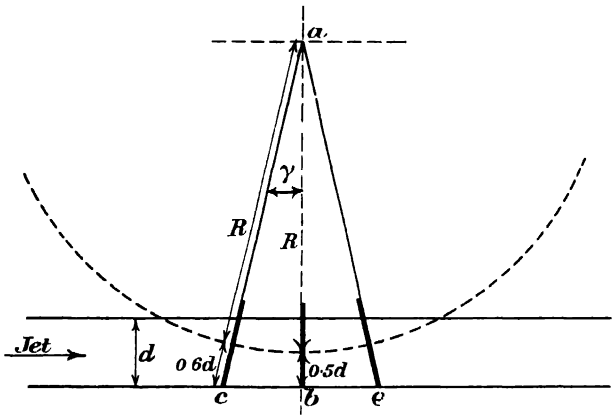


FIG. 307

In practice it is found that the maximum efficiency is when the speed of the wheel is $0{\cdot}46V$.

Putting $v = V/2$ in the above efficiency equation,

$$\text{maximum efficiency} = \frac{2\left[\frac{V^2}{2} \quad V^2 \cos\phi \quad \frac{V^2}{4}\right]}{V^2}$$

$$= \frac{1}{2}(1 \quad \cos\phi)$$

When $\phi = 0$, the efficiency is equal to unity. It will be noticed that the deviation of the jet is $180° - \phi$.

The following rules are used for the proportions of the buckets.

Let d = diameter of jet.

Depth of bucket = $1{\cdot}2d$

Width of bucket = $5d$

The number of buckets may be obtained by arranging them so that the jet is always completely intercepted by a bucket.

Let R be the mean radius of bucket circle and γ be the angle subtended by two adjacent buckets (Fig. 307). If the jet is to be

always intercepted, one bucket will be just about to move out of the jet as another has just moved in.

Let b, c and e be adjacent buckets. As jet is moving at twice the speed of the buckets, a section of jet will move from c to e in the same time as bucket b moves to e. Hence, for jet to be always intercepted, the buckets will be as shown in Fig. 307.

Consider triangle abc.

$$ac = R + \tfrac{1}{2} \text{ depth of bucket} = R + 0{\cdot}6d$$

$$ab = R + \tfrac{1}{2} \text{ diameter of jet} = R + 0{\cdot}5d$$

Then
$$\cos\gamma = \frac{R + 0{\cdot}5d}{R + 0{\cdot}6d}$$

from which equation γ is obtained. Then

$$\text{number of buckets*} = \frac{360}{\gamma}$$

Pelton wheels are in use with heads as large as 5,000 ft.

EXAMPLE 7

A cup, similar to that in a Pelton wheel, deflects a jet of water through an angle of 120°. Determine the speed of the cup in terms of the velocity of the jet so that the work done by the jet on the cup shall be a maximum and express this work as a percentage of the energy of the jet.

Show how the speed necessary for maximum efficiency would be affected if the friction of the water in passing over the surface of the cup were considerable. (*Lond. Univ.*)

Referring to Figs. 305 and 306,

$$\phi = 180° - 120° = 60°$$

Work done per pound of water

$$= E = \frac{V_w v}{g} - \frac{V_{w_1} v_1}{g}$$

$$= \frac{Vv}{g} + \frac{v(V - v)\cos 60° - v^2}{g}$$

Differentiating for a maximum,

$$\frac{dE}{dv} = V + V\cos 60° - 2v\cos 60° - 2v = 0$$

Hence
$$V(1 + \cos 60°) - 2v(1 + \cos 60°) = 0$$

Therefore
$$v = \frac{V}{2}$$

* This equation does not hold in practice as there is not sufficient space around the wheel perimeter for this number of buckets to be inserted. Actually, the number of buckets is about two-thirds of that given by the equation.

Then,

maximum work done per pound of water

$$= \frac{\frac{1}{2}V^2 + \frac{1}{4}V^2 \cos 60^\circ - \frac{1}{4}V^2}{g}$$

$$= \frac{\frac{3}{8}V^2}{g}$$

$$\text{Energy supplied} = \frac{V^2}{2g}$$

$$\text{Efficiency} = \frac{\frac{\frac{3}{8}V^2}{g}}{\frac{V^2}{2g}} = 0{\cdot}75$$

If there is no friction over the cup, the relative velocity at exit equals relative velocity at entrance. Let friction reduce relative velocity at exit to $k \times V_r$.

Then,

relative velocity at exit $= k(V - v)$

and $V_{w_1} = k(V - v) \cos 60^\circ - v$

Work done per pound of water

$$= E = \frac{Vv}{g} + \frac{[k(V - v) \cos 60^\circ - v]v}{g}$$

$$= \frac{Vv}{g} + \frac{vk(V - v) \cos 60^\circ - v^2}{g}$$

Differentiating for a maximum,

$$\frac{dE}{dv} = V + Vk \cos 60^\circ - 2vk \cos 60^\circ - 2v = 0$$

or $V(1 + k \cos 60^\circ) - 2v(1 + k \cos 60^\circ) = 0$

from which $v = \dfrac{V}{2}$

Therefore, the speed for maximum efficiency is not affected by the friction of the water passing over the cup.

Example 8

A Pelton wheel is required to work under a head of 130 ft, and to develop 100 h.p. at 250 r.p.m. Assuming an efficiency of 80 per cent and a coefficient of velocity of 0·98, find the jet diameter, the diameter of the bucket circle, the size of the buckets, and the number of buckets required. (*Lond. Univ.*)

$$V = 0{\cdot}98\sqrt{2gH}$$

$$= 0{\cdot}98\sqrt{64{\cdot}4 \times 130} = 89{\cdot}5 \text{ ft/sec}$$

For maximum efficiency,

$$v = 0{\cdot}46V \quad \text{(practical value)}$$
$$= 0{\cdot}46 \times 89{\cdot}5 = 41{\cdot}3 \text{ ft/sec}$$
$$\text{Horse-power} = \frac{0{\cdot}8WH}{550}$$

Hence
$$W = \frac{100 \times 550}{130 \times 0{\cdot}8} = 530 \text{ Lb/sec}$$

Let d = diameter of jet
and D = diameter of bucket circle.

Then,
$$v = \pi D \frac{n}{60}$$

from which
$$D = \frac{41{\cdot}3 \times 60}{\pi \times 250} = 3{\cdot}16 \text{ ft}$$

Quantity of water flowing per second

$$= \frac{\pi}{4} d^2 V = \frac{W}{w}$$

Hence
$$d^2 = \frac{530 \times 4}{\pi \times 62{\cdot}4 \times 89{\cdot}5}$$

from which
$$d = 0{\cdot}347 \text{ ft}$$
$$= 4{\cdot}17 \text{ in.}$$
$$\text{Depth of bucket} = 1{\cdot}2d$$
$$= 1{\cdot}2 \times 4{\cdot}17 = 5 \text{ in.}$$
$$\text{Width of bucket} = 5d$$
$$= 5 \times 4{\cdot}17 = 20{\cdot}8 \text{ in.}$$

For number of buckets,

$$\cos\gamma = \frac{R + 0{\cdot}5d}{R + 0{\cdot}6d}$$
$$= \frac{18{\cdot}96 + 2{\cdot}08}{18{\cdot}96 + 2{\cdot}5} = 0{\cdot}982$$

from which
$$\gamma = 11°$$

Then,
$$\text{number of buckets} = \frac{360}{11} = 33$$

22.7. Specific Speed of a Turbine. The specific speed of a water turbine is the speed at which a geometrically similar turbine would run if producing 1 h.p. under a head of 1 ft of water. This is sometimes called the *type characteristic* of the turbine. Within certain

limits each type of turbine will have its own value for the specific speed; hence, if the specific speed is known it is possible to judge the type of turbine.

An equation for the specific speed of a turbine can be obtained by applying the principle of similarity to water turbines. It will be assumed that all turbines are geometrically similar; that is, that all their linear dimensions are in proportion, and the blade angles are constant.

Let D = diameter of turbine in feet,
n_s = specific speed of turbine in revolutions per minute,
P = horse-power developed.

Then, using the notation of § 22.2,

$$v = \frac{\omega D}{2}$$

from which $$D \propto \frac{v}{\omega}$$

But $$\omega \propto n$$

Also, from inlet diagram of any turbine,

$$v \propto V$$

that is $$v \propto \sqrt{H} \qquad (\text{as } V \propto \sqrt{H})$$

Hence $$D \propto \frac{\sqrt{H}}{n} \qquad . \quad . \quad . \quad . \quad . \quad (5)$$

Assuming linear dimensions of turbines to be similar,

$$b \propto D$$

Hence, from eq. (5), $$b \propto \frac{\sqrt{H}}{n} \qquad . \quad . \quad . \quad . \quad . \quad (6)$$

From inlet diagram of any turbine,

$$V_f \propto V$$

that is, $$V_f \propto \sqrt{H} \qquad (\text{as } V \propto \sqrt{H}). \quad . \quad (7)$$

Quantity per second passing through turbine

$$= \text{radial area of flow} \times \text{vel. of flow}$$
$$= \pi D b \times V_f$$

Substituting from eqs. (5), (6) and (7),

$$\text{Quantity per second} \propto \frac{\sqrt{H}}{n} \times \frac{\sqrt{H}}{n} \times \sqrt{H}$$
$$\propto \frac{H^{3/2}}{n^2}$$

Weight of water per second $= W = w \times$ quantity per second,

or
$$W \propto \frac{H^{3/2}}{n^2} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

Now, horse-power of turbine $= \dfrac{WH}{550}$

Hence, from eq. (8),
$$P \propto \frac{H^{3/2}}{n^2} \times H$$
$$\propto \frac{H^{5/2}}{n^2}$$

Or
$$n \propto \frac{H^{5/4}}{\sqrt{P}}$$

That is,
$$n = k\,\frac{H^{5/4}}{\sqrt{P}}$$

where k is a constant depending on the type of turbine. It will be noticed that the constant k is a function of the constants of eqs. (5), (6), (7) and (8); hence k is a function of the linear proportions of the turbine.

When the turbine is developing 1 h.p. under a head of 1 ft, it will be noticed that k is equal to n which, under these conditions, is known as the specific speed n_s.

Hence,
$$k = n_s$$

then
$$n_s = \frac{n\sqrt{P}}{H^{5/4}} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

It is found that for impulse turbines n_s lies between 3 and 10, and for reaction turbines the value of n_s is between 10 and 140; for radial-flow type between 10 and 100, and for propeller type up to 140. The n_s for a Pelton wheel is between 3 and 7.

It should be noticed that, as $n_s = k$, then the specific speed is a function of the linear proportion of the turbine. Also, the specific speed is not a non-dimensional factor; it is a function of the head and is thus in feet or metre units.

Suppose it is required to install a water turbine to work at a given speed, under a given head, and to produce a given horse-power. Then, putting these quantities in eq. (9) the specific speed is obtained. If this has a value of between 10 and 100, a radial-flow reaction turbine should be used; if the value is less than 10, an impulse turbine or Pelton wheel should be used. If the value is more than 100, then two or more reaction turbines would be required.

In Fig. 308 are shown the proportion of the runners of three reaction turbines of the Francis type, each of different specific

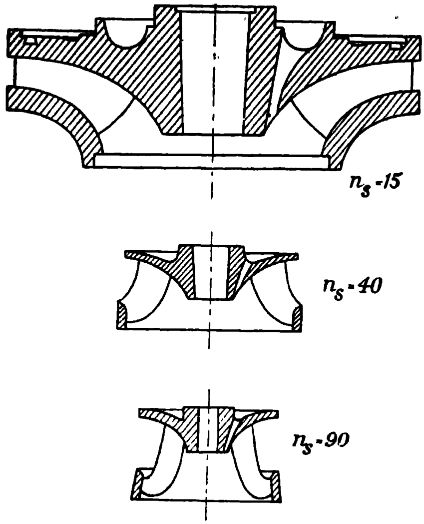


FIG. 308. TYPES OF REACTION-TURBINE RUNNERS
Showing comparative sizes of runners for same output under unit head.
(*Courtesy of Armstrong-Whitworth*)

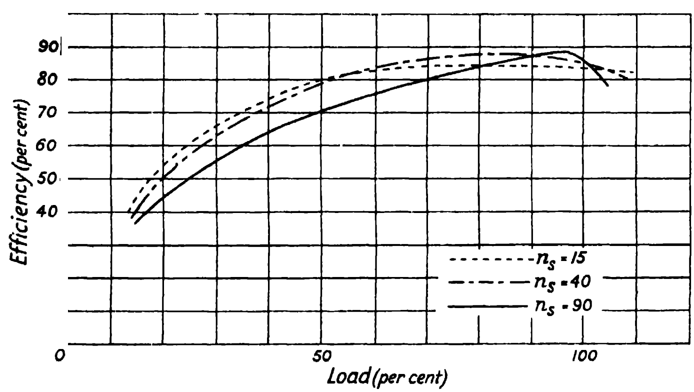


FIG. 309. EFFICIENCY CURVES FOR REACTION-TURBINE RUNNERS
(*Courtesy of Armstrong-Whitworth*)

speeds; the efficiency curves of the same turbines are shown in Fig. 309.

EXAMPLE 9

Deduce an expression for the specific speed of a reaction turbine. Under a head of 40 ft the maximum feasible specific speed is 100. If, under this head,

an installation of 20,000 h.p. is required, and if the speed is to be 150 r.p.m., how many units should be used? (*A.M.I.Mech.E.*)

Using eq. (9),

$$n_s = \frac{n\sqrt{P}}{H^{5/4}}$$

that is,

$$100 = \frac{150\sqrt{P}}{40^{5/4}}$$

from which

$$P = 4{,}500 \text{ per unit}$$

$$\text{No. of units} = \frac{20{,}000}{4{,}500} = 5$$

22.8. Characteristic Curves for Turbine. There is a large variation in the efficiency of a turbine when the gate and speed are varied; for small gate openings and low speeds the efficiency is very low. To obtain the conditions for the maximum efficiency for a turbine a diagram is plotted showing the efficiencies for all conditions of running; from this diagram the condition for maximum efficiency may be obtained; such a diagram is known as a *characteristic curve.* The points on the diagram are obtained by testing the turbine for various gate openings and at various speeds; the diagram holds for that particular turbine only.

Before plotting the diagram certain characteristics for the turbine are calculated from the results of the tests; these characteristics are known as *unit power*, *unit speed* and *unit quantity.*

UNIT POWER. The unit power of any particular turbine may be defined as the power developed under a head of 1 ft, or under unit head if any other system of dimensions be used.

Let P = horse-power developed.

Then $\quad P \propto WH$

But $\quad W = 62{\cdot}4aV$

and $\quad V \propto \sqrt{2gH}$

Hence $\quad W \propto \sqrt{H}$

Then $\quad P \propto H^{3/2}$

or $\quad P = k_1 H^{3/2}$

where k_1 is a coefficient which will vary with the efficiency of the turbine; that is, with the gate opening and speed.

When $H = 1$ ft,

$$P = k_1 = \text{unit power}$$

Hence, the unit power of a turbine $= k_1 = \dfrac{P}{H^{3/2}}$

UNIT SPEED. The unit speed for a particular turbine is the speed when running under a head of 1 ft. For a given turbine,

$$n \propto \sqrt{H}$$

or

$$n = k_2\sqrt{H}$$

where k_2 is a coefficient which will vary with the conditions of running.

When $H = 1$ ft, $n = k_2 =$ unit speed. Hence,

$$\text{unit speed of a turbine} = k_2 = \frac{n}{\sqrt{H}}$$

UNIT QUANTITY. This is the volume of water passing through the turbine when the head is 1 ft.

$$Q = aV$$

or

$$Q \propto a\sqrt{H}$$

Hence

$$Q \propto \sqrt{H}$$

or

$$Q = k_3\sqrt{H}$$

where k_3 is a coefficient depending on the condition of running.

When $H = 1$ ft, $Q = k_3 =$ unit quantity. Therefore

$$\text{unit quantity for a turbine} = k_3 = \frac{Q}{\sqrt{H}}$$

THE CHARACTERISTIC DIAGRAM. This is a chart showing the efficiencies of a particular turbine under all conditions of running. The turbine is first tested for a particular gate opening; the speed and head are varied, and the quantity and brake horse-power are measured. From these results values of the efficiency, unit power and unit speed are calculated for the various speeds and heads at that gate opening. A curve is then plotted for this gate opening with unit power and unit speed as ordinates; the efficiency for each point obtained is written on the curve at that point (Fig. 310). These tests are repeated for various gate openings, and the efficiencies plotted as before. By examining the efficiencies written at each point, lines of equal efficiency can be drawn by interpolation; these lines correspond to the contour lines on a map.

From this chart it can be seen at a glance what the speed of the turbine should be, at any gate opening, in order to give the best efficiency for that gate opening. It also shows clearly the maximum efficiency of the turbine for all conditions, and the gate opening and speed which produce this maximum efficiency can be read off the chart; this should be the normal condition of running for the turbine.

It should be noticed that plotting unit power and unit speed, instead of the actual power and speed of the tests, has reduced all results to unit head; thus, the variation of the head is eliminated. This gives a simpler diagram because one of the variables has been eliminated.

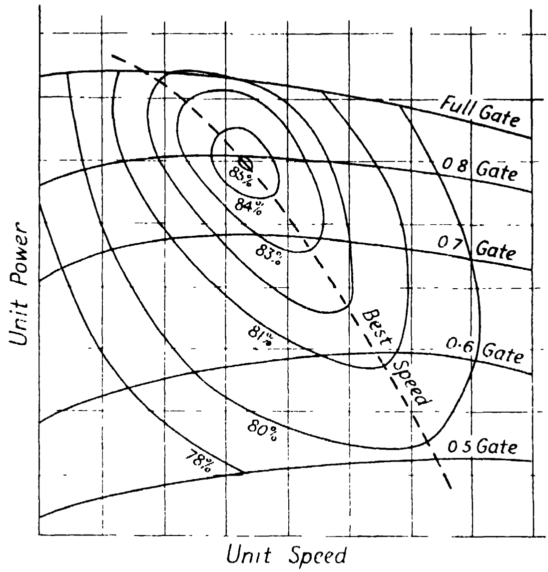


FIG. 310

22.9. Principle of Similarity Applied to Turbines. The principle of similarity may be applied to turbines in order to predict the performance of a future design from the tests on a model. A small model is made similar to the actual turbine and, by means of a test, its horse-power is measured under a known head and at a known speed; the quantity of water supplied is also measured. From these results it is possible to calculate the performance of the actual turbine.

Let D = diameter of a turbine,

P = horse-power of turbine.

For all similar turbines all the velocities such as V, v, V_r, V_f, etc. will be proportional to $\sqrt{H}$. This is obvious from the velocity diagrams, the blade angles being constant.

Hence, $$v \propto \sqrt{H}$$

and $$V_f \propto \sqrt{H}$$

But $$v = \pi D n$$

Hence $$\pi D n \propto \sqrt{H}$$

from which $$D \propto \frac{\sqrt{H}}{n} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Also, $$P = \frac{WH}{550}$$

$$= \frac{w(\pi Db)V_f H}{550}$$

But $$b \propto D$$

and $$V_f \propto \sqrt{H}$$

Hence $$P \propto D^2 H^{3/2}$$

or $$P = k_1 D^2 H^{3/2} \quad . \quad . \quad . \quad . \quad (11)$$

where k_1 is a constant for the type of turbine considered. By combining eqs. (10) and (11) the equation for specific speed may be obtained, as in § 22.7.

Also, $$Q = \pi D b V_f$$

Then $$Q \propto D^2\sqrt{H}$$

or $$Q = k_2 D^2 \sqrt{H} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where k_2 is a constant for the type considered.

It was shown in § 22.7 that

$$n = k\frac{H^{5/4}}{\sqrt{P}}$$

Hence, by measuring the values of P, H, n and D from the model test the values of k, k_1 and k_2 may be calculated. These values will also hold for the large turbine; hence, as its diameter is known, and as the head under which it will run is known, its horse-power, speed and quantity may be calculated. It should be noted that the horse-power used for eq. (10) is the water horse-power supplied, whilst that measured in the model test is the brake horse-power; hence, the efficiency of the runner has been assumed to be the same for both model and large turbine. This is not quite true, as the efficiency of the model is slightly less than the large turbine due to the friction of the water being greater in the small passages of the model.

EXAMPLE 10

Tests on a model turbine, 1 ft in diameter, give a maximum efficiency of 82 per cent at 900 r.p.m. and at ¾ gate opening, under a head of 64 ft. The output was then 38·4 b.h.p. A similar turbine is required to develop 500 b.h.p. at ¾ gate under a head of 81 ft. Calculate its diameter and speed of rotation. How would you expect its efficiency to compare with that of the model? (*Lond. Univ.*)

Assume efficiency is the same for model and turbine, and use the equation for specific speed given in § 22.7.

Then $$n = k\frac{H^{5/4}}{\sqrt{P}}$$

from which $$k = \frac{n\sqrt{P}}{H^{5/4}}$$

$$= \frac{900\sqrt{38{\cdot}4}}{64^{5/4}} = 30{\cdot}75$$

Next apply this equation to the large turbine; the value of k will be the same as for model.

$$n = 30{\cdot}75\frac{H^{5/4}}{\sqrt{P}}$$

$$= 30{\cdot}75 \times \frac{81^{5/4}}{\sqrt{500}}$$

$$= 334 \text{ r.p.m.}$$

From eq. (10),

$$D = c\frac{\sqrt{H}}{n}$$

where c is a constant for model and large turbine.

Hence, applying this equation to model,

$$c = \frac{Dn}{\sqrt{H}}$$

$$= \frac{1 \times 900}{\sqrt{64}} = 112{\cdot}5$$

Using this value of c and applying the equation to the large turbine,

$$D = \frac{112{\cdot}5\sqrt{H}}{n}$$

$$= \frac{112{\cdot}5\sqrt{81}}{334}$$

$$= 3{\cdot}04 \text{ ft}$$

The efficiency of the turbine should be slightly larger than that of the model on account of the relative increase in the frictional resistance coefficient of the model due to its narrow passages for the fluid.

22.10. Inertia of Water in Supply Pipe. Another difficulty in water turbine governing is the regulation of the increase of pressure due to the inertia effect of the column of water in the supply pipe.

On the governor partly closing the gate there will be a slowing down of the water in the supply pipe; this will cause an increase of pressure at the guide vanes which may tend to speed up the turbine. In order to prevent this, a pressure regulator in the form of a spring relief valve is fitted at the turbine end of the supply pipe.

Another method of overcoming the inertia effect of the water column in the supply pipe is to fit a vertical pipe and tank, known as a "surge tank" or stand pipe, on the supply pipe as near to the turbine as possible (Fig. 311). This tank is open to the atmosphere

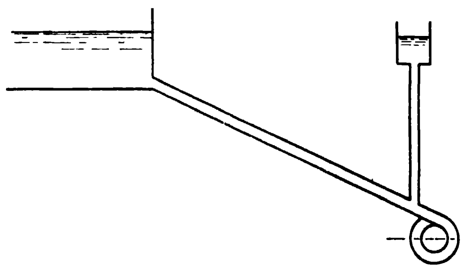


FIG. 311

at the top. When the turbine gates are closing, the slowing down of the water column in the supply pipe will cause a rise of pressure, and water will flow into the surge tank, thus reducing the shock. When the turbine gates are opening, water will flow from the surge tank into the turbine whilst the water column in the supply pipe is accelerating.

For turbines with very large heads the surge tank is closed at the top, the air trapped in being compressed and expanded by the closing and opening of the turbine gates. This is the same in principle as the air vessel on a reciprocating pump.

22.11. Modern Turbines. Modern turbine installations* now consist of the following types of turbine only.

(1) PELTON WHEEL. This is an axial-flow impulse wheel used for very high heads. Heads up to 5,000 ft are used, although many Pelton wheels are in use at medium heads. Overall efficiencies are obtained up to 88 per cent, and single wheels have been constructed to produce 50,000 b.h.p. The specific speed varies between 3 and 5 in feet units.

(2) TURGO IMPULSE WHEEL. This is a modern type of impulse wheel which uses a free jet in place of guide vanes, in the same manner as in the Pelton wheel. This is a high-speed wheel used for medium heads and for small and medium output. Its specific speed

* The diagrams and photographs shown in Figs. 312 to 324 are due to the courtesy of Gilbert Gilkes and Gordon, Ltd., Kendal, Westmorland; and the author has drawn freely on their descriptions and water turbine experience.

varies between 3 and 5. Overall efficiencies of 85 per cent are obtained from this turbine.

(3) FRANCIS TURBINE. This is an inward-radial-flow reaction turbine and is used for intermediate heads and large output. As much as 50,000 b.h.p. can be obtained from a single runner. The specific speed varies between 15 and 90. Overall efficiencies up to 90 per cent are obtained.

(4) KAPLAN TURBINE. This is a propeller turbine, and is of the axial-flow reaction type. It is used for low heads, varying between 9 ft and 60 ft, and has overall efficiencies up to 88 per cent. A single unit can produce an output of 50,000 b.h.p. The specific speed varies between 80 and 140.

All the earlier types of turbine described in § 22.6 have been superseded by the above four types which produce the highest efficiencies and are controlled by the most efficient governing devices.

Descriptions of these four modern types are given in § 22.13 to § 22.16.

22.12. The Governing of Turbines. Modern governors are usually worked by high-pressure oil which is produced by an oil pump driven by the turbine shaft. The governor is operated by the usual method of revolving weights.

An outside view of a Gilkes oil-pressure governor is shown in Fig. 312 and a sectional view, showing its working units, can be seen in Fig. 313. It consists essentially of a speed-sensitive pendulum driven from the turbine shaft and operating a distribution valve. A gear pump supplies oil under pressure to this valve, which regulates the flow of oil to a servomotor cylinder. The servomotor piston is connected to the turbine guide vanes or deflector gear.

If the load on the turbine is reduced and the speed tends to rise, the oil pressure moves the servomotor in the closing direction. An increase in load tends to lower the speed, operating the servomotor in the opening direction. A compensation mechanism ensures the complete absence of hunting.

The problem of accurately controlling the speed of a water turbine with variations in the load has concerned manufacturers for over a hundred years, and frequently it still dictates important features of civil and mechanical engineering design. The main difficulty arises when a heavy load is rejected from a reaction turbine supplied by a pipeline the length of which is more than three or four times the height of the fall. The sudden rejection of load involves very rapid closure of the turbine guide-vanes by the governor in order to limit the speed rise, and this rapid closure sets up water-hammer waves in the pipeline which may become dangerous if they are not carefully controlled. In the same way big pressure surges can also take place when heavy loads are thrown on,

due to the sudden reduction in pipeline pressure which is followed by a high rise in pressure as the water accelerates. It is, however, usually possible to limit these "load on" surges by increasing the

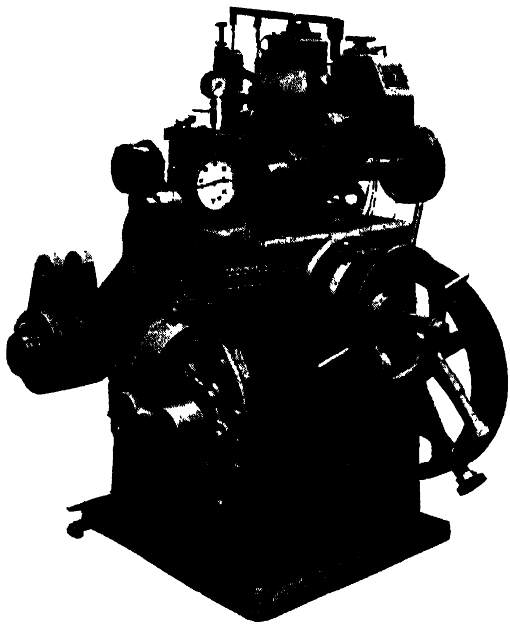


FIG. 312. OIL-PRESSURE GOVERNOR FITTED WITH ELECTRIC SPEED CONTROL
(*Courtesy of Gilbert Gilkes and Gordon, Ltd.*)

"opening time" of the governor, or by controlling the rate at which load is added.

The conventional method of dealing with the problem of the long pipeline is to build a large surge tower close to the turbine house, or to incorporate a governor operated relief valve, which bypasses a large volume of water as the turbine guide vanes are closed, the bypass valve subsequently closing under the action of a dashpot. Often both are required. Such bypass or relief valves are expensive and complicated. Their seatings require constant maintenance and serious leakage rapidly reduces the efficiency of the plant.

The Giljet Governing System provides a method of governing a reaction turbine supplied by a long pipeline in a manner as simple and effective as the governing of an impulse turbine by deflector and slow closing spear control. It consists of a flywheel-brake which contains curved vanes and is mounted on the turbine shaft, as shown in Fig. 314. The vanes are so shaped that, when a high-velocity water jet impinges on them, they tend to rotate the brake

wheel in the opposite direction to that of the turbine. The water is admitted to the brake through an inlet pipe with a spear controlled nozzle similar to the type used on Pelton wheels. After impinging

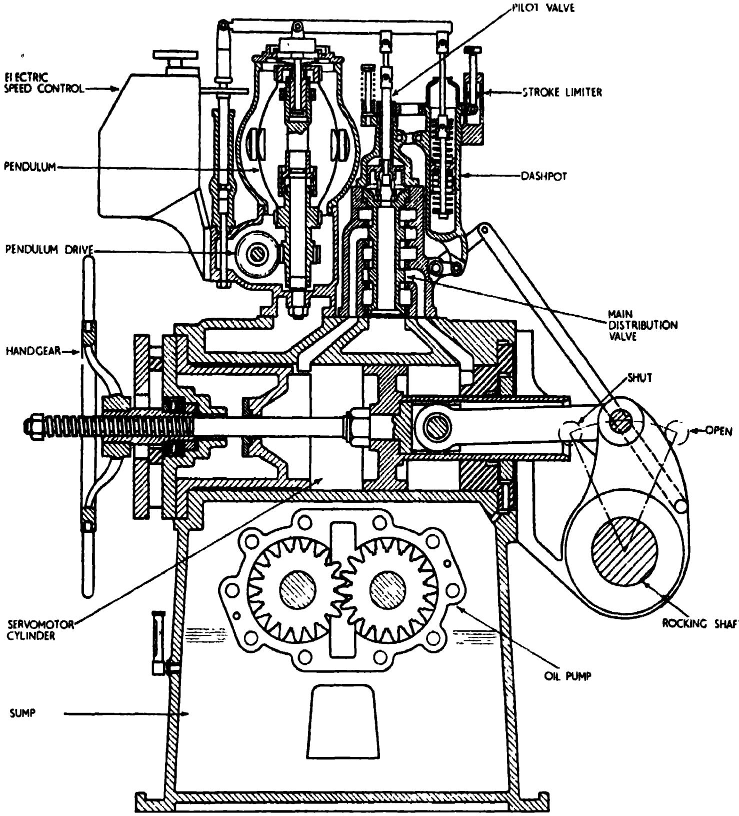


FIG. 313. DIAGRAMMATIC SECTION OF OIL-PRESSURE GOVERNOR

(*Courtesy of Gilbert Gilkes and Gordon, Ltd.*)

on the brake disc, the water is discharged into the tail race through a passage in the foundations, care being taken that its kinetic energy is dissipated harmlessly.

An oil-pressure governor controls the position of the turbine guide vanes, but coupled to its main rocking shaft is a set of levers connected through a dashpot mechanism to the spear valve (Fig. 314). The dashpot piston is free to move rapidly when the turbine speed falls, and the governor operates to *open* the guide vanes, but its movement is restricted by means of an orifice when the turbine guide vanes close due to a rejection of load and a rise in speed.

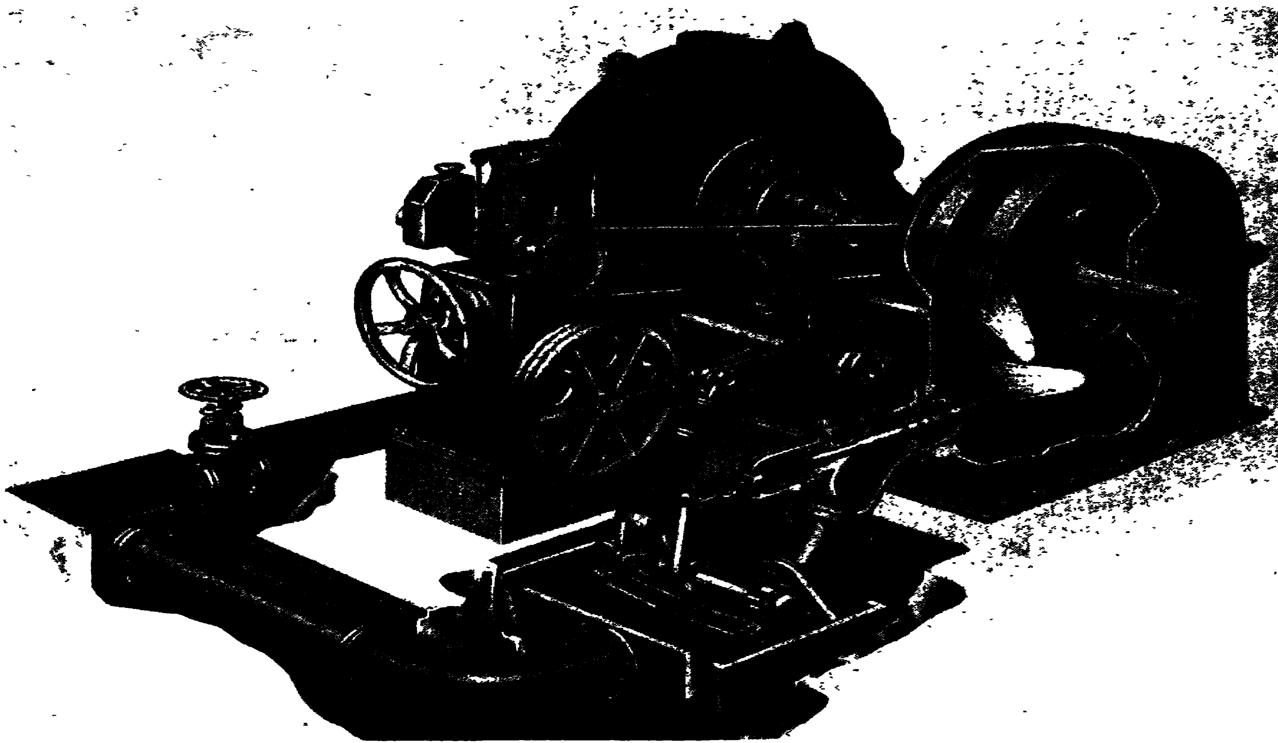


Fig. 314. Francis Turbine Equipped with "Giljet" Governing Gear

(Courtesy of Gilbert Gilkes and Gordon, Ltd.)

Assume that the set is operating steadily at full load, and the breaker suddenly trips, throwing the load off. The turbine speed will rise, causing a closing movement of the guide-vane operating mechanism, and an opening movement of the Giljet spear. When the guide vanes have closed through about 10 per cent of their travel, the spear nozzle will be full open and the retarding torque exerted on the flywheel-brake will be such that approximately nine-tenths of the power of the turbine is immediately absorbed. There will be a slight rise in speed, but the rate of flow of water through the pipeline will remain almost unchanged. If the linkage between the rocking shaft of the governor and the spear were solid, the turbine would continue to operate indefinitely at this setting of the guide vanes and "Giljet" unless there were an increase in the load on the alternator, in which case the fall in speed would cause a movement of the governor rocking shaft which would slightly close the spear, and re-proportion the load between the alternator and the hydraulic brake. Due to the incorporation of a dashpot in the mechanism the spear will slowly close the nozzle, thus reducing the braking jet and the retarding torque on the flywheel-brake. The gradual reduction in the load allows the governor to close the turbine guide vanes, the rate of closing being controlled by the setting of the dashpot orifice.

22.13. The Pelton Wheel. A diagrammatic view of a Pelton wheel is shown in Fig. 315. The runner consists of 24 buckets, made of cast steel. The whole runner is a single casting containing the central disc and the buckets, thus preventing failure by the buckets becoming loose. A view of a cast steel runner is shown in Fig. 316. This runner developed 9,000 b.h.p. under a head of 2,600 ft.

In the self-contained design of Pelton Wheel which has its own shaft and bearings, the turbine shaft is of carbon steel and the coupling flange may either be forged integral or a separate keyed-on coupling may be used. The shaft is carried in two bearings of the "Michell" or "Kingsbury" tilting pad type. One of these is a journal bearing and the other a combined unit having double thrust faces to locate the shaft axially.

Centrifugal water throwers are fixed to the turbine or generator shaft where it passes through the casing. These prevent leakage of water along the shaft under all operating conditions.

The spear and nozzle are manufactured of stainless steel or special bronze depending on the head and the quality of the water. The spear rod is either automatically controlled or hand operated. It is carried in a streamlined bronze spear support which also corrects the flow as it approaches the nozzle, removing any whirl component emanating from the preceding bend. The spear tip and nozzle are designed to give a clean efficient jet at all openings and both are easily renewable. The jet deflector is a stainless steel or manganese

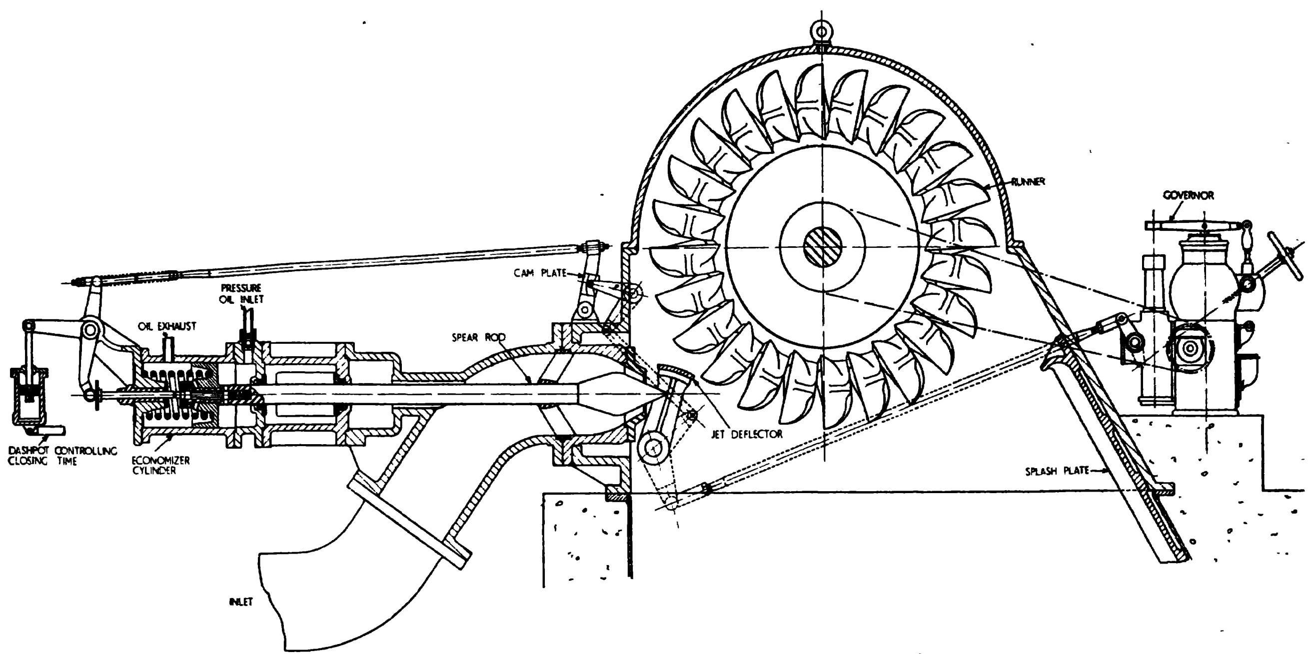


FIG. 315. SECTIONAL DRAWING OF A TYPICAL SINGLE-JET PELTON WHEEL

(Courtesy of Gilbert Gilkes and Gordon, Ltd.)

bronze casting which can be readily replaced when worn. It is mounted on a deflector shaft carried in grease lubricated bearings.

The governing of medium and large powered Pelton wheels is usually carried out by an oil-pressure governor. The shaft-type governor is restricted to turbines of relatively low output.

A typical Gilkes oil-pressure governor is shown in Fig. 312. The governor, which is driven from the turbine shaft by an endless laminated leather belt controls the position of the jet deflector. When the turbine is operating with the "run of the river," hand regulation of the spear is usually considered satisfactory on medium-powered units. The spear is set in accordance with the volume of water available or the anticipated load, and speed control is carried out entirely by the jet deflector without varying the size of the jet.

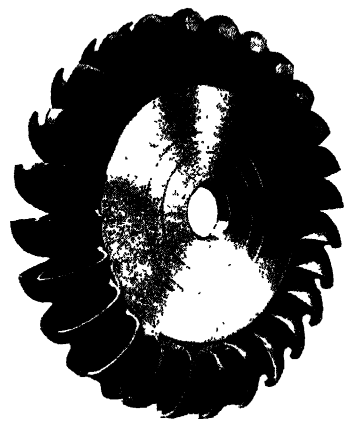


FIG. 316. CAST STEEL RUNNER FOR A 9,000 B.H.P. PELTON WHEEL OPERATING ON 2,600 FT (800 M) HEAD

(Courtesy of Gilbert Gilkes and Gordon, Ltd.)

Where the flow of water is limited and the turbine is drawing from storage, automatic control of the flow of water at all loads is required, and this is arranged by an oil operated servomotor known as an "Economizer." The general arrangement of an economizer can be seen in Fig. 315 and the principle of operation is as follows.

If the turbine is carrying load and a part or whole of that load is rejected, the speed tends to rise, and the governor immediately diverts the jet of water from the runners by means of a jet deflector. At the same time the spear is gradually closed to its new position by the servomotor cylinder, the speed of closure being controlled by a dashpot. This slow closing obviates any undue pressure rise which might occur if the governor were directly connected to the spear rod. When load is accepted, the servomotor opens the spear rapidly, giving the required quantity of water to maintain the turbine speed.

The oil pressure for the servomechanism is supplied by a separate oil pump driven either from the turbine shaft or by an electric motor. Auxiliary starting and braking jets are fitted where required. Where the maximum output of the turbine exceeds 1,000/2,000 b.h.p. it is usual to provide an economizer even when the turbine is not drawing from storage. This is to reduce wear on the deflector and the noise of the interference of the deflector with the jet, both of which may be severe if the unit runs for long periods at low loads with the spear fully open.

A view of a 4,250 b.h.p. Pelton wheel installation, working under a head of 1,170 ft, is shown in Fig. 317.

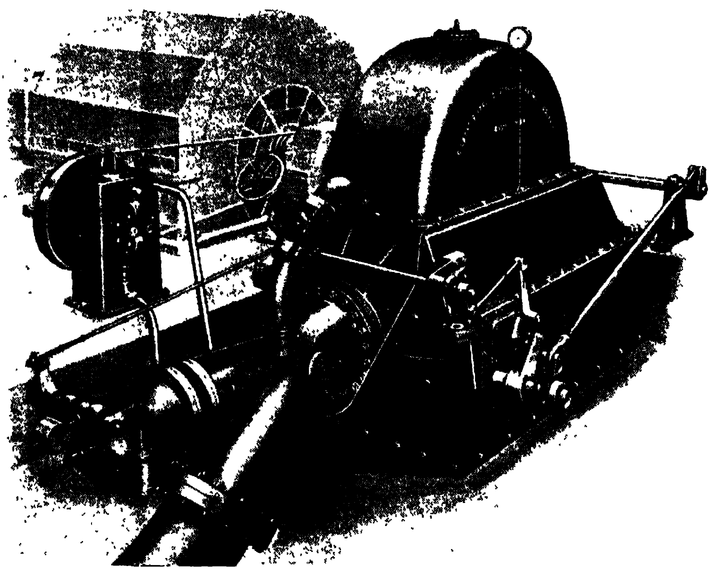


FIG. 317. A 4,250 B.H.P. PELTON WHEEL WORKING ON A HEAD OF 1,170 FT

This wheel was supplied to the Bolivian Power Co., Ltd., South America. The runner is overhung on the extended alternator shaft.

(*Courtesy of Gilbert Gilkes and Gordon, Ltd.*)

22.14. Turgo Impulse Wheel. This is a free jet impulse wheel used for small and medium outputs. A sectional view of a typical Turgo impulse wheel is shown in Fig. 318. They are constructed to run at speeds up to 2,000 r.p.m. and outputs up to 8,000 b.h.p.; heads up to 900 ft are used. It will be noticed that there are no fixed guide vanes, and the water enters the runner vanes as a free jet issuing from a nozzle, as in a Pelton wheel. In Fig. 319 is shown a diagram of the shaft governor and jet deflector fitted to a small power Turgo impulse wheel.

Turgo impulse wheels of larger output require an oil-pressure governor for control purposes. A typical arrangement of a Turgo impulse wheel with oil-pressure governor and automatic spear control is shown in Fig. 320. The automatic operation of this governor is the same as that described for the Pelton wheel in § 22.13.

The turbine casing is split on the horizontal centre-line and made of cast iron for the smaller sizes or of welded steel for larger machines.

The bottom half is specially designed to allow free discharge of water from the runner and is normally arranged for grouting into

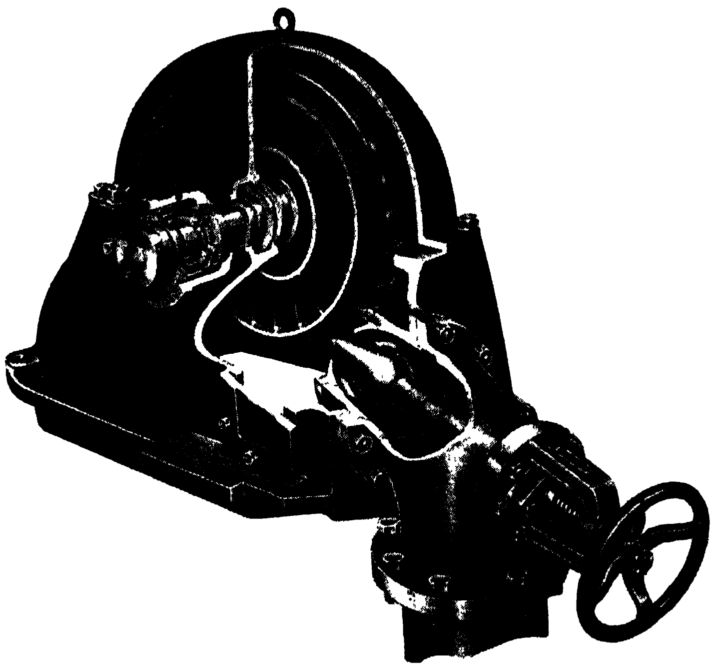


FIG. 318. SECTIONAL VIEW OF A TYPICAL TURGO IMPULSE WHEEL

(*Courtesy of Gilbert Gilkes and Gordon, Ltd.*)

concrete foundations. The top half is easily removable for access to the runner and dismantling. The runner is formed from a single casting with integral buckets or blades. It is usually of cast iron for heads up to 300 ft and bronze or cast steel for higher heads. The inlet side of the runner can be seen in Fig. 318.

The turbine shaft is of carbon steel and the half coupling may either be forged on the shaft end or a separate keyed-on coupling can be used. The shaft is carried in two bearings of the well-known "Michell" tilting pad type. One of these is a journal bearing and the other a combined thrust and journal unit taking the hydraulic thrust load from the runner. Cooling water pipes and fittings are supplied where necessary. On smaller sizes of turbines, generally below 200 b.h.p., grease lubricated ball and roller bearings are used.

The construction of the spear, nozzle, jet deflector and the Gilkes oil-pressure governor used is the same as that described in § 22.13 for the Pelton wheel.

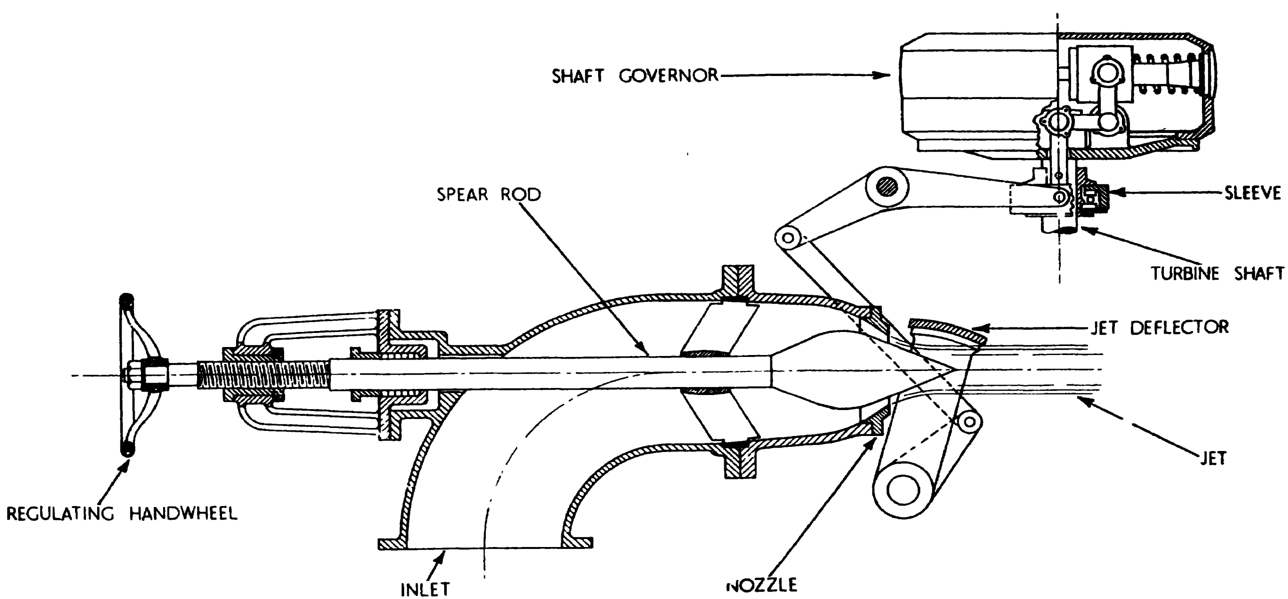


Fig. 319. Diagram of Shaft Governor and Jet Deflector as Fitted to Turgo Impulse Wheels

(Courtesy of Gilbert Gilkes and Gordon, Ltd.)

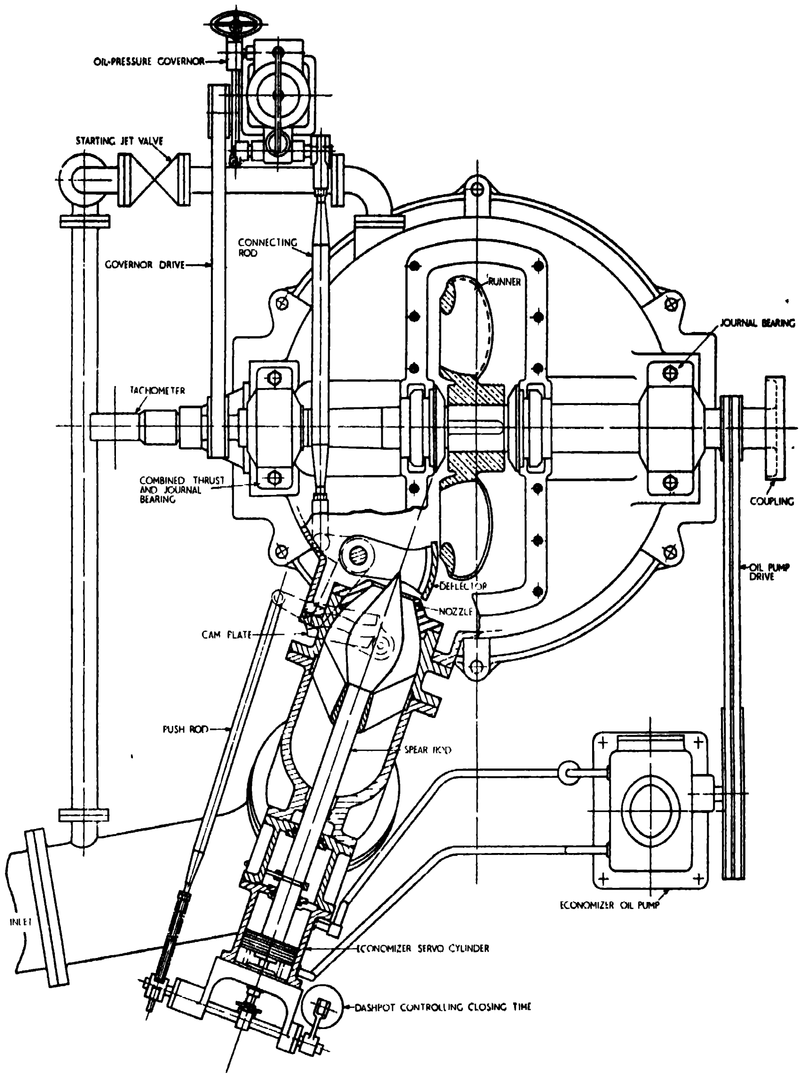


FIG. 320. TYPICAL TURGO IMPULSE WHEEL WITH OIL-PRESSURE GOVERNOR AND AUTOMATIC SPEAR CONTROL

(Courtesy of Gilbert Gilkes and Gordon, Ltd.)

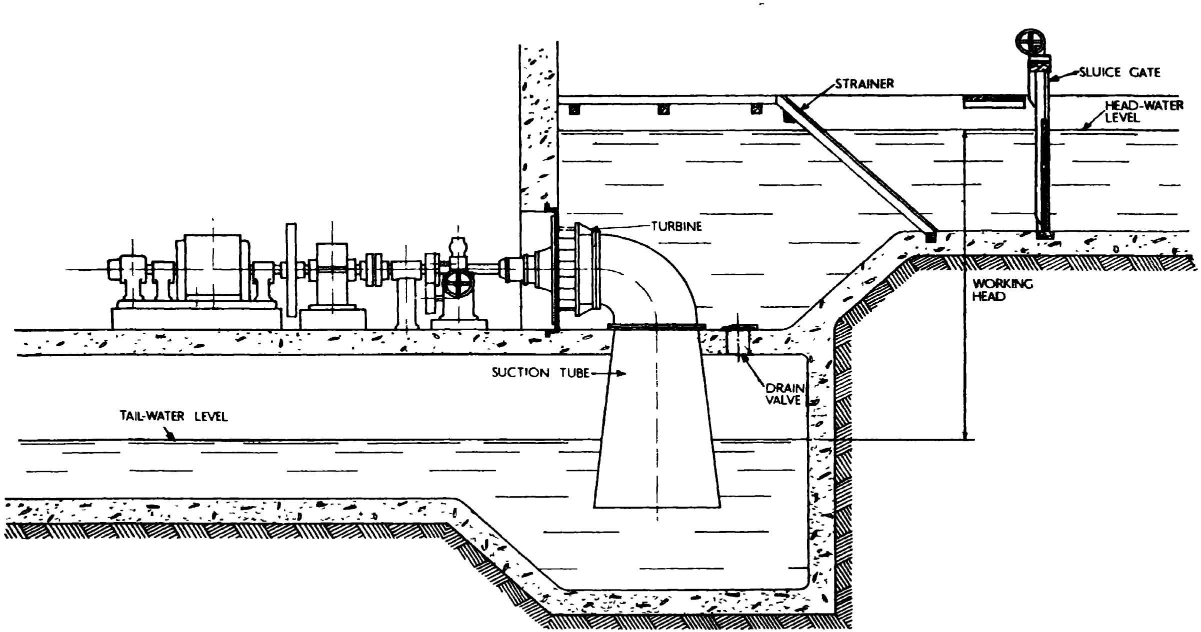


FIG. 321. OPEN TYPE OF FRANCIS TURBINE INSTALLATION

(Courtesy of Gilbert Gilkes and Gordon, Ltd.)

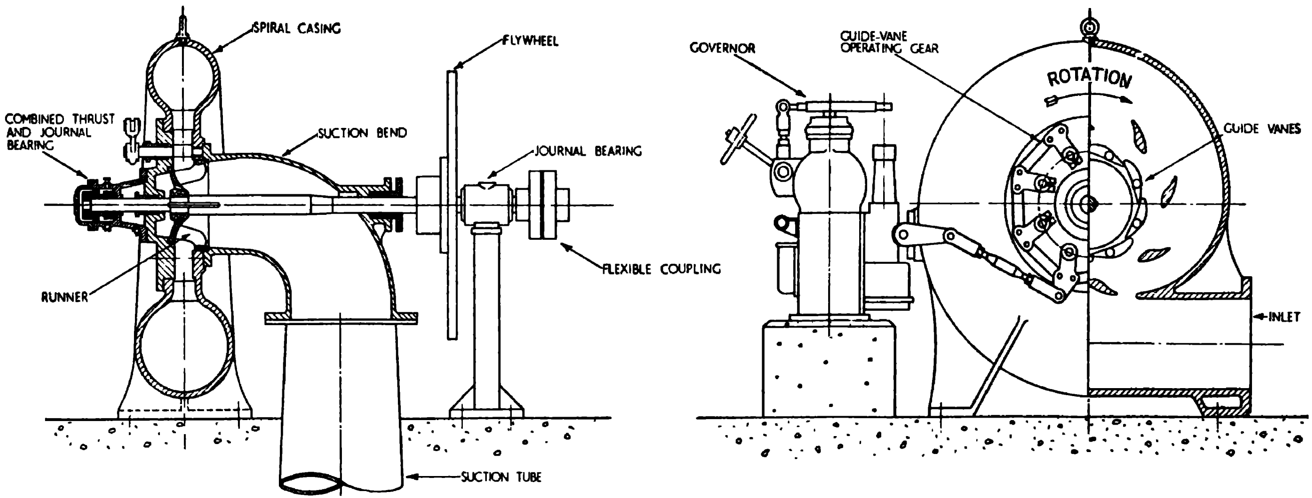


FIG. 322. SECTION OF TYPICAL FRANCIS TURBINE

(Courtesy of Gilbert Gilkes and Gordon, Ltd.)

22.15. The Francis Turbine. This turbine is an inward-radial-flow reaction type and is now generally used for medium heads of between 30 and 600 ft. Francis turbines of the "open" type are used for low heads and small output. A view of an open type of Francis turbine installation is shown in Fig. 321. It will be noticed that there is no external casing surrounding the vanes at inlet. The turbine is fitted with a suction, or draught, tube at outlet.

A sectional view of a larger type of Francis turbine is shown in Fig. 322. In this case the turbine is fitted with an outer casing forming a spiral-shaped passage for the entering water. A view of the turbine runner is shown in Fig. 323; the outlet end is at the left. The turbine is governed by altering the angle of the swivelled guide vanes by the method shown in the right-hand view of Fig. 322; the movement is operated by an oil-pressure governor. A detailed view of the guide-vane operating gear is shown in Fig. 324.

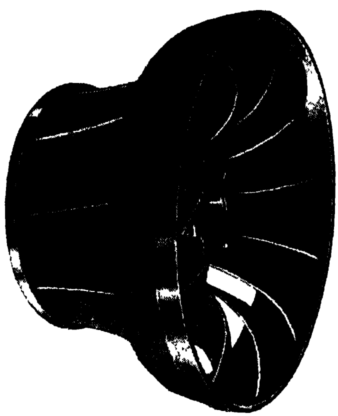


FIG. 323
FRANCIS TURBINE RUNNER
Courtesy of Gilbert Gilkes and Gordon, Ltd.)

A view of a Francis turbine showing the oil-pressure governor gear of the "Giljet" type is shown in Fig. 314. To the right of the installation can be seen the special braking device, consisting of a vaned brake disc operated by a jet of water from the main supply (§ 22.12).

The spiral case is normally constructed of cast iron, or of cast steel for the higher heads. When the inlet diameter is greater than 33 in., fabricated welded steel cases are usually adopted. Integrally cast fixed guides in the cast cases, or welded vanes in the fabricated cases, reinforce the structure against distortion from hydraulic pressure and also direct the flow of water on to the adjustable guide vanes. Where turbines operate on high heads, high-tensile steel staybolts are also fitted. The angular setting of the inlet to the spiral casing and the direction of rotation of the turbine can be arranged to suit the layout of the power station.

The material used for the guide vanes depends largely upon the working head and the condition of the water, and may be grey cast-iron, spheroidal graphite cast-iron, cast-steel, bronze or stainless steel. The spindles are cast integral and extended through glands in the turbine chamber. They are connected to the central regulating ring by links and levers, Fig. 324 showing a close-up view of the normal arrangement. The levers are not keyed to the guide-vane

spindles, but are clamped on to them, as can be seen in this illustration. This ensures that should any debris become jammed between two vanes, one of the levers will slip, avoiding damage to the governor gear and connexions. The turbine governor operates the guide-vane regulating ring through an adjustable connecting rod, the arrangement being clearly shown in Fig. 322.

The turbine runner is formed from a single casting and carefully balanced. The runner vanes are ground smooth, and their profile ensures efficient performance and freedom from cavitation. Choice

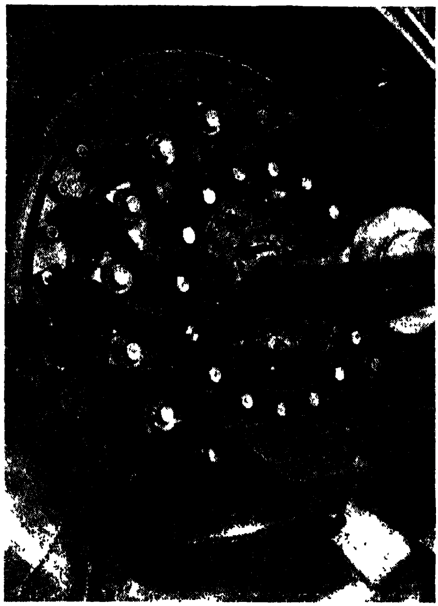


FIG. 324. A CLOSE-UP VIEW OF THE GUIDE-VANE OPERATING GEAR OF A SPIRAL-CASED FRANCIS TURBINE

(Courtesy of Gilbert Gilkes and Gordon, Ltd.)

of material for the runner again depends on the site conditions, and may be cast-iron, cast-steel, bronze or stainless steel. On high heads or where the water contains silt, renewable clearance rings may be fitted.

The turbine shaft is normally of forged carbon steel with integral thrust collar and half coupling. It is carried in a single "Michell" type combined thrust and journal bearing.

The speed control of Francis turbines is usually carried out by an oil-pressure governor, and Fig. 314 shows a typical Gilkes governor of this type. Many problems are involved in the governing of water turbines, especially if the pipeline is a long one.

22.16. Propeller Turbines. A type of water turbine which has attained great popularity within recent years is known as the propeller turbine. It is an axial-flow reaction type having a small number of blades, usually from four to six. The runner is fitted horizontally (see Fig. 325) and the blades resemble, in appearance, the propeller blades of a ship. Propeller turbines have a high specific speed, varying between 80 and 140.

A very efficient type of propeller turbine is the Kaplan turbine; this make is fitted with swivel blades by means of which the blade

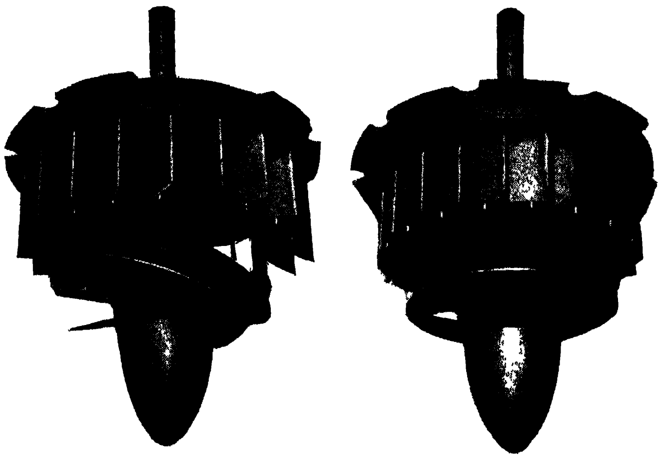


FIG. 325. KAPLAN TURBINE RUNNER

Left: Blades open. *Right:* Blades closed.

(*Courtesy of Escher Wyss Engineering Works, Ltd.*)

angle can be varied with the output. By this method a high efficiency is maintained at all gates.

Views of the runner and guide blades of a Kaplan turbine are shown in Fig. 325. The left-hand view shows the runner blades and guide vanes fully open; in the right-hand view they are closed. The blades are swivelled by a mechanism contained in the boss; this is operated by the turbine governor through the action of a servo-motor. By this means the blade angles are automatically adjusted, according to the power developed by the turbine. This prevents the falling off of the efficiency at part gate.

In Fig. 326 are shown the efficiency curves of a Kaplan turbine with swivel blades, and a propeller turbine with fixed blades, plotted to the same scale on a base representing the output or gate opening; both turbines have the same specific speed. It will be noticed from the curves how the Kaplan swivel-bladed turbine maintains its high efficiency through most of its range of output.

Kaplan turbines are pure axial flow, with four to six blades having no outside rim; the blades are made of stainless steel. They

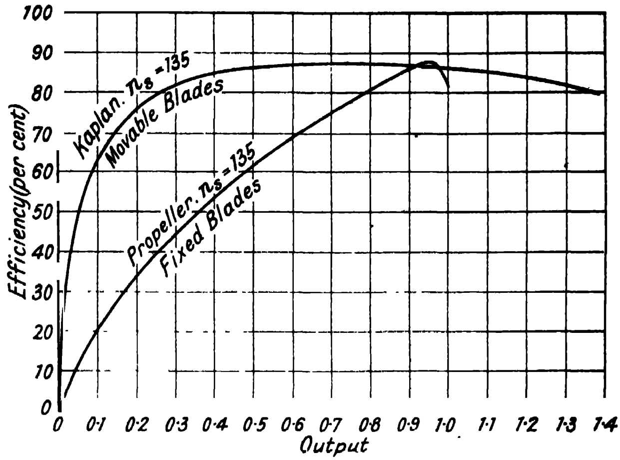


FIG. 326

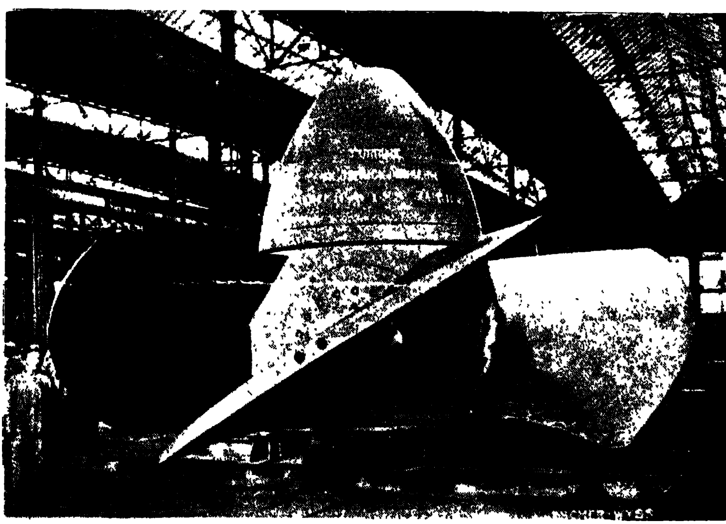


FIG. 327. KAPLAN TURBINE ROTOR
(Courtesy of Escher Wyss Engineering Works, Ltd.)

are constructed to run at speeds varying between 60 and 220 r.p.m. and to work under maximum heads varying from 9 ft to 50 ft. They are very efficient at low heads. The runner may have a very

large diameter; one runner is 26 ft 3 in. in diameter, and weighs 150 tons. Horse-powers of 40,000 per unit have been reached.*

A view of a large Kaplan turbine rotor is shown in Fig. 327. It will be noticed that the rotor has four blades only and that the cross-section of the blade is in the shape of an aerofoil.

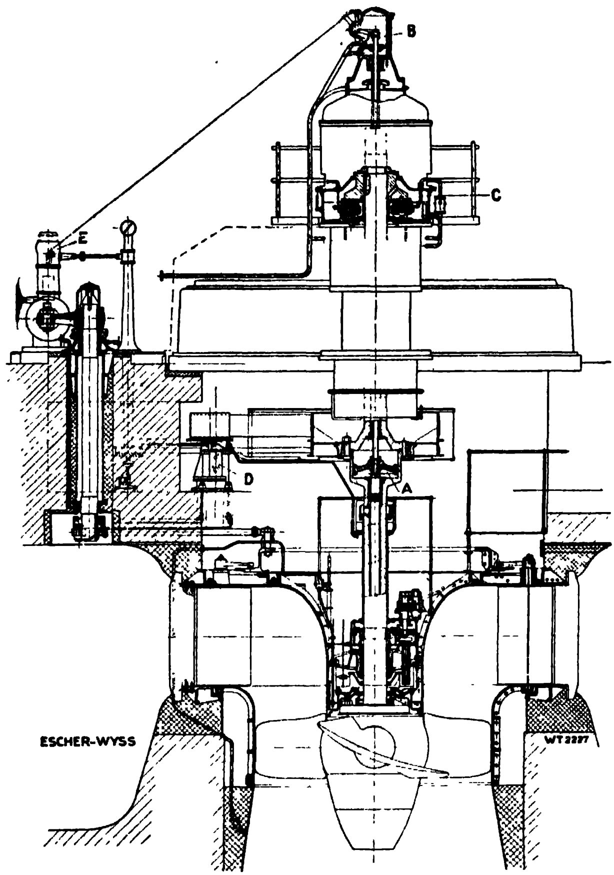


FIG. 328. LONGITUDINAL SECTION OF A TURBINE OF THE DIETIKON POWER STATION

(*Courtesy of Escher Wyss Engineering Works, Ltd.*)

A sectional view through a Kaplan turbine installation is shown in Fig. 328. The oil-pressure servomotor is shown at A; the high-pressure oil admission is at B. C is the thrust bearing, D the geared oil pump, and E the speed control governor.

* For aerofoil blading solution of propeller turbine see § 15.7.

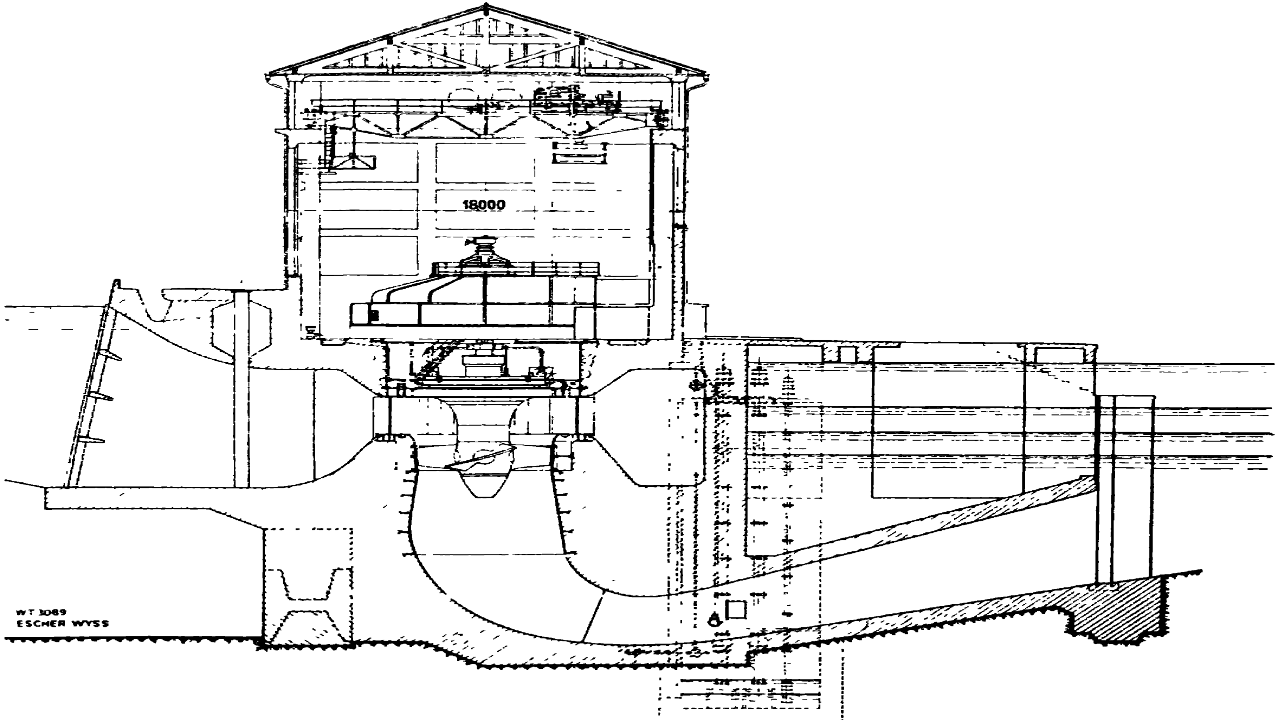


FIG. 329. CRO: TION ROUGH THE REC GEN PO PLANT

y of E Wyss Engineering W

A cross-sectional view through a power station containing this turbine is shown in Fig. 329. The small difference of head between the two water levels should be noticed.

The Kaplan turbine blading cannot be designed by the application of the momentum equation (eq. (4), Chapter 6), as in other types of turbine, on account of its large pitch/chord ratio (§ 15.6). Satisfactory results are obtained only by the application of the aerofoil blading solution given in § 15.7.

EXERCISES 22

1. An inward-flow reaction turbine has an external diameter of 2 ft. If the breadth of the wheel at inlet is 6 in. and the velocity of flow at inlet is 5 ft/sec, find the weight of water passing through the turbine per second.
Ans. 980 Lb.

2. If the turbine of Question No. 1 has a speed of 192 r.p.m., and if the guide blade makes an angle of 10° to the wheel tangent, draw the velocity diagram at inlet and find the runner blade angle, the velocity of whirl, the absolute velocity of the water leaving the guide vane, and the relative velocity of the water entering the runner blade.
Ans. $\theta = 31°$; $V_w = 28{\cdot}4$ ft/sec; $V = 28{\cdot}8$ ft/sec; $V_r = 9{\cdot}8$ ft/sec.

3. If the turbine of Question No. 1 has an inner diameter of 1 ft, find the breadth of the wheel at outlet in order to keep the velocity of flow 5 ft/sec. Find also the runner blade angle at outlet if the discharge is radial and draw the velocity diagram at outlet. *Ans.* 12 in.; 26°.

4. Find the work done per pound of water for the turbine in Question No. 1; find also the head supplied, the horse-power produced, and the hydraulic efficiency.
Ans. 17·7 ft-Lb; $H = 18{\cdot}12$ ft; h.p. = 31·6; eff. = 98 per cent.

5. The lead-on angle of the guide vanes in an axial-flow impulse turbine is 20°; the wheel vane angle at entrance is 30°; the head 400 ft. If the velocity at discharge is axial, and if the coefficient of velocity for the guide vanes is 0·98, determine the work done per second when passing 10 ft³/sec and running under maximum efficiency conditions. [Sin 20° = 0·342.] (*A.M.I.Mech.E.*)
Ans. 229,000 ft-Lb.

6. In an outward-flow turbine supplied with 180 ft³/sec and making 200 r.p.m., the internal and external diameters of the wheel are 6 ft and 7 ft 6 in. respectively and the effective width of the wheel-face at inlet and outlet is 9 in. The head on the wheel is 115 ft and the discharge is free and radial. Neglecting the thickness of the vanes and friction losses, determine the angles of the vanes at entrance and exit, and sketch a vane showing these angles. (*A.M.I.C.E.*) (Assume turbine to be a reaction.) *Ans.* 109·8°; 7·4°.

7. The peripheral velocity of the wheel of an inward-flow turbine is 70 ft/sec. The velocity of whirl of the inflowing water is 55 ft/sec, and the radial velocity of flow 7 ft/sec. If the flow is 24 ft³/sec and the hydraulic efficiency 80 per cent, find the head on the wheel, the horse-power of the turbine, and, by drawing to scale the triangle of velocities, the inlet angle of the vanes. The discharge is radial. (*A.M.I.C.E.*) *Ans.* 149·5 ft; 325 h.p.; 155°.

8. Describe the working of, and a method of governing, an axial-flow Girard turbine.

If for such a turbine the angle of the guide blades is 30°, and the angle of the rotor vanes is 25° at outlet, find the maximum hydraulic efficiency, and the best speed of the turbine.

The available head is 100 ft, and the mean diameter of the rotor 6 ft. (*Lond. Univ.*) [Assume axial discharge for max. eff.]

Ans. 93·7 per cent; 137 r.p.m.

9. An inward-flow turbine, having an overall efficiency of 75 per cent, is required to give 175 h.p. The head H is 20 ft; velocity of periphery of the wheel is $0{\cdot}95\sqrt{(2gH)}$; and the radial velocity of flow is $0{\cdot}35\sqrt{(2gH)}$. The wheel is to make 230 r.p.m., and the hydraulic losses in the turbine are 20 per cent of the available energy. Determine (*a*) the angle of the guide blade at inlet; (*b*) the wheel vane angle at inlet; (*c*) the diameter of the wheel; (*d*) the width of the wheel at inlet. [Assume turbine is reaction and radial discharge.] (*Lond. Univ.*) *Ans.* (*a*) 39·8°; (*b*) 146½°; (*c*) 2·83 ft; (*d*) 11·02 in.

10. A Pelton wheel has a mean bucket speed of 40 ft/sec, and is supplied with water at the rate of 150 gal/sec under a head of 100 ft. If the buckets deflect the jet through an angle of 160°, find the horse-power and efficiency of the wheel. *Ans.* 264·2 h.p.; $e = 97$ per cent.

11. Obtain an expression for the theoretical efficiency of a Pelton wheel when the angle of the bucket at exit makes an angle of θ with the direction of the jet. Show by a diagram how the efficiency of the wheel will vary as the relation of the velocity of the jet to the velocity of the bucket is varied.

Describe, with carefully drawn sketches, at least one method of governing a Pelton wheel. (*Lond. Univ.*)

12. Briefly describe an inward-flow turbine. Show that in a turbine with radial vanes at the receiving circumference the theoretical hydraulic efficiency is $2/(2 + \tan^2 \alpha)$ where α is the angle made by the guide blade with a tangent to the point where it cuts the receiving circumference, the velocity of radial flow being constant. (*Lond. Univ.*)

(Assume turbine is reaction with radial discharge.)

13. In an outward-flow reaction turbine the rim speed at inlet is 40 ft/sec, and the ratio of the radii is 0·8. The turbine is placed 3 ft below the water surface in the tail race, and the wheel vane angles are 90° and 20° at inlet and outlet respectively. The radial velocity of flow at inlet is 14 ft/sec. Neglecting frictional losses, and taking velocity of outflow as radial, find the guide vane angle, pressure head at inlet to the wheel, speed of flow from guides, the total head, and hydraulic efficiency. (*Lond. Univ.*)

Ans. 19·3°; 29·8 ft of water (gauge); 42·3 ft/sec; 54·5 ft.
$e = 91{\cdot}2$.

14. An inward-flow turbine, having an overall efficiency of 75 per cent, is required to give 180 h.p. The head H is 30 ft. The velocity of the periphery of the wheel is $6\sqrt{H}$, and the radial velocity of flow is $2\sqrt{H}$. The wheel is to make 120 r.p.m. The hydraulic losses in the turbine are 20 per cent of the available energy. Determine: (*a*) the guide blade angle at inlet; (*b*) the wheel vane angle at inlet; (*c*) the diameter of the wheel; (*d*) the width of the wheel at inlet. [Assume turbine is reaction and radial discharge.] (*Lond. Univ.*) *Ans.* (*a*) 25·1°; (*b*) 130·5°; (*c*) 5·23 ft; (*d*) 4·8 in.

15. An inward-flow pressure turbine has a runner whose vanes are radial at inlet and inclined backwards at 30° to the tangent at discharge. The diameter at entry is twice that at discharge, and the width at entry is one-half that at discharge. The guide vanes are inclined at 15° to the tangent. The velocity of the water leaving the guide is 80 ft/sec. Determine the correct velocity for the runner, and the absolute velocity of the water at the point of discharge. (*A.M.I.C.E.*) *Ans.* $v = 77{\cdot}3$ ft/sec; $V_1 = 20{\cdot}9$ ft/sec.

16. The following data were obtained from a test on a Pelton wheel—

Area of jet	= 12·0 in.^{2}
Discharge	= 6·35 ft^3/sec
Head at nozzle	= 100·0 ft
Brake horse-power	= 56·0
H.P. absorbed in friction and windage	= 3·0

Determine the energy lost in the nozzle, and also the energy absorbed due to losses in the wheel at discharge. (*A.M.I.Mech.E.*)
Ans. 7·05 h.p. and 5·8 h.p.

17. Given the formula $N_s = N\sqrt{P} \div H^{5/4}$ for the specific speed of a water-turbine, in which N is revolutions per minute, P is the brake horse-power, and H is the available head in feet, prove that the specific speed of a single jet Pelton wheel is $55d/D$, in which d is the diameter of the jet in feet, and D is the diameter of the mean bucket circle in feet. Assume that the coefficient of velocity of the jet is unity, and that when the maximum efficiency is 85 per cent the mean bucket speed is $0{\cdot}46\sqrt{(2gH)}$. (*Lond. Univ.*)

18. Define the terms *specific speed*, *unit speed* and *unit power* as applied to a hydraulic turbine. Describe the method of preparation of, and the use of, the characteristic diagram for a turbine, the co-ordinates being *unit speed* and *unit power*. (*Lond. Univ.*)

19. Show that in a given turbine the peripheral speed of the runner for maximum efficiency is proportional to $\sqrt{H}$ where H is the available head, and that under these conditions the quantity of water consumed is proportional to $\sqrt{H}$, and the power developed to $H^{3/2}$.

Hence, show how the performance of a turbine may be predicted from that of a geometrically similar model. (*Lond. Univ.*)

20. What factors determine whether a turbine of the Francis, Kaplan, or Pelton type would be used in a hydroelectric power scheme? Determine the horse-power of a machine suitable for a head of 500 ft, the quantity available being 30 ft^3/sec. If the speed is to be 375 r.p.m., what type of turbine would be used, and what would be its leading dimensions? (*I.Mech.E.*)
Ans. 1,700 h.p.; $n_s = 6{\cdot}55$ (use Pelton wheel), $d = 5{\cdot}5$ in.
$D = 4{\cdot}6$ ft.

21. Deduce an expression for the specific speed of a hydraulic turbine. A turbine develops 10,000 h.p. under a head of 81 ft at 120 r.p.m. What is its specific speed? What would be its normal speed and output under a head of 64 ft? (*I.Mech.E.*) *Ans.* 49·4; 106·7 r.p.m.; 7,024 h.p.

22. Determine the diameter, speed, and specific speed of a Kaplan turbine runner to develop 7,500 h.p. under a head of 17 ft, given: velocity of flow = 20 ft/sec; diameter of boss = 0·36 × external diameter; and mechanical efficiency = 87 per cent. Velocity of outer edges of blades, 66 ft/sec. How is a Kaplan turbine governed? (*I.Mech.E.*)
Ans. 22·8 ft; 55·3 r.p.m.; $n_s = 138{\cdot}5$.

23. Obtain from first principles an expression for the power P, developed by a hydraulic turbine, in the form

$$P = \rho N^3 D^5 . \phi\left(\frac{N^2 D^2}{gH}\right)$$

where ρ is the mass density of the fluid, g the acceleration due to gravity, N the rotational speed, D the diameter of the rotor, and ϕ denotes an arbitrary function.

Hence, or otherwise, deduce the expression for the "specific speed" in terms of N, H, and P.

Explain briefly the application of specific speed to preliminary design of a turbine. (*Lond. Univ.*)

24. An axial-flow reaction turbine of the propeller type has a speed of 30 r.p.m. The mean diameter of the runner blades is 18 ft and the effective axial area of flow is 300 ft². The water enters the runner in an axial direction, and the inlet and outlet angles of the runner blades are 150° and 20° respectively to the direction of motion of the wheel periphery. The velocity of flow is constant throughout.

Calculate the quantity of water required per second, the theoretical h.p. of the turbine, the hydraulic efficiency, and its specific speed. (*Lond. Univ.*)

Ans. $Q = 4{,}890$ ft³/sec; 8,070 h.p.; eff $= 63{\cdot}9$ per cent; $n_s = 54$.

25. A Pelton wheel is supplied with water from a mountain lake situated 1,500 ft above the nozzle of the wheel. The pipe line conveying the water to the wheel is 1 ft in diameter and 3 miles long; the pipe line and nozzle are designed for the maximum transmission of power. Calculate (i) the diameter of the nozzle in inches, (ii) the quantity of water used, in cubic feet per second, (iii) the speed of the wheel in r.p.m., assuming it to be designed for maximum efficiency and to have a mean bucket circle diameter of 4 ft, and (iv) the theoretical horse-power developed by the wheel. $f = 0{\cdot}01$ for the pipe line. Neglect all frictional losses in the nozzle and the wheel. (Assume $v = 0{\cdot}46V$ and $\phi = 30°$.) (*Lond. Univ.*)

Ans. (i) $d = 2$ in.; (ii) $Q = 5{\cdot}57$ ft³/sec; (iii) 558 r.p.m.; (iv) h.p. $= 500$.

CHAPTER 23

CENTRIFUGAL PUMPS

23.1. Centrifugal Pumps. The action of a centrifugal pump is that of a reversed reaction turbine, except that special arrangements must be made in order to increase the efficiency. All centrifugal pumps are outward-flow, as the radial velocity of the water in the pump is then increased by the centrifugal head impressed on it by the rotating vanes. The pump must be full when starting; for this reason, it should not be allowed to drain. The pump is driven by power from an external source, by which means the vanes are rotated. This gives a centrifugal head to the water in the pump, and the water will leave the vanes at the outer circumference with a high velocity and pressure. A partial vacuum will form in the centre, into which the water from the suction pipe flows. The high pressure of the leaving water is utilized in overcoming the delivery head of the pump. In earlier types of centrifugal pumps the high velocity of the leaving water was wasted in eddies in the circular chamber which surrounded the vanes; but this is now transformed into pressure head by causing the leaving water to flow through a passage of gradually increasing area known as a *diffuser*. The kinetic energy of the leaving water is thus converted into pressure energy, which is utilized in increasing the delivery head of the pump. The efficiency is thus considerably increased.

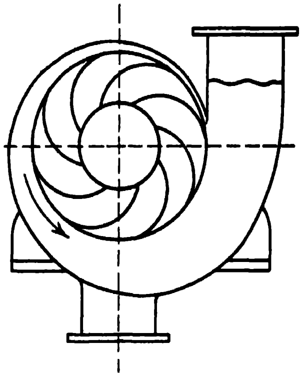


FIG. 330

Centrifugal pumps are usually of the radial-flow type, but pumps having a mixed flow and axial flow are also made. Axial-flow pumps are known as propeller pumps and are used for low heads.

The following are the methods adopted to convert the kinetic energy of the leaving water into pressure energy.

1. VOLUTE CHAMBER. The vane wheel, or impeller, is surrounded by a spiral casing known as a volute chamber (Fig. 330). The leaving water flows inside this chamber circumferentially, the velocity decreasing with the increasing area of flow. When the water reaches the delivery pipe, the velocity will be small and the pressure will have correspondingly increased. It has been found

from tests that this type of chamber only slightly increases the efficiency of the pump; a considerable loss takes place in eddies due to the continually increasing quantity of water flowing through the chamber.

2. VORTEX OR WHIRLPOOL CHAMBER. Professor James Thomson improved on the volute chamber by combining a circular chamber with a spiral chamber (Fig. 331); such a casing is known as a vortex

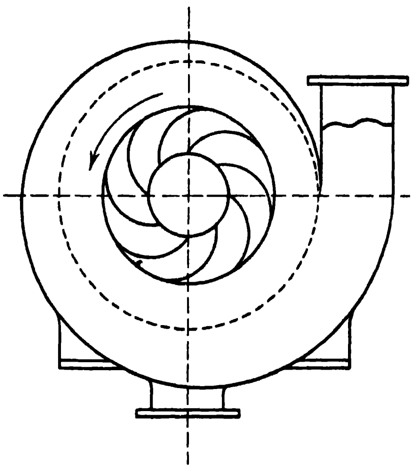


FIG. 331

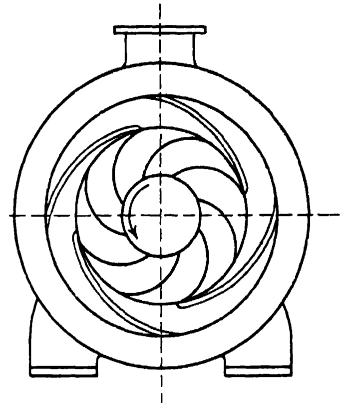


FIG. 332

or whirlpool chamber. An increased efficiency is obtained by means of this type of casing.

3. GUIDE BLADES. Another method of converting the velocity head of the leaving water into pressure head is by causing the water to flow through passages of increasing area formed by guide vanes (Fig. 332). A pump fitted with such vanes is known as a turbine pump, and is similar in principle to a reversed inward-flow reaction turbine. The guide blades are placed at such an angle that the water enters without shock and is surrounded by a volute chamber, by which the water reaches the delivery pipe. The ring of guide blades is called a diffuser, and is found to be very efficient.

There is much looseness in the use of the above terms. Some authorities apply the terms vortex chamber, whirlpool chamber, and diffuser to all types of casings surrounding the impeller.

23.2. Work Done and Efficiency of Centrifugal Pump. The blade angles and work done by a centrifugal pump may be found from the velocity triangles in the same way as for a turbine, except that the inlet diagram now becomes the outlet and *vice versa*. It is usual to assume the water enters the wheel radially.

Using the same notation as used for turbines (§ 22.2), centrifugal head impressed on water

$$= \frac{v_1^2}{2g} - \frac{v^2}{2g} \qquad (\S\ 3.9)$$

Let H = total theoretical lift of pump, or design head. Then,

$$\text{theoretical gross lift} = H + \frac{v_d^2}{2g}$$

where v_d = velocity of discharge from delivery pipe.

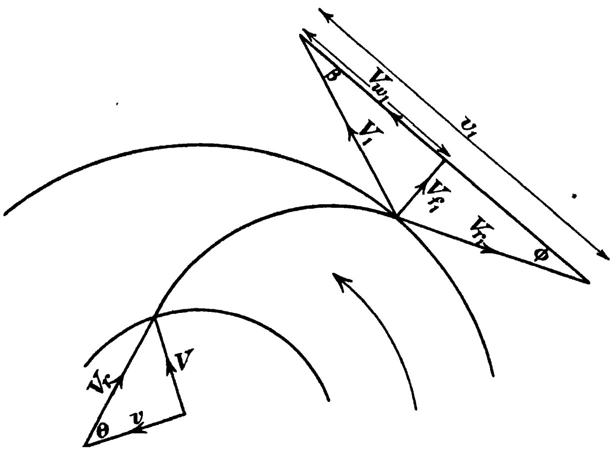


FIG. 333

The term $v_d^2/2g$ is small and may usually be neglected.
The velocity diagrams for inlet and outlet are shown in Fig. **333**.

$v_1 \quad r_1$

Applying the momentum equation [eq. (4), Chapter 6,] work done by impeller per pound of water

$$\frac{V_{w_1} v_1}{g} \qquad (\text{as } V_w = 0)$$

$$H + \frac{v_d^2}{2g}$$

Therefore
$$\frac{V_{w_1} v_1}{g} = H + \frac{v_d^2}{2g}$$

Let h_f = head lost in friction in delivery and suction pipes,
h = actual height water is lifted by pump,
W = weight of water pumped per second.

Then, $$\text{gross lift} = h + h_f + \frac{v_d^2}{2g}$$

Actual efficiency

$$= \frac{\text{actual lift}}{\text{energy supplied to pump shaft per pound of water}}$$

$$= \frac{Wh}{\text{horse-power} \times 550}$$

The manometric efficiency

$$= \frac{\text{gross lift}}{\text{theoretical gross lift}}$$

$$= \frac{h + h_f + \dfrac{v_d^2}{2g}}{H + \dfrac{v_d^2}{2g}} = \frac{h + h_f + \dfrac{v_d^2}{2g}}{\dfrac{V_{w_1}v_1}{g}}$$

Hydraulic efficiency

$$= \frac{\text{gross lift}}{\text{energy supplied to impeller per pound of water}}$$

The energy supplied to the impeller is less than that supplied to the shaft by the mechanical losses in bearings, etc.

23.3. Minimum Starting Speed of Centrifugal Pump. In starting a centrifugal pump, there will be no flow through the wheel until the pressure difference in the impeller is large enough to overcome the total lift. If the impeller is rotating and there is no flow, the water is rotating in a forced vortex and the pressure head caused by the centrifugal force on the rotating water will be $v_1^2/2g - v^2/2g$.

Flow will not commence until this amount is greater than h, as $v_d = 0$ when flow commences.

As $$h = \frac{V_{w_1}v_1}{g} \times \text{efficiency}$$

the least theoretical speed for flow to commence will be when

$$\frac{v_1^2}{2g} - \frac{v^2}{2g} = \frac{V_{w_1}v_1}{g} \times \text{efficiency}$$

Actually, flow will commence when

$$\frac{v_1^2}{2g} - \frac{v^2}{2g} = e\,\frac{V_{w_1}v_1}{g}$$

where e = manometric efficiency.

EXAMPLE 1

A centrifugal pump has an impeller 20 in. outer diameter, and, when running at 520 r.p.m., discharges 1,700 gal of water per minute against a head of 28 ft. At that discharge, the water enters the impeller without shock. The inner diameter is 10 in., the vanes are set back at outlet at an angle of 45°, and the area of flow, which is constant from inlet to outlet of the impeller, is 0·65 ft.²

Determine the manometric efficiency of the pump and the vane angle at inlet. Find also the least speed at which the pump commences to work.

The velocity diagrams are the same as shown in Fig. 333.

$$v_1 = \pi d_1 \frac{n}{60} = \pi \times \frac{20}{12} \times \frac{520}{60} = 45{\cdot}4 \text{ ft/sec}$$

$$v = v_1 \times \frac{10}{20} = 22{\cdot}7 \text{ ft/sec}$$

$$V = V_{f_1} = \frac{\text{quantity per second}}{\text{area of flow}}$$

$$= \frac{1{,}700}{60 \times 6{\cdot}24 \times 0{\cdot}65}$$

$$= 7 \text{ ft/sec}$$

$$V_{w_1} = v_1 - \frac{V_{f_1}}{\tan 45^\circ} = 45{\cdot}4 - 7 = 38{\cdot}4 \text{ ft/sec}$$

Work done per pound of water

$$= \frac{V_{w_1} v_1}{g}$$

$$= \frac{38{\cdot}4 \times 45{\cdot}4}{32{\cdot}2} = 54{\cdot}2 \text{ ft-Lb}$$

$$\text{Manometric efficiency} = \frac{\text{gross lift}}{\text{theoretical work done}}$$

$$= \frac{28}{54{\cdot}2} = 51{\cdot}7 \text{ per cent}$$

From inlet velocity triangle,

$$\tan \theta = \frac{V}{v} = \frac{7}{22{\cdot}7} = 0{\cdot}308$$

Then $\theta = 17{\cdot}1^\circ$

Least speed of starting is when

$$h = \frac{v_1^2 - v^2}{2g}$$

$$\frac{\omega^2}{2g} (r_1^2 - r^2)$$

Hence $$28 = \frac{\omega^2}{2g} \left(\frac{10^2 - 5^2}{144} \right)$$

$$\omega = \sqrt{\frac{28 \times 64{\cdot}4 \times 144}{(100 - 25)}} = 58{\cdot}4 \text{ rad/sec}$$

$$\text{Least speed of starting} = \frac{60\omega}{2\pi}$$

$$= \frac{60 \times 58{\cdot}4}{2\pi} = 560 \text{ r.p.m.}$$

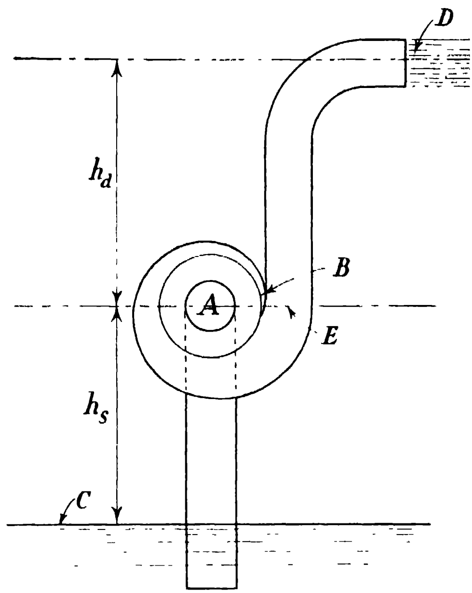


FIG. 334

23.4. Water Pressure in Centrifugal Pumps. The pressure of the water at any section of the stream in a pump may be found by applying Bernoulli's equation to various sections of the pump stream. Consider the pump and piping shown in Fig. 334. Let A be a point at the inlet edge of the impeller on the horizontal centre-line of the pump, let B be a point at the outlet edge of impeller on the same centre-line, let C be a point on the water surface in the sump, and D be a point at the outlet of the discharge pipe. If all losses are neglected, the total energy of the water will be the same at all those points provided that the work done by the impeller is added to these points which are in the stream on the exit side of the impeller.

Let h_s = height of pump above sump in feet,

h_d = height of outlet end of discharge pipe above pump in feet,

p_A = pressure in pounds per square foot absolute at A,
p_B = pressure in pounds per square foot absolute at B.

Apply Bernoulli's equation to points A and C; let water level in sump be datum and neglect all losses.

Total head at A = total head at C

Hence
$$h_s + \frac{V^2}{2g} + \frac{p_A}{w} = 34 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

From this equation p_A may be found.

If the frictional loss in the suction pipe is taken into account the equation becomes—

$$h_s + \frac{V^2}{2g} + \frac{p_A}{w} + h_f = 34 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where h_f is the head lost in friction in the suction pipe.

Next apply Bernoulli's equation to points A and B, using the centre-line of the pump as datum.

Total head at A = total head at B
— useful work done by impeller

Hence
$$\frac{V^2}{2g} + \frac{p_A}{w} = \frac{V_1^2}{2g} + \frac{p_B}{w} - \left(\frac{V_{w_1}v_1}{g} \times \text{eff.}\right) \quad . \quad (3)$$

From this equation p_B may be found.

Next apply Bernoulli's equation to points B and D, using the centre-line of the pump as datum.

Total head at B = total head at D + losses

Hence
$$\frac{V_1^2}{2g} + \frac{p_B}{w} = 34 + h_d + \frac{v_d^2}{2g} + \text{loss in diffuser} + h_f \,. \quad (4)$$

where h_f is the head lost in friction in the delivery pipe.
From this equation the loss in the diffuser may be found.

Theoretical head saved by fitting diffuser to pump

$$= \frac{V_1^2}{2g} - \frac{v_d^2}{2g}$$

Efficiency of diffuser

$$= \frac{\text{theoretical head saved} - \text{loss in diffuser}}{\text{theoretical head saved}}$$

EXAMPLE 2

A centrifugal pump has a total lift of 50 ft from well to delivery tank. The wheel is 5 ft above the well water surface. The velocity of delivery from the uptake is 5 ft/sec; the radial velocity of flow through the wheel is 10 ft/sec; the tangent to the vane at exit from the wheel makes an angle of 120° with the direction of motion; the water enters the wheel radially. Find (1) the velocity

of the wheel at exit; (2) the pressure head at exit from the wheel; (3) the velocity head at exit from the wheel; (4) the desirable direction for the fixed guide vanes. Neglect friction and other losses. (*Lond. Univ.*)

$$\text{Total head} = 50 + \frac{v_d^2}{2g}$$

$$= 50 + \frac{5^2}{2g} = 50{\cdot}39 \text{ ft}$$

Referring to velocity diagrams in Fig. 333,

$$\phi = 180° - 120° = 60°$$

$$V_{w_1} = v_1 - \frac{10}{\tan\phi} = v_1 - 5{\cdot}77$$

(1) $$\frac{V_{w_1}v_1}{g} = H + \frac{v_d^2}{2g}$$

or $$\frac{v_1(v_1 - 5{\cdot}77)}{g} = 50{\cdot}39$$

from which $$v_1 = 43{\cdot}23 \text{ ft/sec}$$

(2) $$V_{w_1} = v_1 - \frac{10}{\tan 60°}$$

$$= 43{\cdot}23 - 5{\cdot}77$$

$$= 37{\cdot}46 \text{ ft/sec}$$

$$V_1 = \sqrt{V_{w_1}^2 + V_{f_1}^2}$$

$$= \sqrt{37{\cdot}46^2 + 10^2}$$

$$= 38{\cdot}8 \text{ ft/sec}$$

Apply Bernoulli's equation to the pump at outlet and to the outlet end of delivery pipe, and taking the level of the centre-line of the pump as datum,

$$H_p + \frac{V_1^2}{2g} = 50 + \frac{v_d^2}{2g} - 5$$

where H_p = pressure head at vane outlet.

Therefore, $$H_p = 50 + 0{\cdot}39 - 5 - \frac{(38{\cdot}8)^2}{2g}$$

$$= 50 + 0{\cdot}39 - 5 - 23{\cdot}3$$

$$= 22{\cdot}1 \text{ ft of water}$$

(3) $$\frac{V_1^2}{2g} = \frac{(38{\cdot}8)^2}{2g} = 23{\cdot}3 \text{ ft of water}$$

(4) The fixed guide vanes will be parallel to the absolute velocity of water at outlet, i.e. will be parallel to V_1. From velocity diagram at outlet,

$$\tan \beta = \frac{V_{f_1}}{V_{w_1}}$$

where β = inclination of guide vanes. Then,

$$\tan \beta = \frac{10}{37{\cdot}46} = 0{\cdot}267$$

from which $\beta = 15°$

23.5. The Specific Speed of a Centrifugal Pump. There are two definitions in use for the specific speed of a centrifugal pump. One is based on unit quantity of discharge; the other is based on unit power, as in water turbines (§ 22.7). Each method will be dealt with in turn.

1. Specific Speed Based on Unit Quantity. This specific speed of a centrifugal pump is the speed at which a geometrically similar pump would deliver 1 gal of water per minute under a head of 1 ft. It may be found by applying the principle of similarity to centrifugal pumps; the method is the same as that used for finding the specific speed of water turbines in § 22.7. The specific speed is useful as an index for denoting the type of pump; the value varies between 500 and 8,000 for a single impeller.

In working out the equation for the specific speed the assumption is made that all pumps of a given type are geometrically similar; then all linear dimensions will be in proportion to the diameter of the impeller. Also, the velocity diagrams for all pumps of this type are assumed to be similar, and all velocities are proportional to the square root of the total head.

Using the same notation as used for turbines (§ 22.2),

let d = external diameter of impeller,
n = speed in revolutions per minute,
n_s = specific speed in revolutions per minute.
h = total head or lift in feet,
Q = discharge in gallons per minute.

Then $$b \propto d$$

Since $v = \omega d/2$ and $\omega \propto n$,

$$v \propto nd$$

or $$d \propto \frac{v}{n}$$

But $$v \propto \sqrt{h}$$

Hence $$d \propto \frac{\sqrt{h}}{n} \qquad \text{. . . . . (5)}$$

Now $$Q \propto \text{area of flow} \times \text{vel. of flow}$$

that is $$Q \propto \pi d b \times V_f$$

But $$V_f \propto \sqrt{h}$$

Hence $$Q \propto d^2\sqrt{h}$$

Substituting for d from eq. (5),

$$Q \propto \frac{h}{n^2} \times \sqrt{h}$$

that is $$Q \propto \frac{h^{3/2}}{n^2} \quad . \quad . \quad . \quad . \quad (6)$$

Hence $$n \propto \frac{h^{3/4}}{\sqrt{Q}}$$

This may be written, $n = k\,\dfrac{h^{3/4}}{\sqrt{Q}}$

where k is a constant.

When h equals 1 ft and Q is 1 gal/min it will be noticed that $n = k$, and also $n = n_s$ from definition of specific speed.

Hence, $$k = n_s = \frac{n\sqrt{Q}}{h^{3/4}} \quad . \quad . \quad . \quad . \quad (7)$$

2. Specific Speed Based on Unit Power. This is defined as the speed of a geometrically similar pump when absorbing 1 h.p. and working under a head of 1 ft.

Let P = horse-power required to drive pump considered.

Then $$P = \frac{wQh}{550}$$

Substituting for Q from eq. (6),

$$P \propto \frac{h^{5/2}}{n^2}$$

Hence $$n = \frac{k_1 h^{5/4}}{P^{1/2}}$$

where k_1 is a constant for the given pump.

When $h = 1$ and $P = 1$, $n = n_s = k_1$.

Hence $$k_1 = n_s = \frac{nP^{1/2}}{h^{5/4}} \quad . \quad . \quad . \quad . \quad (8)$$

When stating the specific speed of a pump it should be indicated on which definition it is based.

estimate the performance of the pump under any condition of working at the speed for which it was designed. Characteristic curves of a single impeller low-lift pump are shown in Fig. 335; in Fig. 336 are shown the characteristic curves of a multi-stage high-lift pump. Both these sets of curves were taken from Worthington pumps, from data supplied by the makers. The dotted vertical line

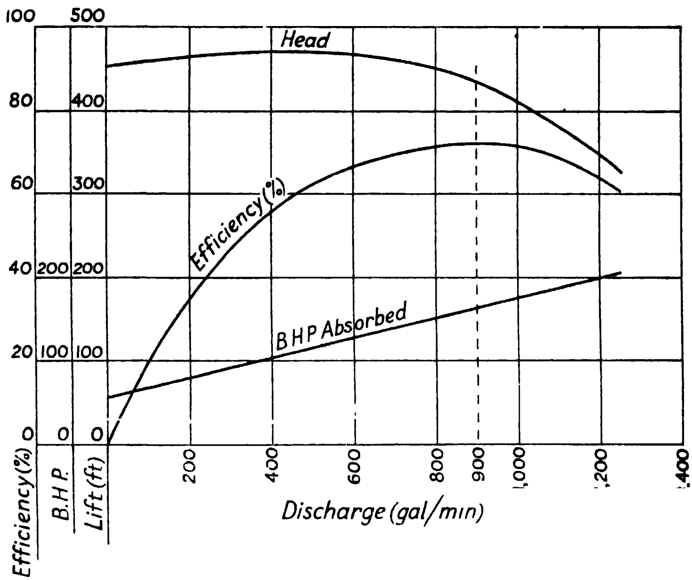


FIG. 336. PERFORMANCE OF MULTI-STAGE HIGH-LIFT PUMP
Output, 900 gal/min: head, 430 ft: speed, 1,450 r.p.m.
(*Courtesy of Worthington-Simpson, Ltd.*)

on each diagram is drawn through the point of maximum efficiency; from the points of intersection of this line with the other curves, the best conditions of running may be read off.

23.8. The Least Diameter of Impeller. It is usual to make the outside diameter of an impeller to be twice the inner diameter. On this assumption, it is possible to obtain an expression for the minimum diameter of an impeller which will enable it to start pumping when running at its normal speed. The solution is based on the result of § 23.3 which showed that, for the pump to start pumping, the centrifugal head must equal the actual lift.

From the equation of § 23.3,

$$\text{Actual lift} = h - \frac{v_1^2}{2g} \quad \frac{v^2}{2g}$$

As $v = \omega r$ and $v_1 = \omega r_1$,

$$h = \frac{\omega^2}{2g}(r_1^2 - r^2)$$

Let d_1 = outer diameter of impeller,

d = inner diameter of impeller.

Then
$$h = \frac{\omega^2}{8g}(d_1^2 - d^2)$$

But
$$d_1 = 2d$$

Hence
$$h = \frac{3\omega^2 d_1^2}{32g}$$

from which
$$d_1 = \frac{18{\cdot}54}{\omega}\sqrt{h} \text{ ft}$$

But
$$\omega = \frac{2\pi n}{60}$$

where n = no. of r.p.m. Hence

$$d_1 = \frac{18{\cdot}54 \times 60}{2\pi n} \times \sqrt{h}$$

$$= \frac{177\sqrt{h}}{n} \text{ ft}$$

$$2{,}120\sqrt{h} \text{ in}$$

Assuming a manometric efficiency of 0·75, the actual lift h will equal 0·75 of the theoretical lift H. Then

$$d_1 = \frac{1{,}840\sqrt{H}}{n} \text{ in.}$$

This equation is used in practice for the design of impellers; the outside diameter should be at least this amount, otherwise the impeller will be unable to start pumping at its normal speed.

23.9. The Design of a Turbine Pump. The following is a rough outline of the design of the piping, impeller, and diffuser of a turbine pump, and is based on the matter already dealt with in this chapter. It is assumed that the required discharge, actual lift and speed are

the absolute velocities V and V_1 can be obtained from the velocity diagrams; the lengths of the path of the water at various points in the pump may be estimated from an existing pump of a similar design.

v_s is plotted at the point where the suction pipe is attached to the pump casing.

V is plotted at the beginning of the blade.

V_1 is plotted at the tip of the blade.

v_d is plotted at the point where the delivery pipe is attached to the casing.

In a well-designed pump, the absolute velocity of the water should increase smoothly from the suction pipe to the outlet of the impeller blade; it should then fall smoothly in the diffuser until the delivery pipe is reached, as shown in Fig. 337. Any abrupt change in the absolute velocity will cause a loss of head and should be avoided. If, after plotting these velocities, a suitable curve is not obtained, the assumed values of θ and ϕ must be altered.

The number of blades in the impeller vary with the size; six would be sufficient for a small pump and twelve for a large pump.

THE DIFFUSER. The diffuser should have about the same number of blades as the impeller, and, as these blades are at rest, the relative velocity of the water to the blade will be the absolute velocity of the water. Hence, in order that the water will glide over the diffuser blade without shock, the blade at inlet of diffuser must be parallel to the absolute velocity V_1 of the water leaving the impeller; that is, to the angle β (Fig. 333). The water will glide over the diffuser blade, and, as the area of flow becomes larger with the increased radius of the diffuser, the velocity will become smaller. The water will flow from the diffuser blade into the whirlpool chamber in a direction parallel to the diffuser blade at outlet, and, as the flow of the water in the whirlpool chamber is circumferential it follows, therefore, that the diffuser blade angle at outlet should be as small as possible, which is about 10° to 15°. Hence, assume an angle of 10° for the diffuser blade at outlet.

After leaving the diffuser blade the water will pass through the whirlpool chamber into the discharge pipe. The object of the whirlpool chamber is to collect the water from the diffuser blades and to provide a passage to the discharge pipe; hence, it is spiral in shape owing to its cross-sectional area increasing uniformly up to the discharge pipe diameter; this allows for the increasing volume of water flowing through it. The full reduction of velocity from V_1 to v_d takes place in the diffuser, but as there is an unavoidable loss of about 2 ft/sec in the whirlpool chamber, the water should leave the diffuser blade with a circumferential velocity of $v_d + 2$. Then the outlet velocity triangle for the diffuser blade will be as shown in Fig. 338; it is a 10° right-angled triangle with a base of $v_d + 2$. The hypotenuse represents the actual velocity when

leaving the blade; the radial component of this will be the velocity of the flow V_{f_2} which is lost in shock in the whirlpool chamber.

Let b_2 = breadth of diffuser at outlet, in feet,

d_2 = diameter of diffuser at outlet, in feet,

Then $$Q = \pi d_2 b_2 V_{f_2}$$

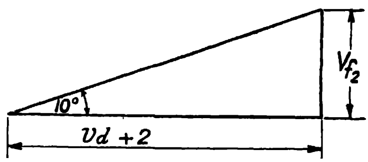


FIG. 338

From this equation b_2 can be found if d_2 is first assumed, as V_{f_2} is already known.

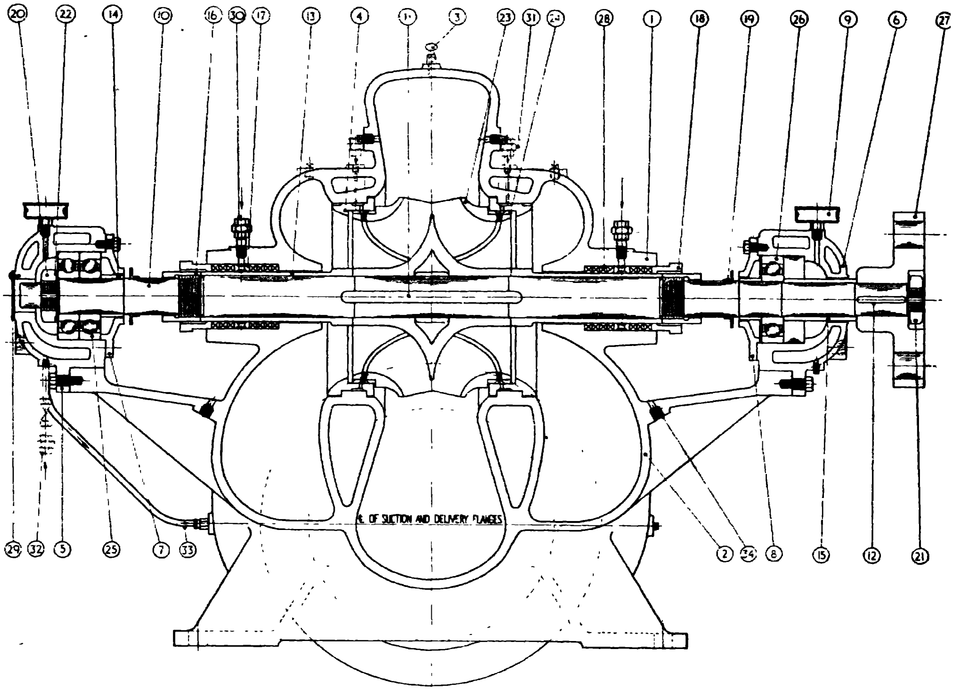
As a first attempt, assume d_2 to be 6 in. greater than d_1; if this does not give a suitable value for b_2 another value for d_2 must be chosen.

A view of a single impeller Worthington-Simpson pump, having double entry, is shown in Fig. 339.

23.10. The Multi-stage Pump. A single impeller will produce a head of not more than 120 ft; if a larger head than this is required other impellers are fitted in series, so that the discharge from the first impeller is guided into the inlet of the second impeller. This is repeated with the third impeller, and so on, until the required head is reached; each impeller will increase the water pressure by the same amount. A pump of this type is called a multi-stage pump, and may be a two-stage, three-stage, etc., according to the number of impellers fitted in the casing. A view of a four-stage pump is shown in Fig. 340.

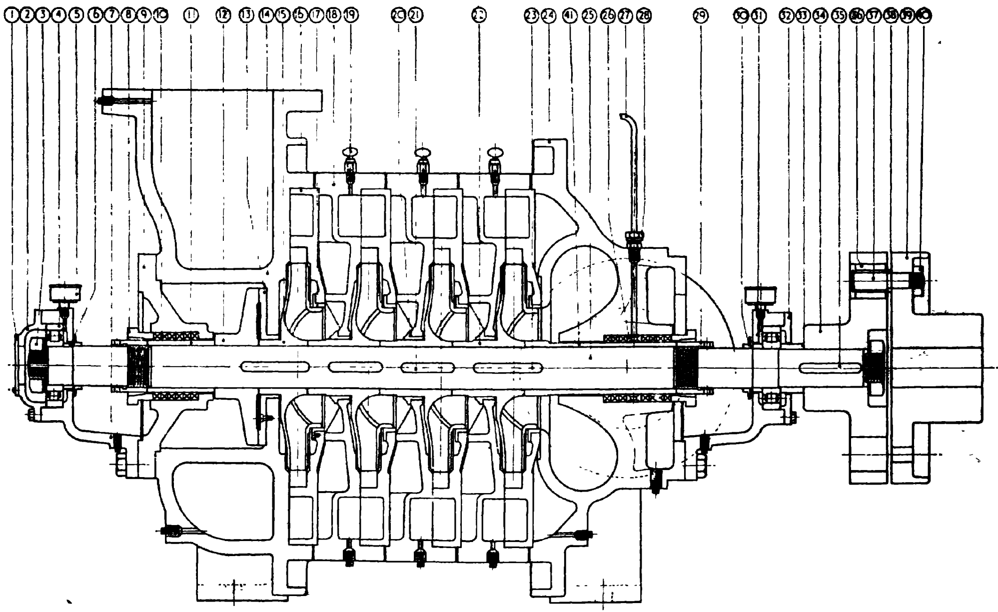
All the impellers are keyed to the same shaft, and usually all impellers and diffusers of one pump are identical; this has the advantage of reducing the labour in manufacture. The discharge from each diffuser is either circumferential or radial; this is collected by vanes attached to the casing which deflect the water into the eye of the next impeller. The last diffuser will discharge into the delivery pipe.

The design of an impeller and diffuser of a multi-stage pump is the same as for a single-stage pump (§ 23.9), the head used for design being the head per impeller.



1. Top half casing.
2. Bottom half casing.
3. $\frac{1}{4}$ in. P.T. air valve.
4. Casing renewable ring.
5. Far side bearing housing.
6. Driving side bearing housing.
7. Far side bearing cover.
8. Driving side bearing cover.
9. $\frac{3}{8}$ in. P.T. Stauffer lubricator.
10. Pump shaft.
11. Impeller feather.
12. Coupling feather.
13. Shaft sleeves.
14. Shaft collars.
15. Coupling distance piece.
16. Shaft nut.
17. Water seal cage.
18. Gland.
19. Water shield.
20. Thrust nut.
21. Coupling nut.
22. Locking plates.
23. Impeller.
24. Impeller hub ring.
25. H.J.T. 75 ball bearing.
26. H.J. 75 ball bearing.
27. 15 in. pump half-coupling.
28. $\frac{5}{8}$ in. graphited packing.
29. Name plate.
30. $2\frac{1}{2}$ in. B.S.P.T. union for external sealing water supply.
31. $2\frac{1}{4}$ in. B.S.P.T. union for flushing water supply.
32. $\frac{1}{2}$ in. B.S.P.T. union and valve for external cooling water supply to bearing.
33. Cooling water return to pump suction.
34. $\frac{1}{2}$ in. B.S.P.T. drain.

FIG. 339. SECTIONAL ARRANGEMENT OF HORIZONTAL CENTRIFUGAL PUMP
(*Courtesy of Worthington-Simpson, Ltd.*)



1. Nameplate.
2. Far side BRG. cover.
3. Shaft nut.
4. 2½ in. M.R.J. roller BRG.
5. Lubricator ¼ in. B.S.P.T.
6. Shaft collars.
7. Bearing bracket (far side).
8. Gland.
9. Balance chamber.
10. Packing.
11. Far side shaft sleeve.
12. Balance disc.
13. Balance disc face.
14. Delivery cover.
15. Impeller (last stage).
16. Guide vane.
17. Impeller renewable ring.
18. Inner casing.
19. Air valve.
20. Inner casing ren bush.
21. Impeller key.
22. Impeller.
23. Impeller key.
24. Suction cover.
25. Pump shaft.
26. Water seal cage.
27. Water seal tube.
28. Water seal union.
29. Sleeve nut.
30. Water shield.
31. D.S. bearing bracket.
32. D.S. bearing cover.
33. Coupling distance piece.
34. Pump half-coupling.
35. Coupling key.
36. Coupling pad.
37. Coupling pin.
38. Collar.
39. Motor half-coupling.
40. Coupling nut.
41. D.S. shaft sleeve.

FIG. 340. SECTIONAL ARRANGEMENT OF 8 IN. FOUR-STAGE CENTRIFUGAL PUMP
(*Courtesy of Worthington-Simpson Ltd.*)

EXERCISES 23

In the following examples assume the liquid pumped to be water, unless otherwise stated.

1. A centrifugal pump is required to deliver 6,300 gal of water per minute, against a head of 20 ft, at a speed of 600 r.p.m. Assuming that all the velocity head is lost, and that the actual head is 75 per cent of the theoretical head, find the diameter and breadth of the impeller at outlet. The velocity of flow, taken as constant, is 10 ft/sec, and the blades are curved back 30° to the tangent at outlet. Also determine the inlet blade angles, if the inlet diameter is made half the outlet diameter. (*Lond. Univ.*)
Ans. $d = 1{\cdot}25$ ft; $b = 5{\cdot}15$ in.; $\theta = 27°$.

2. A centrifugal pump is employed to pump water from a river into a canal. The pump is fixed with its centre at a height of 20 ft above the level of the surface of the water in the river, and the mouth of the delivery pipe is 30 ft above the level of the surface of the water in the river. The velocity of flow through the delivery pipe is 5 ft/sec. If the angle made by the tangent to the blade with the tangent to the wheel at the discharge edge is 125°, and if the radial velocity of flow through the wheel is 5 ft/sec, determine (1) the pressure head at the inlet circumference of the wheel, and (2) the pressure at the outlet circumference of the wheel. (*Lond. Univ.*)
Ans. (1) 13·61 ft of water (abs.); (2) 30·39 ft of water (abs.).

3. A centrifugal pump is placed with the centre of the impeller at a height of 12 ft above the water in the suction well. The suction pipe is 5 in. in diameter, and the discharge is 350 gal/min. The total head through which the water is lifted is 75 ft. The vanes of the impeller at exit are set back and make an angle of 150° with the tangent to the wheel. The radial velocity at exit from the wheel is 10 ft/sec, and the efficiency of the pump is 70 per cent. Determine (*a*) the velocity of the rim of the wheel; (*b*) the pressures at the inlet and outlet of the wheel, on the assumption that the whole loss of head takes place after the water leaves the wheel. (*Lond. Univ.*)
Ans. (*a*) 68·86 ft/sec; (*b*) 21·27 and 86·3 ft of water (abs.).

4. A centrifugal pump having a wheel 1 ft outside diameter rotates at 1,000 r.p.m. The vanes are radial at exit and are 3 in. wide. The velocity of radial flow through the wheel is 10 ft/sec. The velocities in the suction and delivery pipes are 8 and 5 ft/sec respectively. Neglecting frictional losses, determine (1) the height through which the pump lifts; (2) the horse-power of the pump. (*Lond. Univ.*) *Ans.* (1) 84·61 ft; (2) 75·6 h.p.

5. A centrifugal pump wheel is 20 in. external and 10 in. internal diameter. It runs at 950 r.p.m. The vanes are set back at an angle of 35° to the outer rim. If the radial velocity of the water through the wheel be maintained constant 6 ft/sec, find the angle of the vanes at inlet, the velocity and direction of the water at outlet, and the work done by the wheel per pound of water. (*Lond. Univ.*) *Ans.* $8\frac{1}{2}°$; 74·4 ft/sec; $4\frac{1}{2}°$; 191 ft-Lb.

6. A centrifugal pump of 4 ft diameter runs at 200 r.p.m., and pumps 66·5 ft^3/sec, the average lift being 20 ft. The angle which the vanes make at exit with the tangent to the impeller is 26°, and the radial velocity of flow is 8 ft/sec. Determine the useful horse-power and the efficiency. Find also the lowest speed to start pumping against a head of 20 ft, the inner diameter of the impeller being 2 ft. (*Lond. Univ.*)
Ans. 151 h.p.; 60·6 per cent; 198 r.p.m.

7. Give a short account of the various methods which have been adopted to increase the efficiency of a centrifugal pump by altering the shape of the casing or chamber surrounding the impeller. (*Lond. Univ.*)

8. A centrifugal pump running at 390 r.p.m. discharges 4 ft^3/sec. The impeller is 10 in. in diameter at inlet and 21 in. at outlet; the inlet width is 5 in., and the outlet width $3\frac{1}{2}$ in. Neglecting friction losses and the thickness of the vanes, what is the head pumped against if the vanes at outlet are curved back to give a discharge angle of 28°? (*A.M.I.C.E.*) *Ans.* 34·4 ft.

9. A centrifugal pump has an impeller 4 ft in diameter, whose peripheral speed is 30 ft/sec. Water enters the eye of the pump radially and is discharged with a velocity whose radial component is 5 ft/sec. The vanes are curved backward at exit and make an angle of 30° with the periphery. If the pump discharges 120 ft^3/min, what will be the turning moment on the shaft? (*A.M.I.C.E.*) *Ans.* 166 Lb-ft.

10. The impeller of a centrifugal pump has an external diameter of 12 in. and an internal diameter of 6 in. If full of water, with the discharge pipe closed, what would be the difference of pressures at the outer and inner periphery, corresponding to a speed of 300 r.p.m.? (*A.M.I.C.E.*)
Ans. 2·87 ft of water.

11. A propeller pump 9 in. in diameter was found to be most efficient when delivering 2·8 ft^3/sec at 1,200 r.p.m. against a head of 18 ft of water. A similar pump is required to deliver 50 ft^3/sec at 700 r.p.m. Calculate the pump diameter and the head it will develop. (*Lond. Univ.*)
Ans. $d = 28{\cdot}2$ in.; $H = 59{\cdot}6$ ft.

12. A five-stage centrifugal pump consists of five identical impellers in series and has a total lift of 450 ft when running at 1,500 r.p.m.; its discharge under these conditions is 3,000 gal/min. The impeller diameter at outlet is 2 ft and is twice that at inlet, and the theoretical efficiency of the pump is 72 per cent. Assuming a constant velocity of flow of 30 ft/sec throughout, and neglecting the energy lost at discharge, calculate (i) the angles of the impeller blade at inlet and outlet, (ii) the angle of the diffuser blade at the inlet of the diffuser, (iii) the blade widths of the impeller at inlet and outlet if the blade area factor is 0·7, and (iv) the theoretical horse-power required to drive the pump. (*Lond. Univ.*)
Ans. (1) 20° 55′ and 12° 50′; (2) 49·5°; (3) 1·46 in. and 0·73 in.; (4) 567·5 h.p.

13. In order to predict the performance of a large centrifugal pump, a scale model of $\frac{1}{4}$ size is constructed and tested under a head of 25 ft, whilst running at its best speed of 500 r.p.m. It was found that 10 h.p. was required to drive the model under these conditions.

Calculate the speed and horse-power required by the large pump when pumping under a head of 144 ft. What is the ratio of the quantities pumped by the large pump and by the model under these conditions? (*Lond. Univ.*)
Ans. 300 r.p.m.; 2,210 h.p.; 38 : 1.

14. In order to predict the performance of a large centrifugal pump, a geometrically similar model was made having linear dimensions one-fifth of those of the large pump. The model was tested at its best speed of 1,250 r.p.m. and was found to require 2·87 h.p. when pumping under a head of 30 ft; the quantity discharged being 37·9 ft^3/min.

Estimate from these results the speed, horse-power required, and the quantity delivered, for the large pump which is to be designed for a head of 120 ft of water. (*Lond. Univ.*)
Ans. 500 r.p.m.; 574 h.p.; 1,895 ft^3/min.

15. A centrifugal pump delivers 30 gal of fuel oil per second against a pressure of 30 Lb/in.^{2}; the oil has a specific gravity of 0·85. The inner and outer diameters of the pump rotor are 9 in. and 18 in. respectively, and the corresponding widths of the rotor blades are 2 in. and 1 in. respectively. The speed of the rotor is 1,000 r.p.m. and the blade area coefficient is 0·9 at inlet and outlet.

If the manometric efficiency of the pump is 70 per cent, find the rotor blade angles at inlet and outlet, the diffuser blade angle at its inlet edge, and the theoretical horse-power required to drive the pump. (*Lond. Univ.*)

Ans. $\theta = 19{\cdot}1°$; $\phi = 24°$; $\beta = 15{\cdot}85°$; 54 h.p.

CHAPTER 24

FLOW OF GASES THROUGH DUCTS AND TURBINE BLADES

24.1. Introduction. Equations for the high-speed, frictional flow of gases through lagged, tapering ducts, of non-circular cross-section, can be deduced by the method used for a circular pipe in Chapter 18. In this case, the length of the duct should be divided into short sections and the equation of flow can then be applied, successively, to each short length in turn. The process must commence at the inlet end of the duct and continue throughout its length until the outlet end is reached. By this method, the final velocity, temperature and pressure of the gas can be calculated providing the inlet conditions are known.

The problem of a gas flowing through the blade passages of the guide and rotor blades of a turbine, or the blade passages of the impeller and diffuser of a rotary compressor, can be solved by this method. Pressure jumps are liable to occur within the blade passages, and each section must also be checked for a decrease of entropy.

The condition of the gas should be plotted on its temperature-entropy diagram, as the flow proceeds from section to section. It should be remembered that no decrease in entropy can occur during frictional reheating in a lagged duct; if it is found to occur in the solution, it denotes a pressure jump at a previous section. A pressure jump occurring in the blade passages of a gas turbine denotes a bad design of passage and should be avoided.

It has been found that the fluid flow through a rectangular-sectioned duct approximates to the same Reynolds number condition of flow as in a round pipe, if the hydraulic mean depth of the rectangular duct is the same as that of the pipe.

Consider a rectangular duct of breadth b and depth t. Let

$$\begin{aligned} d_e &= \text{equivalent diameter of rectangular duct} \\ A &= \text{area of cross-section of duct} \\ P &= \text{length of perimeter of its cross-section} \\ &= 2(b + t) \end{aligned}$$

Then,

hydraulic mean depth of duct

= hydraulic mean depth of round pipe

or $$\frac{A}{P} = \frac{d_e}{4} \quad (\S\,7.6)$$

Hence $$\frac{4A}{P}$$

and $$R_e \text{ for duct} = \frac{\rho v d_e}{\eta}$$

This value of R_e can be used for obtaining the frictional coefficients, f or C_f, from the curves obtained from pipe flow experiments.

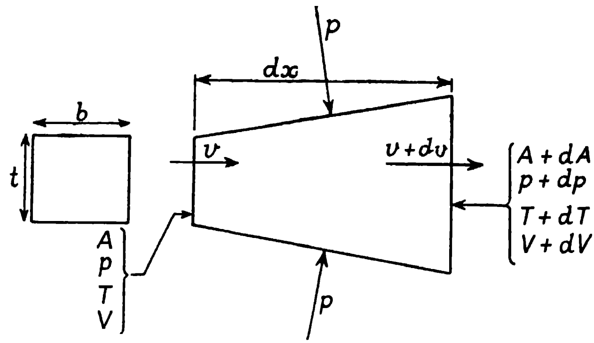


FIG. 341

24.2. Frictional Flow through a Lagged Tapering Duct. The problem of the high-speed flow of a gas through a lagged, non-circular, tapering duct can be solved by the application of the four fundamental equations of fluid flow—

(1) the momentum equation,
(2) the energy equation,
(3) the gas equation,
(4) the equation of continuity of flow.

Consider the short length dx of the tapering duct, of rectangular cross-section, shown in Fig. 341. Let A, p, v, T and V represent the cross-sectional area, pressure, velocity, absolute temperature and specific volume of the gas stream at the left-hand end, and let these increase to $A + \mathrm{d}A$, $p + \mathrm{d}p$, $v + \mathrm{d}v$, $T + \mathrm{d}T$ and $V + \mathrm{d}V$ at the right-hand end.

Let P = length of perimeter of cross-section of duct
$= 2(b + t)$, for a rectangular duct.

Applying eq. (2), chapter 15, for the frictional drag on the gas stream due to the sides of the duct,

$$D_f = \frac{C_f \times \rho \times \text{surface area} \times v^2}{}$$

$$\frac{C_f}{2} \times \frac{w}{g} \times P\,\mathrm{d}x \times v^2$$

Substituting for $w = 1/V$,
$$D_f = \frac{C_f P v^2 \,\mathrm{d}x}{2Vg} \quad . \quad . \quad . \quad (1)$$

As shown in § 16.11,
$$C_f = \text{Darcy's } f$$

Resolving the horizontal forces on the fluid stream of Fig. 341,

$$pA + p\,\mathrm{d}A - [(p + \mathrm{d}p)(A + \mathrm{d}A)] = \frac{wAv\,\mathrm{d}v}{g} + D_f$$

Substituting for D_f from eq. (1), and reducing,

$$-\mathrm{d}p = \frac{v\,\mathrm{d}v}{Vg} + \frac{C_f P v^2\,\mathrm{d}x}{2AVg} \quad . \quad (2)$$

Applying the energy equation to each end of the short length of the duct,

$$T + \frac{v^2}{2gJc_p} = (T + \mathrm{d}T) + \frac{(v + \mathrm{d}v)^2}{2gJc_p}$$

from which,
$$-Jc_p\,\mathrm{d}T = \frac{v\,\mathrm{d}v}{g}$$

Hence
$$\mathrm{d}T = -\frac{v\,\mathrm{d}v}{Jc_p g} \quad . \quad . \quad . \quad (3)$$

Applying the gas equation to 1 pound of gas,

$$pV = RT$$

Differentiating,
$$p\,\mathrm{d}V + V\,\mathrm{d}p = R\mathrm{d}T$$

Hence,
$$-\mathrm{d}p = p\frac{\mathrm{d}V}{V} - R\frac{\mathrm{d}T}{V} \quad . \quad . \quad (4)$$

From the equation of continuity of flow,

$$W = \frac{Av}{V} = \text{constant}$$

Differentiating,
$$\frac{\mathrm{d}V}{V} = \frac{\mathrm{d}A}{A} + \frac{\mathrm{d}v}{v} \quad . \quad . \quad (5)$$

Substituting eq. (5) in eq. (4),

$$-\mathrm{d}p = p\frac{\mathrm{d}A}{A} + p\frac{\mathrm{d}v}{v} - \frac{R\,\mathrm{d}T}{V} \quad (6)$$

Equating eq. (2) and (6),

$$p\frac{\mathrm{d}A}{A} + p\frac{\mathrm{d}v}{v} - \frac{R\,\mathrm{d}T}{V} = \frac{v\,\mathrm{d}v}{Vg} + \frac{C_f P v^2\,\mathrm{d}x}{2AVg} \quad . \quad (7)$$

Substituting for $\mathrm{d}T$ from eq. (3),

$$p\frac{\mathrm{d}A}{A} + p\frac{\mathrm{d}v}{v} + \frac{Rv\,\mathrm{d}v}{VJc_p g} = \frac{v\,\mathrm{d}v}{Vg} + \frac{C_f P v^2\,\mathrm{d}x}{2AVg}$$

Multiplying throughout by V and substituting for $pV = RT$,

$$RT\left(\frac{dA}{A} + \frac{dv}{v}\right) + \frac{Rv\,dv}{Jc_pg} = \frac{v\,dv}{g} + \frac{C_fPv^2\,dx}{2Ag}$$

Substituting for $\frac{R}{Jc_p} = \frac{\gamma - 1}{\gamma}$,

$$RT\left(\frac{dA}{A} + \frac{dv}{v}\right) - \frac{v\,dv}{\gamma g} = \frac{C_f(P/A)v^2\,dx}{2g} \quad . \quad . \quad . \quad (8)$$

Hence,

$$dv = \frac{\dfrac{C_f(P/A)v^2\,dx}{2g} - RT\dfrac{dA}{A}}{\dfrac{RT}{v} - \dfrac{v}{\gamma g}} \quad . \quad (9)$$

Eq. (9) is known as the momentum equation and can be applied to short lengths of all ducts, of any cross-sectional shape and of any taper.

In order to apply these equations to a short length of a duct, let suffix 1 apply to the inlet end and suffix 2 to the outlet end (Fig. 341). The values of the terms v, P, A and T should then represent the mean values over the length dx. Then,

$$T = \tfrac{1}{2}(T_1 + T_2)$$
$$v = \tfrac{1}{2}(v_1 + v_2)$$
$$A = \tfrac{1}{2}(A_1 + A_2)$$
$$P = \tfrac{1}{2}(P_1 + P_2)$$

Also,

$$dA = A_2 - A_1$$
$$dv = v_2 - v_1$$
$$dT = T_2 - T_1$$

dx = length of short section of duct considered

It should be noticed that if the duct is contracting in cross-sectional area, in the direction of flow, dA is negative.

The value of C_f, or f, used in a diverging duct should be larger than that given by the round pipe experiments, as there are additional losses due to eddies caused by the expanding cross-section.

Applying the energy equation to the two ends of the short length of duct, and assuming that all frictional work be utilized in reheating the gas,

$$c_pT_1 + \frac{v_1^2}{2gJ} = c_pT_2 + \frac{v_2^2}{2gJ}$$

Hence,

$$v_2^2 = v_1^2 + 2gJc_p(T_1 - T_2)$$

or

$$v_2^2 = v_1^2 - 2gJc_p\,dT \quad . \quad . \quad (10)$$

For atmospheric air, this becomes

$$v_2^2 = v_1^2 - 12{,}020\,\mathrm{d}T \quad . \quad . \quad (11)$$

In a problem on a duct of known dimensions, if p_1, v_1, and T_1 are known, the values of v_2 and T_2 can be calculated from eq. (9) and (10). The method of solution is by plotting, and will be demonstrated in § 24.3.

Then, weight of flow per sec $= W = \dfrac{A_1 v_1}{V_1}$

But

$$V_1 = \frac{RT_1}{144p_1}$$

Hence,

$$W = \frac{144p_1 A_1 v_1}{RT_1} \quad . \quad . \quad . \quad (12)$$

In the same way,

$$p_2 = \frac{WRT_2}{144A_2 v_2} \quad . \quad . \quad . \quad (13)$$

24.3. Application of Equations to a Rectangular Tapering Duct. The method of solution of the equations of § 24.2 will now be demonstrated in the following worked-out example.

EXAMPLE 1

A rectangular duct, 2 ft long and of constant depth of 1 in., tapers in width from 2 in. at the inlet end to 3 in. at the outlet. Air at a pressure of 40 Lb/in.^{2} and a temperature of 600°R enters the duct with a velocity of 500 ft/sec. Calculate the weight of air flowing per second and the velocity, temperature and pressure at a section 2 in. from the inlet end of the duct. $C_f = 0{\cdot}01$.

The following are the required dimensions of the duct; suffix 1 applies to the inlet end and suffix 2 to the outlet end of the 2 in. length of the duct.

$$\begin{aligned} A_1 &= (2 \times 1)/144 \\ &= 0{\cdot}01389 \text{ ft}^2 \\ A_2 &= 0{\cdot}01446 \text{ ft}^2 \\ \mathrm{d}A &= A_2 - A_1 \\ &= 0{\cdot}00057 \text{ ft}^2 \\ \text{Mean } A &= \tfrac{1}{2}(A_1 + A_2) \\ &= 0{\cdot}01417 \text{ ft}^2 \\ \frac{\mathrm{d}A}{A} &= \frac{0{\cdot}00057}{0{\cdot}01417} \\ &= 0{\cdot}0402 \end{aligned}$$

$$\begin{aligned} \text{Mean } P &= \frac{2 \quad + 2\tfrac{1}{24})}{12} \text{ ft} \\ &= 0{\cdot}508 \text{ ft} \\ \text{Mean } \frac{P}{A} &= \frac{0{\cdot}508}{0{\cdot}01417} = 36 \\ \mathrm{d}x &= 2 \text{ in.} = 0{\cdot}1667 \text{ ft} \\ C_f &= 0{\cdot}01 \\ v_1 &= 500 \text{ ft/sec} \\ T_1 &= 600°\text{R} \\ p_1 &= 40 \text{ Lb/in.}^2 \end{aligned}$$

First apply the energy equation (eq. (11)).

$$\begin{aligned} v_2^2 &= v_1^2 - 12{,}020\mathrm{d}T \\ &= 250{,}000 - 12{,}020\mathrm{d}T \qquad \text{. . . . (14)} \end{aligned}$$

Assume a value for T_2, then $\mathrm{d}T = T_2 - T_1$. Using this value of $\mathrm{d}T$, calculate the value of v_2 from the energy equation. Then $\mathrm{d}v = v_2 - v_1$ for the assumed value of T_2.

Assume $\quad T_2 = 603°\text{R}$

Then $\quad \mathrm{d}T = 603 - 600 = 3°\text{R}$

Substituting this value of $\mathrm{d}T$ in eq. (14),

$$v_2^2 = 250{,}000 - 36{,}060 = 213{,}940$$

Hence, $\quad v_2 = 463 \text{ ft/sec}$

Then $\quad \mathrm{d}v = 463 - 500 = -37 \text{ ft/sec}$

$$\text{Mean } T = \tfrac{1}{2}(600 + 603) = 601{\cdot}5°\text{R}$$

$$\text{Mean } v = \tfrac{1}{2}(500 + 463) = 481{\cdot}5 \text{ ft/sec}$$

Next apply the momentum equation (eq. (9)) using the above values of T and v.

From eq. (9),

$$\mathrm{d}v = \frac{C_f\left(\dfrac{P}{A}\right)\dfrac{v^2}{2g}\,\mathrm{d}x - RT\dfrac{\mathrm{d}A}{A}}{\dfrac{RT}{v} - \dfrac{v}{\gamma g}}$$

where T and v are the mean values over the length considered. Then

$$\mathrm{d}v = \frac{\left(0{\cdot}01 \times 36 \times \dfrac{v^2}{2g} \times 0{\cdot}1667\right) - (53{\cdot}3 \times 0{\cdot}0402T)}{\dfrac{53{\cdot}3T}{v} - \dfrac{v}{1{\cdot}4 \times 32{\cdot}2}}$$

$$= \frac{9{\cdot}34 \times 10^{-4}v^2 - 2{\cdot}185T}{\dfrac{53{\cdot}3T}{v} - \dfrac{v}{45{\cdot}1}} \qquad \text{. . (15)}$$

Substituting the values of T and v in eq. (15),

$$\begin{aligned} \mathrm{d}v &= \frac{(9{\cdot}34 \times 10^{-4} \times 232{,}000) - (2{\cdot}185 \times 601{\cdot}5)}{\left(\dfrac{53{\cdot}3 \times 601{\cdot}5}{481{\cdot}5}\right) - \left(\dfrac{481{\cdot}5}{45{\cdot}1}\right)} \\ &= -19{\cdot}6 \text{ ft/sec} \end{aligned}$$

The above calculation was then repeated for other assumed values of T_2, and the results are shown entered in the following table.

T_2 (assumed)	dT $(= T_2 - T_1)$	v_2 from eq. (14)	T (mean)	v (mean)	dv from eq. (14)	dv from eq. (15)
603	3	463	601·5	481·5	− 37	− 19·6
602	2	475	601	487·5	− 25	− 20·0
601	1	488	600·5	494	− 12	− 20·1

The values of dv from eq. (14) and the values of dv from eq. (15) are then plotted as separate curves on a base representing T_2. These represent the energy equation and momentum equation

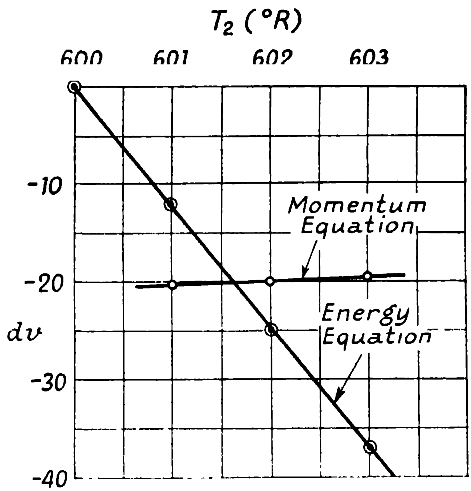


FIG. 342

results respectively. The point of intersection of the two curves gives the correct value of dv and T_2 which satisfies both the energy and momentum equations. The two curves are shown plotted in Fig. 342 from which the correct values are found to be

$$dv = -20 \text{ ft/sec}$$
$$T_2 = 601{\cdot}6°\text{R}$$

Then,
$$v_2 = v_1 + dv$$
$$= 500 - 20 = 480 \text{ ft/sec}$$

From eq. (12),
$$W = \frac{144 p_1 A_1 v_1}{R T_1}$$
$$= \frac{144 \times 40 \times 0{\cdot}01389 \times 500}{53{\cdot}3 \times 600}$$
$$= 1{\cdot}25 \text{ Lb/sec}$$

From eq. (13),

$$p_2 = \frac{WRT_2}{144A_2v_2}$$

$$= \frac{1{\cdot}25 \times 53{\cdot}3 \times 601{\cdot}6}{144 \times 0{\cdot}01446 \times 480}$$

$$= 40{\cdot}1 \text{ Lb/in.}^2$$

If the whole length of the duct be divided into 12 consecutive 2-in. lengths, the above method can be applied to each short length in turn. The inlet condition of the air at the entrance to each section is the same as that of the outlet end of the previous section. By this method the final condition of the air at the outlet end of the duct is obtained.

In the above example the air was supplied at a subsonic speed and as the cross-section of the duct was expanding, the duct was acting as a diffuser and produced a reduction in velocity with an increase of pressure. Hence, no pressure jump could occur as the flow is subsonic throughout. If the flow had been supersonic, it would have been necessary to test each section for a decrease of entropy. If this was found to occur in the calculation, it would indicate that a pressure jump had taken place at a previous section.

24.4. Frictionless Flow of Gases through a Tapering Duct. The frictionless flow of gases through a stationary duct of any cross-sectional shape and of any taper can be solved by the application of the energy equation (eq. (10), § 24.2) and the momentum equation (eq. (9), § 24.2). A frictionless flow does not affect the energy equation because any energy lost due to friction reappears in the form of heat. Consider the duct of Fig. 341. From the momentum equation for a frictional flow (eq. (8), § 24.2),

$$RT\left(\frac{\mathrm{d}A}{A} + \frac{\mathrm{d}v}{v}\right) - \frac{v\,\mathrm{d}v}{\gamma g} - C_f\left(\frac{P}{A}\right)\frac{v^2\,\mathrm{d}x}{2g} = 0$$

If friction be neglected as small, the last term becomes zero; then the equation becomes

$$\frac{\mathrm{d}A}{A} + \frac{\mathrm{d}v}{v} = \frac{v\,\mathrm{d}v}{RT\gamma g} \quad . \quad . \quad (16)$$

But from eq. (3), § 24.2,

$$\frac{v\,\mathrm{d}v}{g} = -Jc_p\,\mathrm{d}T$$

$$= -R\left(\frac{\gamma}{\gamma - 1}\right)\mathrm{d}T$$

Substituting this value in eq. (16),

$$\frac{dA}{A} + \frac{dv}{v} = -\left(\frac{1}{\gamma - 1}\right)\frac{dT}{T}$$

Integrating between the limits A_2, v_2, T_2 and A_1, v_1, T_1,

$$\left[\log_e A\right]_{A_1}^{A_2} + \left[\log_e v\right]_{v_1}^{v_2} = -\left(\frac{1}{\gamma - 1}\right)\left[\log_e T\right]_{T_1}^{T_2}$$

Then $\log_e \dfrac{A_2}{A_1} + \log_e \dfrac{v_2}{v_1} = \log_e \left(\dfrac{T_1}{T_2}\right)^{\frac{1}{\gamma-1}}$

Hence, $\log_e \left(\dfrac{A_2 v_2}{A_1 v_1}\right) = \log_e \left(\dfrac{T_1}{T_2}\right)^{\frac{1}{\gamma-1}}$

Taking antilogs throughout,

$$\frac{A_2 v_2}{A_1 v_1} = \left(\frac{T_1}{T_2}\right)^{\frac{1}{\gamma-1}}$$

Substituting $v_1 = M_{a_1}\sqrt{\gamma g R T_1}$ (§ 19.13)

and $v_2 = M_{a_2}\sqrt{\gamma g R T_2}$

$$\frac{A_2}{A_1} = \frac{M_{a_1}}{M_{a_2}}\sqrt{\frac{T_1}{T_2}} \times \left(\frac{T_1}{T_2}\right)^{\frac{1}{\gamma-1}}$$

$$= \frac{M_{a_1}}{M_{a_2}}\left(\frac{T_1}{T_2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \quad . \quad . \quad . \quad . \quad (17)$$

But from eq. (43), § 19.13,

$$\frac{T_1}{T_2} = \frac{1 + \left(\frac{\gamma - 1}{2}\right) M_{a_2}^2}{1 + \left(\frac{\gamma - 1}{2}\right) M_{a_1}^2}$$

Substituting this value in eq. (17),

$$\frac{A_2}{A_1} = \frac{M_{a_1}}{M_{a_2}}\left[\frac{1 + \left(\frac{\gamma - 1}{2}\right) M_{a_2}^2}{1 + \left(\frac{\gamma - 1}{2}\right) M_{a_1}^2}\right]^{\frac{\gamma+1}{2(\gamma-1)}} \quad . \quad (18)$$

This is the same equation as eq. (46), § 19.13, which was deduced for a circular duct. Eq. (18) applies to any type of duct, of any shape and taper, subjected to a frictionless flow. It can also be applied direct to the whole length of the duct as the expresslón has been intregrated during its solution.

EXAMPLE 2

A uniformly tapering stationary duct, of rectangular cross-section, has a cross-sectional area of 0·954 × 10^{-4} ft² at its inlet end and enlarges to an area of 2·082 × 10^{-4} ft² at the outlet end. Air enters the duct at its sonic velocity; its temperature is 180°R and its pressure 47·9 Lb/in.² Neglecting frictional and other resistances, calculate the velocity, pressure and temperature of the air discharging from the duct, and the weight of air discharged per second. The duct is lagged, so that the flow may be assumed to be adiabatic.

$A_1 = 0{\cdot}954 \times 10^{-4}\ \text{ft}^2 \qquad p_1 = 47{\cdot}9\ \text{Lb/in.}^2 \qquad M_{a_1} = 1$

$A_2 = 2{\cdot}082 \times 10^{-4}\ \text{ft}^2 \qquad T_1 = 180°\text{R}$

From eq. (3), § 17.2,

$$v_1 = v_{s_1} = 49\sqrt{T_1} = 49\sqrt{180} = 658\ \text{ft/sec}$$

Applying eq. (18),

$$\frac{A_2}{A_1} = \frac{1}{M_{a_2}}\left[\frac{1 + \left(\frac{1{\cdot}4 - 1}{2}\right) M_{a_2}{}^2}{1 + \left(\frac{1{\cdot}4 - 1}{2}\right)}\right]^{\frac{1\cdot4+1}{2(1\cdot4-1)}}$$

or,

$$\frac{2{\cdot}082}{0{\cdot}954} = \frac{1}{M_{a_2}}\left[\frac{1 + 0{\cdot}2\ M_{a_2}{}^2}{1{\cdot}2}\right]^3$$

Then,

$$\sqrt[3]{2{\cdot}183\, M_{a_2}} = \frac{1 + 0{\cdot}2 M_{a_2}{}^2}{1{\cdot}2}$$

Hence,

$$1{\cdot}55 M_{a_2}{}^{1/3} - 0{\cdot}2 M_{a_2}{}^2 - 1 = 0$$

Solving this equation by plotting,

$$M_{a_2} = 2{\cdot}28$$

Using eq. (43), § 19.13,

$$\frac{T_1}{T_2} = \frac{1 + \left(\frac{\gamma - 1}{2}\right) M_{a_2}{}^2}{1 + \left(\frac{\gamma - 1}{2}\right) M_{a_1}{}^2}$$

Then

$$\frac{180}{T_2} = \frac{1 + (0{\cdot}2 \times 2{\cdot}28^2)}{1{\cdot}2}$$

from which

$$T_2 = 105{\cdot}8°\text{R}$$

As the flow is an adiabatic process,

$$\frac{T_1}{T_2} = \left(\frac{p_1}{p_2}\right)^{\frac{\gamma-1}{\gamma}}$$

Hence

$$\frac{180}{105{\cdot}8} = \left(\frac{47{\cdot}9}{p_2}\right)^{\frac{0\cdot4}{1\cdot4}}$$

from which
$$p_2 = 7{\cdot}5 \text{ Lb/in.}^2$$
$$\begin{aligned} v_{s_2} &= 49\sqrt{T_2} \\ &= 49\sqrt{105{\cdot}8} \\ &= 502 \text{ ft/sec} \end{aligned}$$

Then
$$\begin{aligned} v_2 &= M_{a_2} \times v_{s_2} \\ &= 2{\cdot}28 \times 502 \\ &= 1{,}148 \text{ ft/sec} \end{aligned}$$

Applying eq. (12), § 24.2,
$$\begin{aligned} W &= \frac{144 p_1 A_1 v_1}{R T_1} \\ &= \frac{144 \times 47{\cdot}9 \times 0{\cdot}954 \times 10^{-4} \times 6{\cdot}58}{53{\cdot}3 \times 180} \\ &= 0{\cdot}0451 \text{ Lb/sec} \end{aligned}$$

24.5. Converging-Diverging Rectangular Nozzle—Subsonic Flow at Throat. The equations of flow deduced in § 24.2 have been applied to the flow of air through the rectangular-sectioned converging-diverging nozzle shown in Fig. 343. This application has been arranged so that the flow through the throat is subsonic.

The centre-line of the nozzle was divided into 9 equal sections, each of length of $\frac{1}{8}$ in. The equations of flow were then applied to each section in turn, commencing at Section 0, the results obtained from one section being used for the inlet conditions at the next section. A known weight of air was assumed to flow through the nozzle, its initial temperature and pressure being known. The calculated results for each section are shown plotted in Fig. 343; curves showing variations of pressure, temperature, velocity and Mach number are plotted.

It will be noticed from the pressure curve that the pressure falls to a minimum just beyond the throat and then increases as it flows through the diverging cone. Hence, for a subsonic flow at the throat, the diverging cone acts as a diffuser and converts the kinetic energy back to a pressure increase.

Under subsonic flow the nozzle is acting in the same manner as a Venturi meter, and produces the same type of distribution of pressure, temperature and velocity.

24.6. Converging-Diverging Rectangular Nozzle—Sonic Flow at Throat. The equations of flow of § 24.2 have also been applied to the same converging-diverging nozzle of Fig. 343, but in this application the mass flow has been adjusted to produce sonic flow at the throat. The calculated results are shown plotted in Fig. 344.

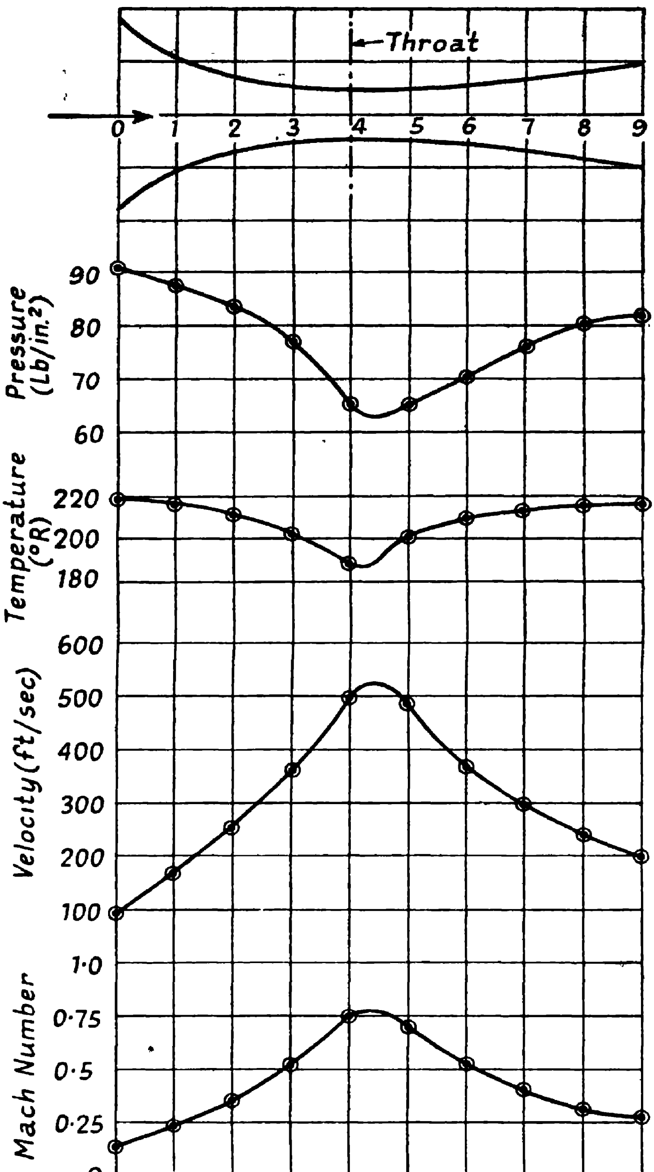


FIG. 343. CONVERGING-DIVERGING NOZZLE: SUBSONIC VELOCITY AT THROAT

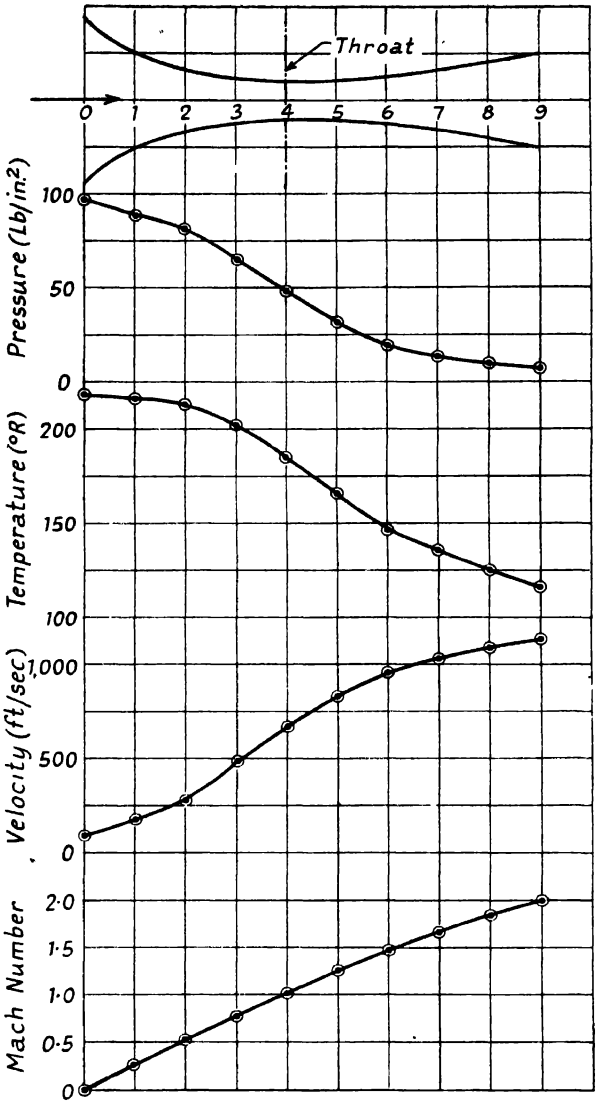


FIG. 344. CONVERGING-DIVERGING NOZZLE: SONIC VELOCITY AT THROAT

It will be noticed that when the flow is sonic at the throat, the diverging cone no longer acts as a diffuser. The pressure now continues to fall and the flow becomes supersonic, the velocity continuing to increase up to the discharge end of the nozzle. This type of flow would not be satisfactory if the nozzle were used as a Venturi meter. In a Venturi meter the flow at the throat must, consequently, be subsonic.

Pressure jumps are liable to occur in the diverging cone when the flow is supersonic.

24.7. Frictional Flow through Rotating Blade Passages. The equations of § 24.2 can be modified to apply to the moving blade passages of turbine rotors and the impellers of centrifugal compressors. As the blades are rotating, the relative velocities between the fluid and the blades must be used in the equations. Hence, v_1 and v_2 are replaced by V_{r_1} and V_{r_2}. The effect of the centrifugal force must also be taken into account.

In radial-flow rotors the effect of the centrifugal force tends to increase the relative velocity in an outward radial-flow type, and to decrease it in an inward radial-flow type. In an axial-flow type of rotor, there is no centrifugal force effect in the direction of fluid flow through the blade passage.

In order to solve the problem, the moving blade passage must be divided into short lengths along the centre-line of the space between two consecutive blades. Then, starting from the inlet end, the method of solution given in § 24.10 is applied to each short length in turn, up to the discharge end of the blade. The initial relative velocity at entry of the first moving section is obtained from the velocity diagram drawn for the inlet of the moving blade, as explained in Chapter 22.

Each short length of passage must be tested for pressure jumps by the method given in § 24.11. The state point of the gas at the outlet of each section should also be plotted on its temperature-entropy diagram before proceeding to the next section. If this denotes a decrease in entropy a pressure jump has probably occurred at a previous section, if the flow is supersonic.

The modifications to the equations of flow for a rotating blade passage will now be dealt with in §§ 24.8, 24.9 and 24.10.

24.8. Effect of Centrifugal Head on Fluid in Rotating Blade Passage. A rotating blade passage of the rotor of a turbine or compressor is shown in Fig. 345; it is rotating at a constant angular velocity of ω rad/sec.

Consider a short length of the passage at a mean radius of r ft and of length dr.

Let suffix 1 and suffix 2 apply to the inlet and outlet ends

respectively of the short length of passage considered, as shown in Fig. 346. Then

$$r = \tfrac{1}{2}(r_1 + r_2)$$

and

$$\mathrm{d}r = r_2 - r_1$$

Let A = mean cross-sectional area of short length of passage, in ft²

F_c = centrifugal force on gas in section considered

$$= \frac{(wA\,\mathrm{d}r)}{g}\,\omega^2 r$$

$$= \frac{wA\omega^2 r\,\mathrm{d}r}{g} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

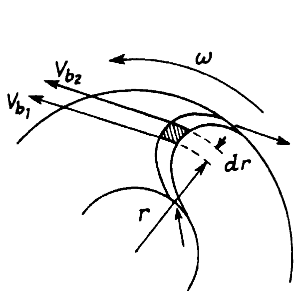


Fig. 345

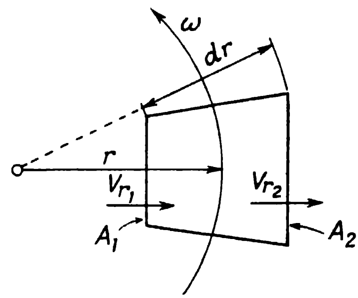


Fig. 346

$$\text{Centrifugal pressure} = p_c = \frac{F_c}{A} = \frac{w\omega^2 r\,\mathrm{d}r}{g}$$

$$\text{Centrifugal head} = h_c = \frac{p_c}{w} = \frac{\omega^2 r\,\mathrm{d}r}{g} \qquad . \quad . \quad . \quad (20)$$

Let V_{b_1} = tangential velocity of short length at inlet

V_{b_2} = tangential velocity of short length at outlet

Substituting for r and $\mathrm{d}r$,

$$h_c = \frac{\omega^2}{g}\,\frac{(r_1 + r_2)}{2}\,(r_2 - r_1)$$

$$= \frac{\omega^2}{2g}\,(r_2^2 - r_1^2)$$

But

$$\omega^2 r_2^2 = V_{b_2}^2 \text{ and } \omega^2 r_1^2 = V_{b_1}^2$$

hence,

$$h_c = \frac{V_{b_2}^2 - V_{b_1}^2}{2g} \qquad . \quad . \quad . \quad . \quad . \quad (21)$$

Equating eq. (20) and (21),

$$\frac{\omega^2 r\,\mathrm{d}r}{g} = \frac{V_{b_2}^2 - V_{b_1}^2}{2g} \qquad . \quad . \quad . \quad . \quad (22)$$

24.9. Application of Energy Equation to Rotating Blade Passage. Consider a short length dx of the blade passage of a radial-flow turbine rotor and let suffixes 1 and 2 apply to the inlet and outlet ends, respectively, of the short length dx.

Let V = absolute velocity of fluid

V_b = tangential velocity of blade

V_r = relative velocity of fluid to blade

V_w = velocity of whirl of fluid (§ 22.2)

V_f = radial velocity of flow of fluid (§ 22.2)

From eq. (4) (§ 6.5),

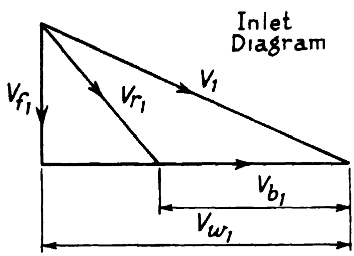


FIG. 347

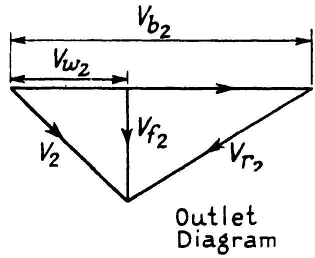


FIG. 348

Work done per pound of fluid

$$= \frac{V_{w_1}V_{b_1} - V_{w_2}V_{b_2}}{g}$$

Applying energy equation to fluid at sections 1 and 2,

$$JH_1 + \frac{V_1^2}{2g} = JH_2 + \frac{V_2^2}{2g} + \text{work done per pound}$$

$$= JH_2 + \frac{V_2^2}{2g} + \frac{(V_{w_1}V_{b_1} - V_{w_2}V_{b_2})}{g} \qquad (23)$$

where $H_1 = c_pT_1$ and $H_2 = c_pT_2$, above absolute zero temperature.

The inlet velocity diagram for the rotating blade passage at section 1 is shown in Fig. 347, and the outlet velocity diagram, at section 2, in Fig. 348.

From Fig. 347,

$$V_{f_1}^2 = V_{r_1}^2 - (V_{w_1} - V_{b_1})^2$$

Also $$V_{f_1}^2 = V_1^2 - V_{w_1}^2$$

Equating these two values of $V_{f_1}^2$,

$$V_1^2 - V_{w_1}^2 = V_{r_1}^2 - (V_{w_1}^2 - 2V_{w_1}V_{b_1} + V_{b_1}^2)$$

from which,

$$V_1^2 = V_{r_1}^2 + 2V_{w_1}V_{b_1} - V_{b_1}^2 \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Next consider the outlet velocity diagram of Fig. 348,

$$V_{f_2}^2 = V_2^2 - V_{w_2}^2$$

Also $$V_{f_2}^2 = V_{r_2}^2 - (V_{b_2} - V_{w_2})^2$$

Equating these two values of $V_{f_2}^2$,

$$V_2^2 - V_{w_2}^2 = V_{r_2}^2 - (V_{b_2}^2 - 2V_{w_2}V_{b_2} + V_{w_2}^2)$$

from which

$$V_2^2 = V_{r_2}^2 - V_{b_2}^2 + 2V_{w_2}V_{b_2} \quad . \quad . \quad . \quad . \quad (25)$$

Substituting in eq. (23) the values of V_1^2 and V_2^2 from eq. (24) and (25),

$$H_1 + \frac{(V_{r_1}^2 + 2V_{w_1}V_{b_1} - V_{b_1}^2)}{2gJ} =$$

$$H_2 + \frac{(V_{r_2}^2 - V_{b_2}^2 + 2V_{w_2}V_{b_2})}{2gJ} + \frac{(V_{w_1}V_{b_1} - V_{w_2}V_{b_2})}{gJ}$$

from which,

$$H_1 - H_2 = \frac{(V_{r_2}^2 - V_{r_1}^2) - (V_{b_2}^2 - V_{b_1}^2)}{2gJ}$$

But from eq. (22), § 24.8,

$$\frac{V_{b_2}^2 - V_{b_1}^2}{2g} \quad \omega^2 r \, dr$$

Hence,

$$Jc_p(T_1 - T_2) = \frac{(V_{r_2}^2 - V_{r_1}^2)}{2g} - \frac{\omega^2 r \, dr}{g} \quad . \quad . \quad (26)$$

This is the energy equation for the flow of gas through a rotating duct, such as the blade passage of a turbine rotor or a centrifugal compressor. In an axial-flow turbine or compressor the term dr is zero, and the equation becomes

$$Jc_p(T_1 - T_2) = \frac{(V_{r_2}^2 - V_{r_1}^2)}{2g}$$

24.10. Flow Equations for Revolving Blade Passage. This problem can be solved analytically by applying the four fundamental equations of fluid flow, in the same manner as in the stationary duct solution (§ 24.2).

Consider a short length dx of the rotating duct shown in Fig. 349, which is rotating with a constant angular velocity ω. The frictional drag D_f is given by eq. (1) (§ 24.2),

$$D_f = \frac{C_f P v^2 \, dx}{2Vg} \quad ,$$

where the velocity v now becomes the relative velocity V_r.

Let F_i = inertia force on fluid

$$= \frac{wAv\,\mathrm{d}v}{g}$$

F_c = centrifugal force on fluid

$$= \frac{wA\omega^2 r\,\mathrm{d}r}{g} \quad \text{(eq. (19), § 24.8)}$$

Using the same notation as in § 24.2, and resolving the forces shown in Fig. 349 horizontally,

$$p(A + \mathrm{d}A) - (p + \mathrm{d}p)(A + \mathrm{d}A) = F_i + D_f - F_c$$

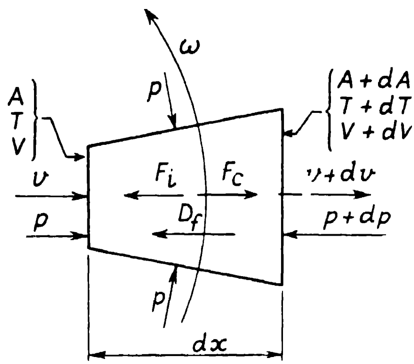


Fig. 349

Substituting for F_i, D_f and F_c,

$$-A\,\mathrm{d}p = \frac{wAv\,\mathrm{d}v}{g} + \frac{C_f P v^2\,\mathrm{d}x}{2Vg} - \frac{wA\omega^2 r\,\mathrm{d}r}{g}$$

Substituting for $w = 1/V$ and dividing throughout by A,

$$-\mathrm{d}p = \frac{v\,\mathrm{d}v}{Vg} + \frac{C_f P v^2\,\mathrm{d}x}{2AVg} - \frac{\omega^2 r\,\mathrm{d}r}{Vg} \quad . \quad (27)$$

Applying the energy equation to the duct of Fig. 349 and using the form given by eq. (26), § 24.9,

$$Jc_p[T - (T + \mathrm{d}T)] = \frac{-v^2 + (v + \mathrm{d}v)^2}{2g} - \frac{\omega^2 r\,\mathrm{d}r}{g} \quad . \quad (28)$$

Hence,
$$-Jc_p\,\mathrm{d}T = \frac{v\,\mathrm{d}v}{g} - \frac{\omega^2 r\,\mathrm{d}r}{g} \quad . \quad (29)$$

The gas equation is not affected by the rotation of the duct; hence, eq. (4) of § 24.2 can be used—

$$-\mathrm{d}p = p\,\frac{\mathrm{d}V}{V} - \frac{R\,\mathrm{d}T}{V}$$

The equation of continuity of flow is also unaffected by the rotation, so that eq. (5) of § 24.2 can be used—

$$\frac{\mathrm{d}V}{V} = \frac{\mathrm{d}A}{A} + \frac{\mathrm{d}v}{v}$$

Combining these two equations,

$$-\,\mathrm{d}p = p\,\frac{\mathrm{d}A}{A} + p\,\frac{\mathrm{d}v}{v} - R\,\frac{\mathrm{d}T}{V} \qquad . \qquad . \quad (30)$$

Equating eq. (30) and (27),

$$p\,\frac{\mathrm{d}A}{A} + p\,\frac{\mathrm{d}v}{v} - R\,\frac{\mathrm{d}T}{V} = \frac{v\,\mathrm{d}v}{Vg} + \frac{C_f P v^2\,\mathrm{d}x}{2AVg} - \frac{\omega^2 r\,\mathrm{d}r}{Vg} \qquad . \quad (31)$$

Using the same substitutions as those used in § 24.2 for stationary ducts, eq. (31) reduces to

$$\mathrm{d}v = \frac{\dfrac{C_f}{2g}\left(\dfrac{P}{A}\right)v^2\,\mathrm{d}x - RT\,\dfrac{\mathrm{d}A}{A} - \dfrac{\omega^2 r\,\mathrm{d}r}{\gamma g}}{\dfrac{RT}{v} - \dfrac{v}{\gamma g}} \qquad . \qquad . \quad (32)$$

where A, P, T, r and v are the mean values as in § 24.2, and $\mathrm{d}r = r_2 - r_1$. In this case v is the relative velocity V_r of fluid to duct, which is found from the velocity vector diagram similar to Fig. 347. Equation (32) is known as the momentum equation.

From eq. (26), § 24.9, the energy equation may be written

$$V_{r_2}{}^2 = V_{r_1}{}^2 + 2\omega^2 r\,\mathrm{d}r + 2gJc_p(T_1 - T_2) \qquad . \qquad . \quad (33)$$

$$= V_{r_1}{}^2 + 2\omega^2 r\,\mathrm{d}r - 2gJc_p\,\mathrm{d}T$$

For air, $\quad V_{r_2}{}^2 = V_{r_1}{}^2 + 2\omega^2 r\,\mathrm{d}r - 12{,}020\,\mathrm{d}T \qquad . \qquad . \qquad . \quad (34)$

It should be noted that

$$r = \frac{r_1 + r_2}{2}$$

$$\mathrm{d}v = V_{r_2} - V_{r_1}$$

$$v = \frac{V_{r_1} + V_{r_2}}{2} \text{ (as } v_1 = V_{r_1} \text{ and } v_2 = V_{r_2}\text{)}$$

Equations (32) and (33) reduce to the corresponding equation deduced in § 24.2 for stationary ducts if ω is zero; hence, they can be regarded as generalized equations for all types of ducts. It should be noted that

$$V_{r_2} = V_{r_1} + \mathrm{d}v$$

$$W = \frac{144p_1A_1V_{r_1}}{RT_1} \quad \text{(eq. (12), § 24.2)}$$

and
$$p_2 = \frac{WRT_2}{144A_2V_{r_2}} \quad \text{(eq. (13), § 24.2)}$$

The method of application of the above equations to a turbine rotor is demonstrated in the worked-out example of § 24.12.

If the blade passages are artificially cooled, as in some gas turbines, the heat extracted per pound of fluid must be allowed for in the energy equation (eq. (33)). In this case, a decrease in entropy is possible if the cooling effect is considerable; this would not indicate the occurrence of a pressure jump at a previous section of the blade passage.

24.11. Pressure Jump in Blade Passage of Turbine Rotor. In order to test for the possibility of a pressure jump at a cross-section of a duct either eq. (34) (§ 19.8) or eq. (36) (§ 19.9) may be used. These equations are identical, as proved in § 19.9.

Using suffix 0 for the gas condition before the jump, and suffix 1 for the condition after the jump, and applying eq. (34) (§ 19.8),

$$\frac{W}{A} = \frac{144\sqrt{b}\,(T' - T_1)^{1/2}\left[k - \frac{\sqrt{b}}{c}(T' - T_1)^{1/2}\right]}{RT_1}$$

Or, alternatively, using eq. (36), § 19.9,

$$\frac{dS}{dT_1} = \frac{c_p}{T_1} - \frac{\frac{R}{J}\frac{\sqrt{b}}{2c}}{k(T' - T_1)^{1/2} - \frac{\sqrt{b}}{c}(T' - T_1)} = 0$$

where
$$b = 2gJc_p$$

$$c = \frac{144Ag}{W}$$

$$k = p_0 + \frac{v_0}{c}$$

T' = total or stagnation temperature (§ 18.13)

$$= T_0 + \frac{v_0^2}{b}$$

$$p_1 = k - \frac{v_1}{c}$$

$$v_1 = V_{r_1} = \sqrt{b}\,(T' - T_1)^{1/2}$$

EXAMPLE 3

The following values apply to a certain cross-section of the blade passage of the rotor of a small, high-speed, cold air turbine. Test this section for a pressure jump by applying eq. (36), § 19.9. If a jump occurs at this section, state the final conditions of the air. $T_0 = 120{\cdot}5°R$, $v_0 = V_{r_0} = 1{,}057{\cdot}6$ ft/sec, $A = 2{\cdot}082 \times 10^{-4}$ ft², $W = 0{\cdot}0212$ Lb/sec, $c_p = 0{\cdot}24$, $p_0 = 8{\cdot}56$ Lb/in.²

Applying the above equations for the values of the constants b, c, k, and T',

$$b = 2 \times 32{\cdot}2 \times 778 \times 0{\cdot}24 = 12{,}020$$

$$\sqrt{b} = 109{\cdot}6$$

$$c = \frac{144 \times 2{\cdot}082 \times 32{\cdot}2}{0{\cdot}0212 \times 10^4} = 45{\cdot}5$$

$$k = 8{\cdot}56 + \frac{1{,}057{\cdot}6}{45{\cdot}5} = 31{\cdot}81$$

$$T' = 120{\cdot}5 + \frac{(1{,}057{\cdot}6)^2}{12{,}020} = 213{\cdot}7$$

$$\frac{\sqrt{b}}{c} = \frac{109{\cdot}6}{45{\cdot}5} = 2{\cdot}41$$

$$\frac{R}{J} \times \frac{\sqrt{b}}{2c} = \frac{53{\cdot}3}{778} \times \frac{2{\cdot}41}{2} = 0{\cdot}0826$$

Substituting these values in eq. (36), § 19.9, and letting y be the value of the equation,

$$y = \frac{c_p}{T_1} - \frac{\dfrac{R}{J}\dfrac{\sqrt{b}}{2c}}{k\,(T' - T_1)^{1/2} - \dfrac{\sqrt{b}}{c}(T' - T_1)} = 0$$

$$= \frac{0{\cdot}24}{T_1} - \frac{0{\cdot}0826}{31{\cdot}81\,(213{\cdot}7 - T_1)^{1/2} - 2{\cdot}41\,(213{\cdot}7 - T_1)}. \qquad (35)$$

Assume several values of T_1 and calculate from eq. (35) the value of y for each assumed value of T_1.

The results are given in the following table.

T_1(°R)	$(213{\cdot}7 - T_1)$	$(213{\cdot}7 - T_1)^{1/2}$	y
120·5	93·2	9·66	0·000981
150	63·7	7·98	0·000778
200	13·7	3·7	0·000224
203	10·7	3·27	0·000126
205	8·7	2·95	0·000037
208	5·7	2·39	− 0·000171
213	0·7	0·837	− 0·002184

The above values of y are shown plotted in Fig. 350 on a base representing the assumed values of T_1. It will be noticed that the curve obtained has a value of $y = 0$ when $T_1 = 206°R$. This proves that a pressure jump may occur, because the resulting value of T_1 would equal T_0 if no jump were possible. Then, assuming the pressure jump has occurred,

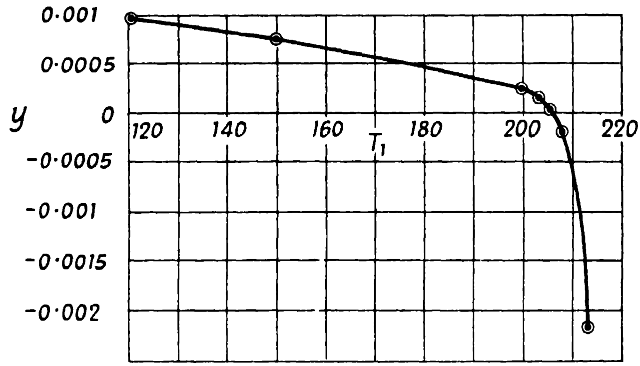


FIG. 350

$$v_1 = \sqrt{b}\,(T'' - T_1)^{1/2}$$

$$= 109{\cdot}6\,(213{\cdot}7 - 206)^{1/2} = 304 \text{ ft/sec}$$

$$p_1 = k - v_1$$

$$= 31{\cdot}81 - \frac{304}{45{\cdot}5} = 25{\cdot}12 \text{ Lb/in.}^2$$

24.12. Application of Equations to a Rotor Blade Passage. The method of application of the equations deduced in § 24.10 will now be demonstrated by applying them to a short length of the blade passage of a radial outward-flow air turbine.

In order to investigate the air flow through the blade passage of the rotor, the whole length of the passage should be divided into short sections, along the centre-line of the passage. The equations should then be applied to each section, in turn, commencing at the inlet end. The inlet conditions of the gas at each section are the same as the discharge conditions from the previous section.

The inlet conditions of the first section correspond to the discharge conditions from the guide vanes and the relative velocity is obtained from the inlet velocity vector diagram (Fig. 347); but a check should be made for a pressure jump at this section. After each section, the change of entropy during the section should be found.

If a decrease of entropy occurs, it follows that a pressure jump has taken place at a previous section. The remainder of the calculations must then be based on the conditions after the jump.

EXAMPLE 4

A small, high-speed, outward radial-flow turbine is used in a refrigeration plant as a turbo-expander. The blade passages through the rotor are rectangular in cross-section. Applying the equations of § 24.10 to the first $\frac{1}{8}$ in. length of the blade, calculate the outlet temperature, relative velocity and pressure of the air discharging from this first section. Use suffix 1 and 2 for the inlet and outlet ends of the section respectively.

Referring to Figs. 341 and 346, the following are the blade passage and air conditions at the entrance to the first section.

$$\begin{aligned}
\text{Mean } b &= 0{\cdot}0246 \text{ ft} \\
\text{Mean } t &= 0{\cdot}0104 \text{ ft} \\
dx &= 0{\cdot}0104 \text{ ft} \\
\text{Mean } P &= 2(b + t) = 0{\cdot}07 \text{ ft} \\
A_2 &= 3{\cdot}02 \times 10^{-4} \text{ ft}^2 \\
A_1 &= 2{\cdot}082 \times 10^{-4} \text{ ft}^2 \\
r_1 &= 0{\cdot}0842 \text{ ft} \\
r_2 &= 0{\cdot}0910 \text{ ft} \\
\frac{r_1 + r_2}{2} &= 0{\cdot}0876 \text{ ft} \\
dr &= r_2 - r_1 = 0{\cdot}0068 \text{ ft} \\
\omega &= 6{,}280 \text{ rad/sec} \\
C_f &= 0{\cdot}01 \\
W &= 0{\cdot}0212 \text{ Lb/sec}
\end{aligned}$$

$$\begin{aligned}
A &= \frac{A_1 + A_2}{2} = 2{\cdot}551 \times 10^{-4} \text{ ft}^2 \\
dA &= A_2 - A_1 = 0{\cdot}938 \times 10^{-4} \text{ ft}^2 \\
\frac{dA}{A} &= \frac{0{\cdot}938}{2{\cdot}551} = 0{\cdot}367 \\
\frac{P}{A} &= \frac{0{\cdot}07}{2{\cdot}551 \times 10^{-4}} = 274{\cdot}5 \\
T_1 &= 132°\text{R} \\
p_1 &= 11{\cdot}5 \text{ Lb/in.}^2 \\
V_{r_1} &= 422 \text{ ft/sec} \\
dv &= (V_{r_2} - V_{r_1}) \\
&= (V_{r_2} - 422) \\
dT &= (T_2 - T_1) \\
&= (T_2 - 132) \\
\gamma &= 1{\cdot}4 \text{ for air}
\end{aligned}$$

ENERGY EQUATION. Applying eq. (34), (§ 24.10),

$$\begin{aligned}
V_{r_2}^2 &= V_{r_1}^2 + 2\omega^2 r\, dr - 12{,}020\, dT \\
&= (422)^2 + 2\,[(6{,}280)^2 \times 0{\cdot}0876 \times 0{\cdot}0068] - 12{,}020\, dT \\
&= 225{,}300 - 12{,}020\, dT \qquad (36)
\end{aligned}$$

Assume several values of T_2 and calculate the corresponding values of V_{r_2} from eq. (36), from which the corresponding values of dv can be found. Then plot the values of dv on a base representing the assumed corresponding values of T_2 (Fig. 351). This curve represents the energy equation. Thus, assume $T_2 = 140°$R, then $dT = 8$. Substituting this value of dT in eq. (36),

$$\begin{aligned}
V_{r_2}^2 &= 225{,}300 - (12{,}020 \times 8) \\
&= 129{,}140
\end{aligned}$$

Hence, $V_{r_2} = 358$ ft/sec
and $dv = 358 - 422 = -64$ ft/sec

MOMENTUM EQUATION. Applying eq. (32), § 24.10,

$$dv = \frac{\frac{C_f}{2g}\left(\frac{P}{A}\right) v^2\,dx - RT\left(\frac{dA}{A}\right) - \frac{\omega^2 r\,dr}{\gamma g}}{\frac{RT}{v} - \frac{v}{\gamma g}}$$

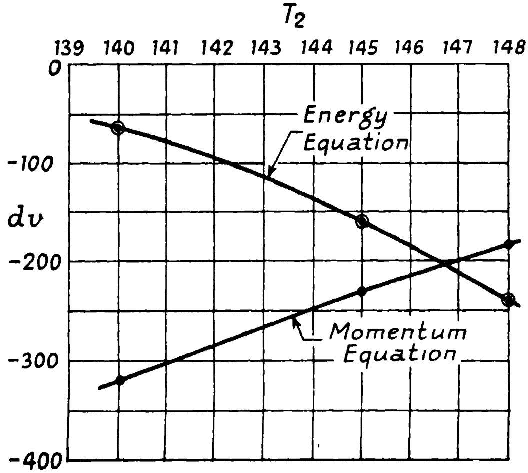


FIG. 351

Using the above assumed value of T_2 and the above values of dv and V_{r_2} from the energy equation,

$T_2 = 140°$R and $dT = 8$,

$dv = -64$ ft/sec

$V_{r_2} = 358$ ft/sec

Then, $T = 136°$R = mean temperature over section

and $v = 390$ ft/sec = mean relative velocity over section

Inserting these values in the momentum equation,

$$dv = \frac{\left[\frac{0\cdot01}{64\cdot4} \times 274\cdot5 \times 0\cdot0104v^2\right] - [53\cdot3 \times 0\cdot367T] - \left[\frac{6{,}280^2 \times 0\cdot0876 \times 0\cdot0068}{1\cdot4 \times 32\cdot2}\right]}{\frac{53\cdot3T}{v} - \frac{v}{1\cdot4 \times 32\cdot2}}$$

$$\frac{0{\cdot}444 \times 10^{-3}v^2 - 19{\cdot}55T - 525}{53{\cdot}3T \qquad \dfrac{v}{45{\cdot}1}} \tag{37}$$

Inserting the values of T and v in eq. (37) for the assumed value of $T_2 = 140°$R,

$$\mathrm{d}v = \frac{67{\cdot}5 - 2{,}658 - 525}{18{\cdot}6 - 8{\cdot}65} \quad \frac{-3{,}115{\cdot}5}{9{\cdot}95} = -313 \text{ ft/sec}$$

These calculations are then repeated for each assumed value of T_2 used in the energy equation. The results obtained are shown in the following table.

T_2 (assumed)	$\mathrm{d}T = T_2 - T_1$	V_{r_2} (eq. (36))	T (mean)	v (mean)	dv Energy eq.	dv Momentum eq.
140°R	8	358	136	390	− 64	− 313
145°R	13	263	138·5	342·5	− 159	− 228
148°R	16	182	140	302	− 240	− 179

The values of dv from the momentum equation are also shown plotted in Fig. 351. The point of intersection of the two curves gives the correct values of T_2 and dv.

Then from curves of Fig. 351,

$$\mathrm{d}v = -203 \text{ ft/sec}$$
$$V_{r_2} = 422 - 203 = 219 \text{ ft/sec}$$
$$T_2 = 146{\cdot}7°\text{R}$$

Applying eq. (13), § 24.2,

$$p_2 \quad \frac{WRT_2}{144A_2V_{r_2}}$$
$$\frac{0{\cdot}0212 \times 53{\cdot}3 \times 146{\cdot}7}{144 \times 3{\cdot}02 \times 10^{-4} \times 219} = 17{\cdot}4 \text{ Lb/in.}^2$$

The above values of V_{r_2}, T_2 and p_2 can now be used for the inlet conditions of the air entering the next section of the blade passage. This is repeated until the outlet end of the blade passage is reached.

24.13. Application to Cold Air Turbine. The conditions of flow of air through the blade passages of an experimental model turbine have been worked out from the application of the equations of flow deduced in §§ 24.2 and 24.10. The turbine was a small outward radial-flow impulse type used in refrigeration processes and known as a turbo-expander.

The following particulars apply to the turbine test.

External diameter of rotor	= 6·25 in.
Width of blades	= 0·125 in.
Inlet pressure of air	= 90·7 Lb/in.^{2} (abs.)
Outlet pressure of air	= 17·5 Lb/in.^{2} (abs.)
Inlet temperature of air	= 218°R
Outlet temperature of air	= 167°R
Speed of rotor	= 60,000 r.p.m.
Weight of air supplied	= 0·2538 Lb/sec
Number of guide blades	= 6
Number of rotor blades	= 12

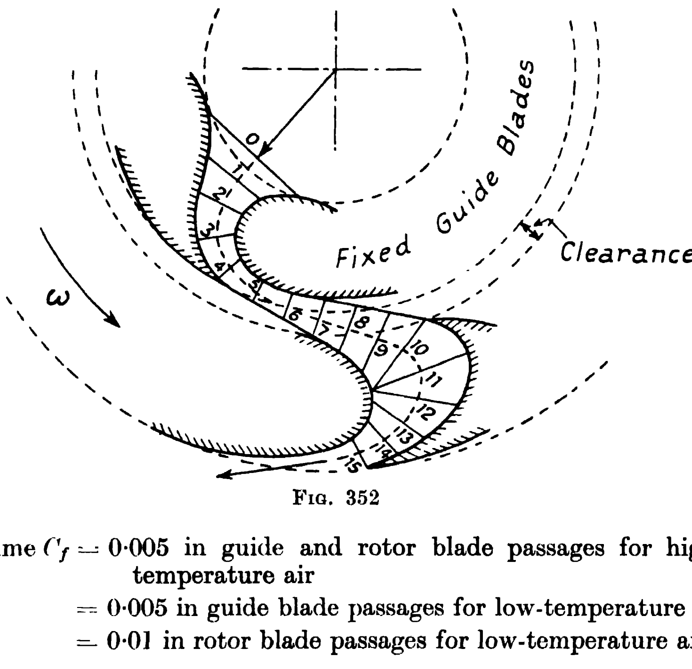


FIG. 352

Assume C_f = 0·005 in guide and rotor blade passages for high-temperature air

= 0·005 in guide blade passages for low-temperature air

= 0·01 in rotor blade passages for low-temperature air

The last two values are high because the passages are acting mainly as diffusers.

A view of the air passage through the guide and rotor vanes is shown in Fig. 352. The rectangular-sectioned guide blade passage acts as the converging cone of a converging-diverging nozzle. The nozzle throat is at section 4 which is at the outlet end of the guide blade ring; the flow at the throat was sonic. The circumferential clearance space between the guide blade and the rotor corresponds to the diverging cone of the nozzle. The centre-line of the flow path through the whole turbine has been divided into 15 equal sections,

each $\frac{1}{8}$ in. long, and the equations of flow have been applied to each section in turn, from 0 to 15.

The expansion of the cold air through the turbine blades was assumed to be a supersaturated flow on account of its high speed.

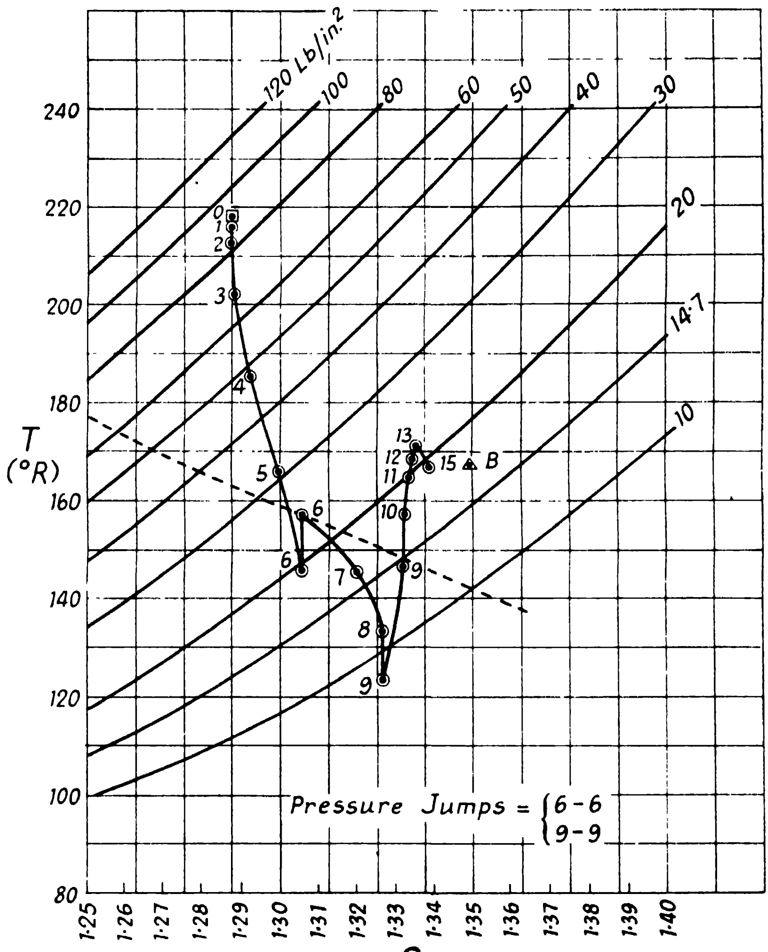
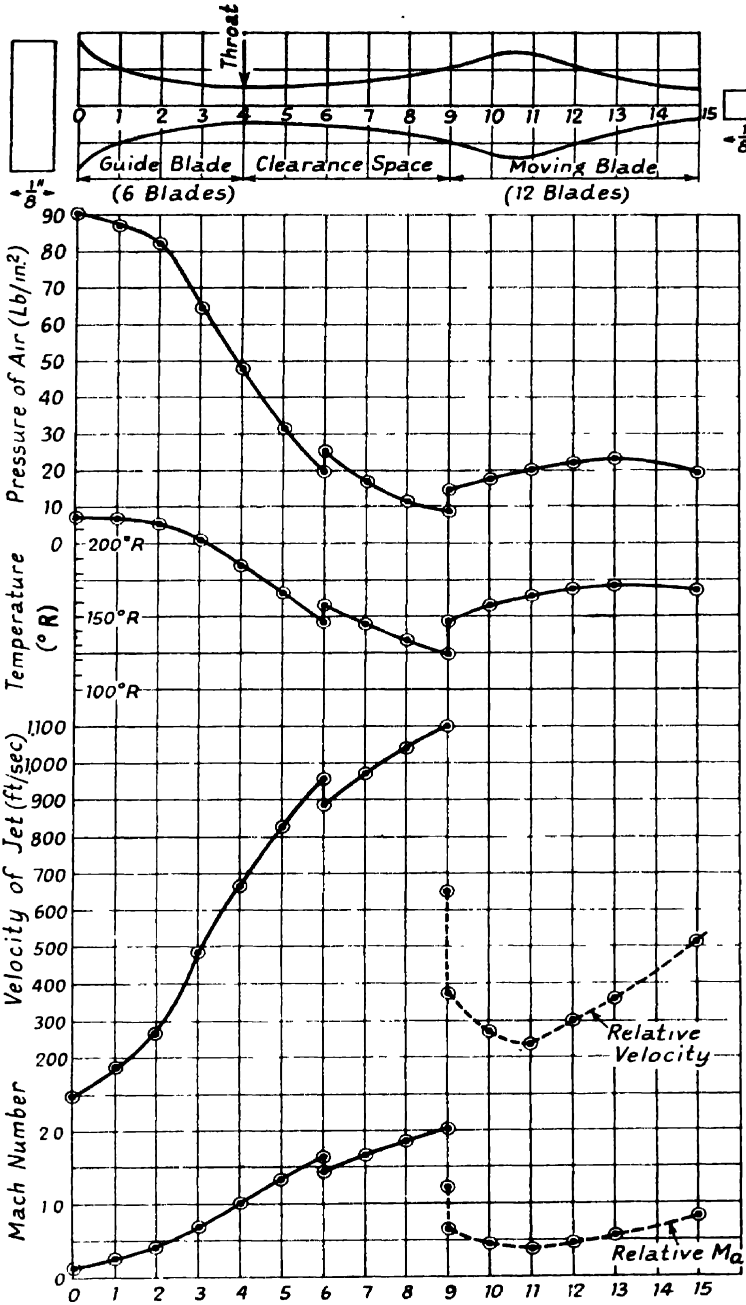


FIG. 353. TEMPERATURE-ENTROPY DIAGRAM OF FLOW

This means that the flow was too rapid to overcome the time lag which occurs in the condensation of the oxygen. The calculated air condition during its flow through the turbine is shown plotted on the temperature-entropy diagram of Fig. 353. It will be noticed that the minimum temperature occurs at section 9, at the inlet of



Throat
Guide Blade
(6 Blades)
Clearance Space
Moving Blade
(12 Blades)
1"/8
Pressure of Air (Lb/in.²)
Temperature (°R)
200°R
150°R
100°R
Velocity of Jet (ft/sec)
Relative Velocity
Mach Number
Relative Ma

FIG. 354

the rotor blades. As its position is only just below the saturation line, shown dotted, the assumption of supersaturated flow appears to be justified.

It will be noticed from Fig. 352 that the flow path is subjected to bends of small radius between sections 2 and 4 in the guide blades, and between sections 10 and 13 in the rotor blades. The centrifugal force due to this cause moves the mass centre of the jet towards the concave face of the passage and causes boundary-layer separation on the inside face of the bend. This has the effect of reducing the effective width of the passage. To compensate for this reduction of area of flow, a coefficient of area of 0·9 was assumed when dealing with the sections affected.

The calculations commenced at section 0–1 and were continued consecutively through all the sections up to the outlet of the rotor. The air condition for each section was plotted on the chart of Fig. 353 before proceeding to the following section. It should be remembered that the entropy must increase at each plotting owing to the frictional reheating (§ 18.4). If a decrease of entropy is registered, the flow is impossible, hence, a pressure jump must have occurred at one of the previous sections. This means that it is necessary to test for a pressure jump (§ 24.11) at each section. If the test at a certain section shows that a pressure jump is possible, it does not necessarily follow that the jump will occur; it will only occur if the flow causes an entropy decrease at a later section.

When section 9 is reached, the relative velocity of the air entering the rotor is required for use in the equations of flow. This is obtained in the usual way by drawing the velocity diagram for the entrance of the rotor blades.

In the early portion of the calculation it was found that there were three possible solutions, due to possible pressure jumps at sections 6 and 9:

(*a*) No jump occurs at 6 but occurs at 9; hence, there is a subsonic flow through the rotor blades.

(*b*) Jumps occur at sections 6 and 9; hence, the flow through the rotor blades is subsonic.

(*c*) No jump occurs at sections 6 or 9; hence, the flow through the rotor blades remains supersonic.

It was found as the calculations proceeded that entropy decreases occurred in the rotor blade passage for cases (*a*) and (*c*), thus eliminating the possibility of these types of flow. Hence, case (*b*) was the only possible solution.

The complete calculated results for case (*b*) are shown plotted in Fig. 354. These curves show the values of the air pressures, temperature, velocity in guide vanes, relative velocity in rotor blades, and Mach numbers of flow. The measured discharge condition of the air is given by point *B* on the chart of Fig. 353, thus showing the closeness of the calculated results with the measured.

It will be noticed from Fig. 353 that the two pressure jumps, 6–6 and 9–9, both bring the state of the air back to the saturation line, which is its natural condition for equilibrium.

The following values were obtained from the calculated results.

Horse-power developed = 3·9

Horse-power lost to frictional reheating = 2·3

Total heat drop = 12·3 B.Th.U. per pound of air

Work produced by rotor = 9·55 B.Th.U. per pound of air

Kinetic energy lost at discharge = 2·88 B.Th.U. per pound of air

Frictional reheat = 8·54 B.Th.U. per pound of air

The frictional reheat was found from the area under the flow path, 0 to 15, of the temperature-entropy chart of Fig. 353, measured down to absolute zero temperature.

It will be noticed from Fig. 353 that the flow through the rotor of this outward radial-flow turbine, from section 9 to 13, corresponds to that obtained from a centrifugal air compressor.

24.14. Application of Equations to an Axial-flow Gas Turbine. The equation of flow deduced in § 24.2 will now be applied to the blade passages of a gas turbine. The turbine selected is an axial-flow type used in the German J.U.M.O. turbo-jet plant. The following are the required particulars and dimensions of the turbine, which consists of a single guide blade ring and rotor.

Speed of rotor = 8,700 r.p.m.

Mean diameter of blade ring = 23 in.

Height of blades = 4 in.

Number of guide blades = 35

Number of rotor blades = 61

Weight of gases flowing = 46·3 Lb/sec

Weight of gases per guide blade passage = 1·32 Lb/sec

Weight of gases per rotor blade passage = 0·76 Lb/sec

Pressure ratio of gases = 3·1

Pressure at entrance to guide blade = 45 Lb/in.^{2}

Temperature at entrance to guide blade = 1,850°R

Fuel/air ratio = 57/1

Mean blade speed = 888 ft/sec

Assume $C_f = 0{\cdot}005$

For gases, $c_p = 0{\cdot}271$

$$R = 53{\cdot}3$$

$$\gamma = 1{\cdot}34$$

$$\text{Sonic velocity} = v_s = 49\sqrt{T}$$

For guide and rotor blades (Fig. 357),

$$\text{Mean } \alpha = 18^\circ$$
$$\text{Mean } \theta = 31{\cdot}5^\circ$$
$$\text{Mean } \phi = 38^\circ$$
$$\text{Length of section} = dx = 0{\cdot}0452 \text{ ft}$$

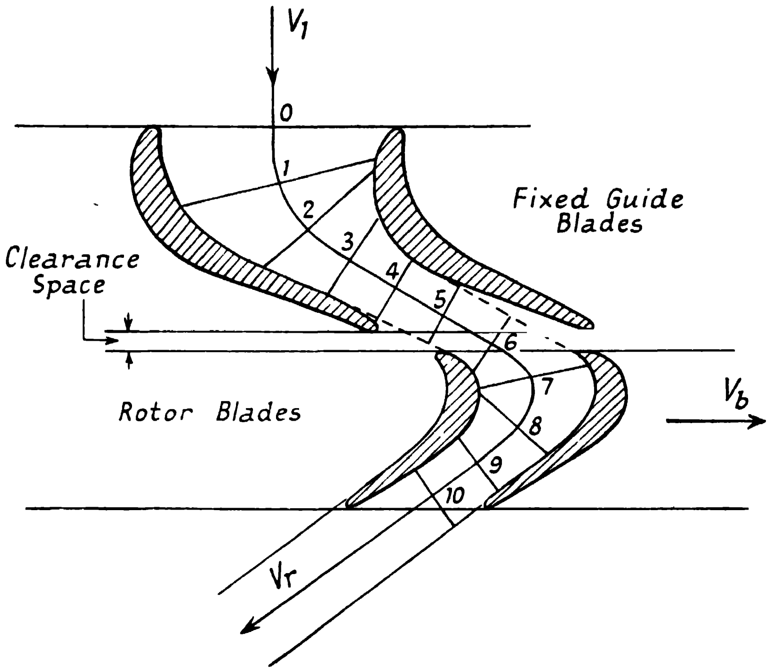


FIG. 355

The shape of the rectangular blade passages through the guide and rotor blades is shown, drawn to scale, in Fig. 355; this represents the shape at the blade tips. A similar diagram, of slightly smaller dimensions, represented the blade roots. The cross-sectional area of the rectangular blade passage at any point was calculated from the mean blade passage widths of the tip and the root. The mean passage was divided into ten equal sections along the centre-line, of length 0·0452 ft, as shown in Fig. 355. The gases enter the guide blade at section 0 and leave at the throat which occurs approximately at section 4. They then traverse the clearance space to section 6, which represents the entrance of the rotor blade passage, and finally exhaust at section 10. The gases are then further expanded to form a high-velocity propulsion jet. An extended view

of the total blade passage, from sections 0 to 10, is shown at the top of Fig. 358; this is shown approximately to scale.

The weight of gases flowing per second, per blade passage, is calculated on 35 blades between sections 0 and 6, and on 61 blades between sections 6 and 10. It should be noted that as the turbine is an axial-flow type, there is no centrifugal effect to take into account in the rotor blade passage.

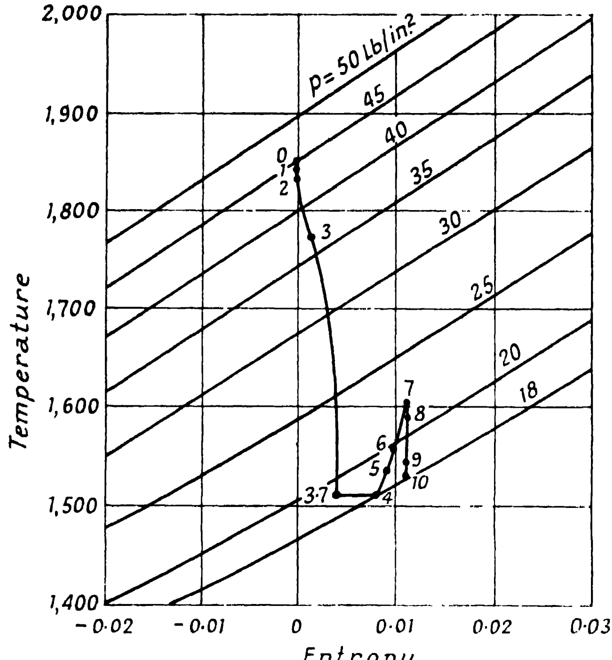


FIG. 356

Applying eq. (12) to the inlet of the guide vane,

$$v_0 = \frac{WRT_0}{144 A_0 p_0} \text{ ft/sec}$$

The energy and momentum equations of § 24.2, eqs. (10) and (9), were applied to each section in turn, by the method given in § 24.3, commencing at section 0–1. At this section $T_0 = 1{,}850°\text{R}$, $p_0 =$ 45 Lb/in.² and $v_0 = 351$ ft/sec. This calculation gave the values of T_1 and v_1. The corresponding value of p_1 was obtained by applying eq. (12).

A temperature-entropy diagram for the gases is shown plotted in Fig. 356. The fluid conditions for sections 0 and 1 should be plotted on this diagram as soon as calculated; this is repeated for each

section in turn. If a decrease in entropy is found to occur at any section, it indicates the occurrence of a pressure jump at a previous section. In this case a test for a pressure jump should be applied to the previous sections, as demonstrated in § 24.11, working backwards until the position of the jump is located.

The outlet conditions at section 6 give the inlet velocity for the first section of the rotor blade. This is used for the construction of

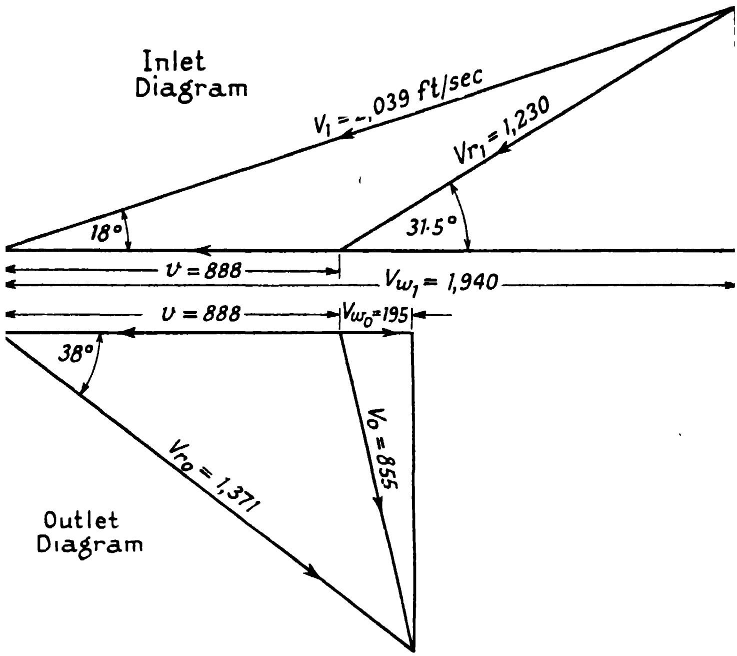
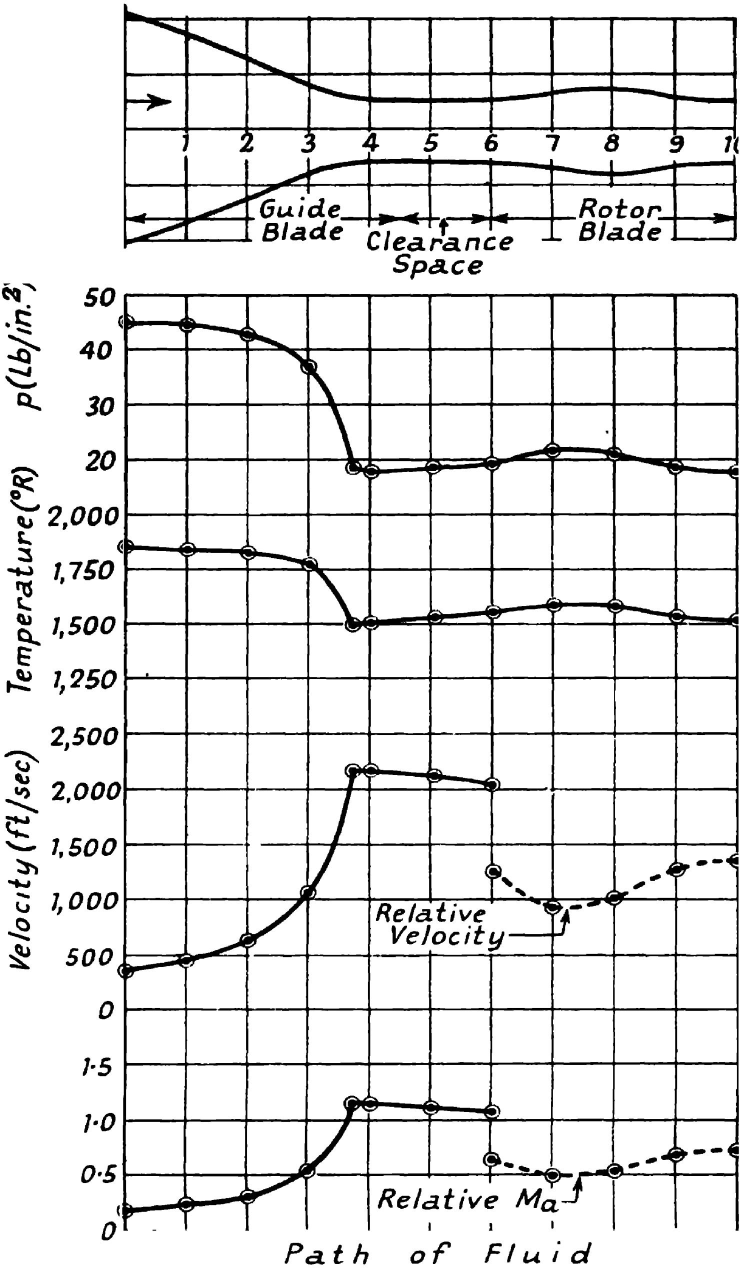


FIG. 357

the inlet velocity vector diagram shown in Fig. 357, the blade angles and blade velocity being already known. From this diagram the relative velocity of the entering gas is obtained and is used for v_6 for section 6–7. The calculations are then continued through the rotor blade passage, being based on the relative velocity.

When the final relative velocity at the outlet of the rotor blade passage (section 10) is obtained, the outlet velocity vector diagram, shown in Fig. 357, can now be drawn. This gives the magnitude and direction of the absolute discharge velocity.

The condition of the gases throughout the whole turbine are shown plotted on the T–S diagram of Fig. 356. It will be noticed that, from the author's calculations, no decrease in entropy is registered, hence, no pressure jump occurred. The total area under the



1 2 3 4 5 6 7 8 9
Guide Blade
Clearance Space
Rotor Blade
p(Lb/in.²)
50
40
30
20
Temperature(°R)
2,000
1,750
1,500
1,250
Velocity (ft/sec)
2,500
2,000
1,500
1,000
500
0
1·5
1·0
0·5
0
Relative Velocity
Relative Ma
Path of Fluid

FIG. 358

T–S diagram gives the amount of frictional reheating from section 0 to 10, due to fluid friction only. This was found to be 18·66 B.Th.U. per pound of gas flowing and is equivalent to 1,237 frictional horse-power.

Assuming momentum blading, the total theoretical net horse-power developed by the turbine is given by the equation

$$\text{h.p.} = \frac{W(V_{w_1} - V_{w_0})\,v}{g \times 550}$$

From the author's results this gives 4,915 horse-power. The actual shaft horse-power produced will be less than this amount on account of windage losses and bearing friction.

From the actual heat drop, shown by the T–S diagram of Fig. 356, and allowing for the increase of kinetic energy between sections 0 and 10, the theoretical horse-power is 5,040; but windage and bearing losses must be deducted from this for the shaft horse-power.

The shaft horse-power of the turbine is utilized in driving the air compressor and fuel pump. From the author's calculations the horse-power required to drive the compressor is 4,830.

In Fig. 358 the values of the pressure, temperature, velocity and Mach number are shown plotted, for all sections, on a base representing the centre-line of the blade passage. It will be noticed that sonic velocity was only exceeded between the throat and section 6; this length is mainly in the clearance space between the guide blade ring and the rotor. The final temperature of the discharging gases was 1,530°R at a pressure of 18·1 Lb/in.^{2} It is this high discharge temperature which causes the low overall efficiency of the turbo-jet plant, as most of it is wasted in the present type of fighter aircraft.

24.15. Gas Conditions at Entrance to an Inlet Duct. In the vicinity of the mouth of a suction nozzle, such as that of a rotary air compressor, the atmosphere is drawn in the nozzle by the suction of the compressor. This causes an adiabatic flow of the surrounding air into the mouth of the duct. At a certain distance from the mouth the atmosphere may be regarded as stagnant air. The suction from the duct causes the air between this distance and the duct to flow towards the mouth; this process approximates to a frictionless adiabatic expansion and is accompanied by a drop in temperature and pressure, and produces a considerable velocity.

Let suffix 0 apply to the distant stagnant air, and suffix 1 apply to the air conditions at the duct's mouth.

Applying the energy equation to positions 0 and 1,

$$c_pT_0 = c_pT_1 + \frac{v_1^2}{2gJ}$$

from which
$$T_0 = T_1 + \frac{v_1^2}{c_pJ2g} \quad . \quad . \quad . \quad . \quad (38)$$

It should be noted that for a frictionless adiabatic flow, the energy equation reduces to the same equation as the momentum equation. As the expansion is adiabatic,

$$\frac{T_1}{T_0} = \left(\frac{p_1}{p_0}\right)^{\frac{\gamma-1}{\gamma}} \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Applying eq. (12) to the mouth of the duct,

$$p_1 = \frac{WRT_1}{144A_1v_1} \quad . \quad . \quad . \quad . \quad . \quad (40)$$

As the atmospheric values of p_0 and T_0 are known, the unknown values of T_1, v_1 and p_1 can be calculated from the above three equations. This is done by assuming several values of T_1 and calculating the corresponding values of v_1 and p_1 for each assumed value of T_1. Thus, applying eq. (38),

$$v_1 = \sqrt{(T_0 - T_1)2gJc_p} \quad . \quad . \quad . \quad (41)$$

from which v_1 may be calculated.

These values can now be inserted in eq. (40) and the corresponding value of p_1 obtained.

Inserting the values of T_1 and p_1 in eq. (39) and taking logs of both sides of the equation,

$$\log\frac{T_0}{T_1} = \frac{\gamma - 1}{\gamma}\log\frac{p_0}{p_1} \quad . \quad . \quad . \quad . \quad (42)$$

This process should be repeated for several values of T_1. Next plot the values of $\log\frac{T_0}{T_1}$ and of $\frac{\gamma - 1}{\gamma}\log\frac{p_0}{p_1}$, as separate curves, on a base representing the assumed values of T_1. The point of intersection of these two curves gives the true value of T_1. By inserting this value in eqs. (40) and (41) the correct values of v_1 and p_1 can be calculated.

Example 5

The mouth of the air intake of a stationary gas turbine plant takes in 630 pound of air per minute. The temperature and pressure of the distant, stagnant atmosphere are 60°F and 14·7 Lb/in.^{2} respectively and the mouth of the intake duct has an area of 0·262 ft^2. Calculate the temperature, pressure and velocity of the air entering the mouth of the intake duct. $c_p = 0{\cdot}24$; $\gamma = 1{\cdot}4$ for air.

$$W = \frac{630}{60} = 10{\cdot}5 \text{ Lb/sec}$$

$$T_0 = 460 + 60 = 520°\text{R}$$

Assume $T_1 = 450°R$; substituting this value in eq. (41),

$$v_1 = \sqrt{(520 - 450)64{\cdot}4 \times 778 \times 0{\cdot}24}$$
$$= 919 \text{ ft/sec}$$

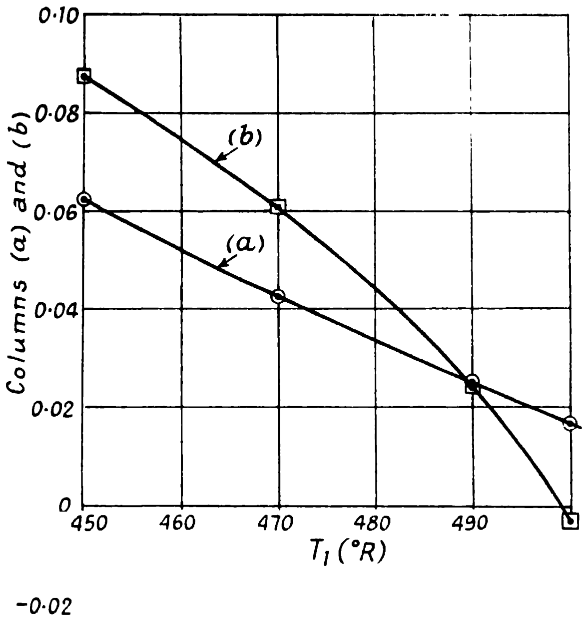


Fig. 359

Inserting these values of T_1 and v_1 in eq. (40),

$$p_1 = \frac{10{\cdot}5 \times 53{\cdot}3 \times 450}{144 \times 0{\cdot}262 \times 919}$$
$$= 7{\cdot}28 \text{ Lb/in.}^2$$

These values of T_1, v_1 and p_1 can now be entered in the table on page 698, and the table completed for this assumed temperature. Repeat this process for other assumed values of T_1, enter the results in the table and complete the table, as shown.

Next plot column (a) on a base representing T_1; also plot column (b) on the same base. The curves obtained are shown plotted in Fig. 359.

From the point of intersection of the two curves, $T_1 = 489°R$. Substituting this value in eq. (41),

$$v_1 = \sqrt{(520 - 489)64{\cdot}4 \times 778 \times 0{\cdot}24}$$
$$= 610 \text{ ft/sec}$$

Substituting these true values of T_1 and v_1 in eq. (40),

$$p_1 = \frac{10\cdot5 \times 53\cdot3 \times 489}{144 \times 0\cdot262 \times 610}$$

$$= 11\cdot9 \text{ Lb/in.}^2$$

T_1 (assumed)	v_1	p_1	$\frac{T_0}{T_1}$	(a) $\log \frac{T_0}{T_1}$	$\frac{p_0}{p_1}$	$\log \frac{p_0}{p_1}$	(b) $\left(\frac{\gamma-1}{\gamma}\right) \log \frac{p_0}{p_1}$
450	919	7·28	1·155	0·0626	2·02	0·3054	0·0872
500	491	15·11	1·04	0·0170	0·973	− 0·0119	− 0·0034
490	601	12·1	1·06	0·0254	1·214	0·0842	0·0240
470	776	9·00	1·106	0·0433	1·634	0·2133	0·0610

EXERCISES 24

1. Air at a temperature of 145°R and a pressure of 16·65 Lb/in.^{2} is supplied to the inlet end of a rectangular-sectioned tapering duct with a velocity of 970 ft/sec. Calculate the temperature, pressure and velocity of the air at a section $\frac{1}{8}$ in. from the inlet end of the duct, taking into account its frictional resistance. Weight of air flowing is 0·0423 Lb/sec. At inlet the cross-sectional area of the duct is $1\cdot43 \times 10^{-4}$ ft^2, breadth = 0·01375 ft, depth = 0·0104 ft. At the outlet end of the $\frac{1}{8}$ in. sectional length, area = $1\cdot736 \times 10^{-4}$ ft^2, breadth = 0·01667 ft, depth = 0·0104 ft. C_f = 0·005. What is the Mach number of the flow at outlet?

Ans. T_2 = 133°R; p_2 = 11·5 Lb/in.^{2}; v_2 = 1,040 ft/sec; M_{a_2} = 1·835.

2. A tapering, diverging, rectangular duct is supplied with 0·8 pound of air per second at a temperature of 100°F, and at a pressure of 60 Lb/in.^{2} The air enters at the smallest end of the duct. Assuming C_f = 0·006, calculate the temperature, pressure and velocity of the air at the end of the first 1 in. length of the duct. The area of the duct's cross-section is 0·5 in.^{2} at inlet and 0·75 in.^{2} at the end of the short length considered. It has a constant depth of $\frac{1}{2}$ in. and a mean breadth of $1\frac{1}{4}$ in.

Ans. T_2 = 595·5°R; p_2 = 75·2 Lb/in.^{2}; v_2 = 450 ft/sec.

3. The guide blade passage of a gas turbine is in the shape of a rectangular-sectioned converging nozzle. Apply the equations of fluid flow to a short section of the passage, commencing at the inlet end of the guide blade. Length of section considered = 0·0333 ft, inlet and outlet areas = 0·0264 ft^2 and 0·0235 ft^2 respectively. Mean length of perimeter = P = 0·6 ft. The gases enter the nozzle at a temperature of 1,850°R and a pressure of 45 Lb/in.^{2} Weight of gases flowing = 0·73 Lb/sec. C_f = 0·005, R = 52·4 ft-Lb Farenheit units, c_p = 0·271 and γ = 1·4.

Calculate the values of v_1, v_2, p_2 and T_2.

Ans. v_1 = 437 ft/sec; v_2 = 497 ft/sec; p_2 = 42 Lb/in.^{2}; T_2 = 1,845°R.

4. Air flows through a duct at a supersonic speed of 570 ft/sec. A pressure jump occurs at a certain section which has an area of flow of $2\cdot082 \times 10^{-4}$ ft^2. The temperature and pressure of the air before the jump are 120·5°R and 8·56 Lb/in.^{2} The weight of air flowing is 0·0212 Lb/sec. Calculate the temperature, pressure and velocity of the air after the jump occurs.

Ans. T = 132°R; p = 11·5 Lb/in.^{2}; v = 422 ft/sec.

5. The following particulars apply to a short section of the rotor blade passage of a cold air expansion turbine.

$W = 0{\cdot}0212$ Lb/sec	Mean $r = 0{\cdot}1135$ ft
Length of section $= 0{\cdot}0104$ ft	$dr = 0{\cdot}007$ ft
Area at inlet $= 2{\cdot}03 \times 10^{-4}$ ft^2	$c_p = 0{\cdot}24$
Area at outlet $= 1{\cdot}663 \times 10^{-4}$ ft^2	Inlet temperature $= 158{\cdot}2$°R
Mean length of perimeter $= 0{\cdot}0592$ ft	Inlet pressure $= 22{\cdot}4$ Lb/in.^{2}
$C_f = 0{\cdot}01$	Inlet relative velocity $= 273$ ft/sec
$\omega = 39{\cdot}6 \times 10^6$ rad/sec	

Calculate the temperature, pressure and relative velocity of the air leaving the section.

Ans. $T_2 = 161$°R; $p_2 = 23{\cdot}85$ Lb/in.^{2}; $v_2 = 318$ ft/sec.

6. Air entering a rotary air compressor is drawn through a diverging duct having an inlet area of $0{\cdot}73$ ft^2 and an exit area of $1{\cdot}25$ ft^2. The stagnant air in the vicinity of the inlet end of the duct has a pressure of $14{\cdot}7$ Lb/in.^{2} and a temperature of 15°C. Neglecting frictional losses, calculate the velocities, temperatures and pressures of the air flowing at the inlet and outlet ends of the duct if the weight of air flow is 30 Lb/sec.

Ans. $\begin{cases} \text{Inlet end: } v_1 = 800 \text{ ft/sec; } T_1 = 258{\cdot}5\text{°K; } p_1 = 8{\cdot}8 \text{ Lb/in.}^2 \\ \text{Outlet end: } v_2 = 374 \text{ ft/sec; } T_2 = 281\text{°K; } p_2 = 11{\cdot}9 \text{ Lb/in.}^2 \end{cases}$

APPENDIX 1

VALUES OF COEFFICIENTS FOR PIPE FLOW, OPEN CHANNELS, ORIFICES, VENTURI METER AND WEIRS

(Compiled from various published results of tests. The values apply to water only.)

1. Values of Coefficient f for Pipe Flow Formula

$$h_f = \frac{4flv^2}{2gd}$$

(Intermediate values of diameters and velocities may be interpolated.)

NEW CAST-IRON PIPES AND SIMILAR SURFACES

Diameter (in.)	Velocity (ft/sec)						
	1·0	2·0	3·0	4·0	6·0	10·0	15·0
1	—	—	0·0075	0·0074	0·0072	0·0070	0·0068
2	0·0071	0·0069	0·0068	0·0067	0·0065	0·0064	0·0062
3	0·0067	0·0065	0·0064	0·0063	0·0061	0·0060	0·0059
6	0·0061	0·0059	0·0058	0·0057	0·0055	0·0054	0·0053
9	0·0057	0·0055	0·0054	0·0053	0·0052	0·0050	0·0049
12	0·0054	0·0052	0·0051	0·0050	0·0049	0·0048	0·0047
18	0·0052	0·0051	0·0050	0·0048	0·0045	0·0041	0·0040
24	0·0050	0·0048	0·0045	0·0043	0·0040	0·0037	0·0036
36	0·0042	0·0040	0·0037	0·0036	0·0035	0·0033	0·0032
48	0·0036	0·0034	0·0032	0·0032	0·0031	0·0029	0·0028
60	0·0032	0·0031	0·0030	0·0030	0·0029	0·0028	0·0027

CLEAN ASPHALTED PIPES

Diameter (in.)	Velocity (ft/sec)					
	1·0	2·0	3·0	4·0	6·0	8·0
6	0·0102	0·0094	0·0090	0·0086	0·0081	0·0079
9	0·0080	0·0074	0·0070	0·0067	0·0064	0·0060
12	0·0067	0·0062	0·0059	0·0056	0·0053	0·0050
18	0·0056	0·0051	0·0048	0·0046	0·0043	0·0042
24	0·0050	0·0046	0·0044	0·0042	0·0039	0·0038
30	0·0047	0·0043	0·0041	0·0039	0·0037	0·0036
36	0·0044	0·0040	0·0038	0·0036	0·0035	0·0034
48	0·0039	0·0037	0·0035	0·0033	0·0031	0·0029

Smooth Pipes

Temperature 60°F (Colebrook and White).* Glass, drawn lead and brass, bitumen-lined steel or concrete pipes over 4 ft in diameter.

Diameter (in.)	Velocity (ft/sec)								
	1·0	2·0	3·0	4·0	5·0	6·0	8·0	10·0	20·0
1	0·0083	0·0072	0·0067	0·0061	0·0058	0·0055	0·0052	0·0049	0·0043
3	0·0066	0·0055	0·0050	0·0047	0·0045	0·0043	0·0042	0·0038	0·0034
6	0·0055	0·0047	0·0043	0·0042	0·0038	0·0038	0·0035	0·0034	0·0030
12	0·0047	0·0042	0·0038	0·0035	0·0034	0·0033	0·0032	0·0030	0·0027
18	0·0044	0·0037	0·0035	0·0033	0·0032	0·0030	0·0029	0·0028	0·0025
24	0·0042	0·0035	0·0033	0·0032	0·0030	0·0029	0·0028	0·0027	0·0024
36	0·0037	0·0033	0·0031	0·0029	0·0028	0·0027	0·0026	0·0025	0·0023
48	0·0035	0·0032	0·0029	0·0028	0·0027	0·0026	0·0025	0·0024	—
60	0·0034	0·0030	0·0028	0·0027	0·0026	0·0025	0·0024	0·0023	—

Old Cast-iron Pipes

Diameter (in.)	Velocity (ft/sec)						
	1·0	2·0	3·0	4·0	6·0	10·0	15·0
3	0·0152	0·0145	0·0139	0·0135	0·0128	0·0122	0·0120
6	0·0135	0·0126	0·0117	0·0114	0·0108	0·0103	0·0100
9	0·0122	0·0119	0·0115	0·0110	0·0100	0·0092	0·0090
12	0·0108	0·0102	0·0096	0·0094	0·0089	0·0084	0·0080
18	0·0087	0·0083	0·0078	0·0076	0·0073	0·0069	0·0067
24	0·0076	0·0072	0·0067	0·0065	0·0063	0·0060	0·0059
36	0·0061	0·0059	0·0056	0·0054	0·0052	0·0050	0·0049
48	0·0057	0·0054	0·0051	0·0050	0·0049	0·0046	0·0045

* "The reduction of carrying capacity of pipes with age," by C. F. Colebrook and C. M. White, *J. Instn. C. Engrs.* (1937–1938), p. 99; and "Experiments with fluid friction in roughened pipes," by Colebrook and White, *Proc. Roy. Soc.*, **161**.

2. Values of Constant for Flow in Open Channels

For flow in open channels

$$v = C\sqrt{mi}$$

where

$$C = \frac{157\cdot5}{1 + \frac{k}{\sqrt{m}}}$$

VALUES OF k FOR CHANNELS

Type of Lining	k
Concrete-lined Channels—	
Smooth finished surface	0 to 0·1
Surface due to planed shuttering	0·1 to 0·45
Rough concrete	1·7 to 2·7
Masonry and Stoneware Channels—	
Vitrified sewer pipe	0 to 0·1
Glazed brickwork	0·1 to 0·4
Brickwork in cement mortar	0·1 to 0·55
Rubble masonry	0·55 to 1·9
Dry rubble	1·9 to 2·9
Timber-lined Channels—	
Smooth planed boards	0 to 0·4
Rough boards	0·1 to 0·45
Old timber	0·45 to 0·6
Steel-plate Channels—	
With countersunk rivets and flush joints	0 to 0·1
Projecting joints and rivets	0·4 to 0·9
Artificial Channels—	
Smooth rock surface	1·9 to 2·9
Rough rock surface	3·0 to 3·7
Gravel	1·8 to 2·0
Sand and gravel	1·7 to 1·9
Smooth earth surface	0·9 to 1·8

3. Coefficient of Discharge for Sharp-edged Circular Orifices

Head (ft)	Diameter of Orifice (in.)										
	0·24	0·48	0·75	1·0	1·5	2·0	2·5	4·8	7·2	9·6	12·0
0·2	—	—	0·684	0·645	0·617	0·611	0·609	—	—	—	—
0·4	—	—	0·675	0·640	0·615	0·609	0·607	—	—	—	—
0·8	0·649	0·626	0·666	0·636	0·613	0·607	0·606	—	—	—	—
1·0	0·644	0·623	0·662	0·635	0·612	0·607	0·606	0·597	0·595	0·593	0·591
1·2	0·642	0·621	0·659	0·634	0·612	0·607	0·606	0·598	0·596	0·594	0·593
1·6	0·638	0·618	0·654	0·632	0·612	0·607	0·606	0·598	0·596	0·595	0·594
2·0	0·635	0·616	0·651	0·630	0·611	0·607	0·606	0·599	0·597	0·596	0·595
4·0	0·624	0·611	0·641	0·627	0·611	0·607	0·606	0·598	0·597	0·597	0·596
6·0	0·618	0·607	0·636	0·626	0·611	0·607	0·606	0·598	0·597	0·596	0·596
8·0	0·613	0·605	0·635	0·626	0·611	0·607	0·606	0·597	0·596	0·596	0·595
10·0	0·609	0·604	0·635	0·625	0·611	0·607	0·606	0·597	0·596	0·596	0·595
20·0	0·601	0·599	0·635	0·625	0·611	0·607	0·606	0·596	0·596	0·595	0·594
100·0	0·593	0·592	0·634	0·624	0·611	0·607	0·606	0·592	0·592	0·592	0·592

4. Coefficient of Discharge for Square Orifices

Head (ft)	Size of Square Orifice (in.)				
	0·24 × 0·24	0·48 × 0·48	0·84 × 0·84	1·44 × 1·44	2·4 × 2·4
0·4	—	0·643	0·628	0·616	—
0·6	0·660	0·636	0·623	0·613	0·605
1·0	0·648	0·628	0·618	0·610	0·605
6·0	0·623	0·612	0·607	0·605	0·604
20·0	0·606	0·604	0·602	0·602	0·602
100·0	0·599	0·598	0·598	0·598	0·598

5. Coefficient of Discharge for Rectangular Weirs

VALUES OF CONSTANT $m\sqrt{2g}$ OF BAZIN'S FORMULA

$$Q = m\sqrt{2g}LH^{3/2}$$

H (ft)	Height of Crest of Weir above Bed of Channel (ft)						
	1·5	2·0	3·0	4·0	6·0	8·0	10·0
0·4	3·53	3·50	3·47	3·46	3·45	3·45	3·45
0·6	3·53	3·48	3·44	3·41	3·40	3·39	3·39
0·8	3·57	3·50	3·30	3·40	3·37	3·36	3·36
1·0	3·61	3·53	3·44	3·40	3·37	3·35	3·34
1·2	3·67	3·57	3·47	3·41	3·37	3·35	3·34
1·4	—	3·62	3·49	3·43	3·37	3·35	3·34
1·6	—	3·66	3·52	3·45	3·38	3·35	3·34
1·8	—	—	3·55	3·47	3·39	3·36	3·34
2·0	—	—	3·58	3·49	3·40	3·36	3·34

6. Value of Coefficient k for Venturi Meter

$$Q = kc\sqrt{h}$$

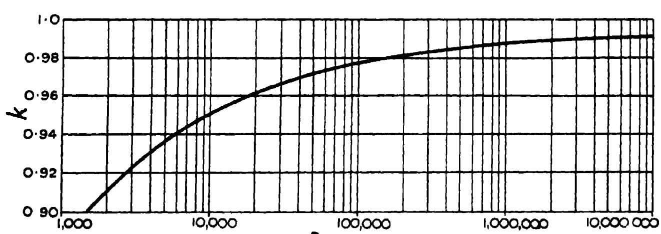


FIG. 360

APPENDIX 2

VALUES OF COEFFICIENT OF VISCOSITY FOR WATER, AIR, CARBON DIOXIDE AND STEAM

(Values of η are in engineers' units)

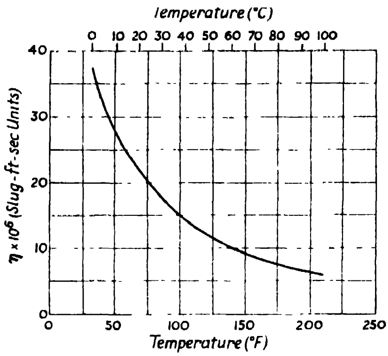


FIG. 361. COEFFICIENT OF VISCOSITY FOR WATER

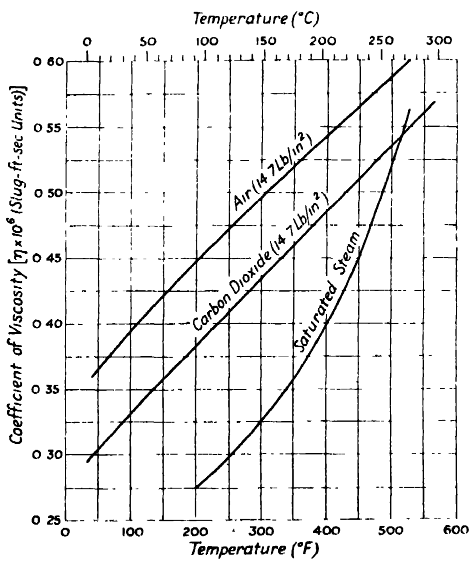


FIG. 362. COEFFICIENTS OF VISCOSITY FOR AIR, CARBON DIOXIDE AND STEAM

24—(T.5167)

APPENDIX 3

PROPERTIES OF ATMOSPHERIC AIR

VALUES OF SPECIFIC HEATS, TOTAL HEAT, INTERNAL

Temperature (°R)	Specific Heat at Constant Pressure c_p		Specific Heat at Constant Volume c_V	
	Instantaneous	Mean from 32°F	Instantaneous	Mean from 32°F
400	0·2390	0·2395	0·1700	0·1705
420	0·2392	0·2396	0·1702	0·1706
440	0·2394	0·2397	0·1704	0·1707
460	0·2396	0·2398	0·1706	0·1708
480	0·2398	0·2399	0·1708	0·1709
492	0·2400	0·2400	0·1710	0·1710
500	0·2400	0·2400	0·1710	0·1710
550	0·2402	0·2403	0·1716	0·1713
600	0·2406	0·2406	0·1721	0·1717
650	0·2412	0·2409	0·1727	0·1721
700	0·2418	0·2412	0·1733	0·1725
750	0·2427	0·2416	0·1742	0·1729
800	0·2436	0·2420	0·1751	0·1734
850	0·2448	0·2424	0·1763	0·1739
900	0·2460	0·2429	0·1775	0·1744
950	0·2474	0·2433	0·1789	0·1750
1,000	0·2488	0·2438	0·1803	0·1756
1,100	0·2518	0·2449	0·1833	0·1769
1,200	0·2549	0·2461	0·1864	0·1783
1,300	0·2581	0·2474	0·1896	0·1798
1,400	0·2613	0·2487	0·1928	0·1813
1,500	0·2644	0·2501	0·1959	0·1828
2,000	0·2775	0·2573	0·2090	0·1890
2,500	0·2866	0·2638	0·2181	0·1952
3,000	0·2931	0·2690	0·2246	0·2000
3,500	0·2980	0·2732	0·2294	0·2050
4,000	0·3018	0·2775	0·2333	0·2090
4,500	0·3049	0·2805	0·2364	0·2125
5,000	0·3075	0·2837	0·2389	0·2150
5,500	0·3097	0·2861	0·2411	0·2178

ENERGY, RATIO OF SPECIFIC HEATS AND SONIC VELOCITY

Total Heat or Enthalpy H above 32°F (B.Th.U.)	Internal Energy U above 32°F (B.Th.U.)	Mean γ from 32°F to T	Instantaneous γ	Sonic Velocity (ft/sec)	Temp. (°K)
− 22·05	− 15·69	1·404	1·405	982	222
− 17·24	− 12·27	1·404	1·405	1,006	233
− 12·45	− 8·88	1·403	1·404	1,030	244
− 7·68	− 5·47	1·403	1·404	1,052	256
− 2·88	− 2·05	1·403	1·403	1,076	267
0	0	1·403	1·403	1,090	273
1·92	1·37	1·403	1·403	1,097	278
13·95	9·80	1·401	1·400	1,147	306
25·98	18·55	1·400	1·398	1,200	333
38·00	27·1	1·399	1·397	1,250	361
50·00	35·90	1·398	1·395	1,291	389
62·05	44·80	1·396	1·393	1,340	416
74·60	53·40	1·395	1·391	1,384	444
86·25	62·20	1·393	1·389	1,428	472
99·10	71·20	1·391	1·386	1,470	500
110·70	80·00	1·390	1·383	1,507	528
123·70	89·30	1·388	1·380	1,545	555
148·70	107·50	1·384	1·374	1,618	611
174·20	126·20	1·380	1·368	1,683	667
200	145·20	1·376	1·361	1,748	722
226	164·50	1·372	1·355	1,807	778
255	184·20	1·367	1·350	1,864	833
388	285	1·360	1·328	2,140	1,110
528	390·50	1·350	1·314	2,389	1,390
674	500	1·345	1·305	2,606	1,665
823	613	1·330	1·299	2,800	1,940
973	730	1·328	1·294	2,985	2,220
1,123	850	1·320	1·290	3,160	2,500
1,275	968	1·318	1·287	3,325	2,775
1,430	1,090	1·314	1·285	3,487	3,060

INDEX